TRAJECTORIES IN NON-INERTIAL FRAMES

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In această lucrare sunt prezentate câteva probleme de dinamică în sisteme neinerţiale. Sunt studiate mişcarea punctelor materiale şi mişcarea corpurilor în câmp gravitaţional. Sunt scrise ecuaţiile Lagrange pentru mişcarea relativă în raport cu un sIstem de referinţă cu traiectorie cunoscută şi este stabilită o ecuaţie de tip Binet pentru mişcarea relative.

Când mişcarea unui sistem de corpuri care compun o staţie orbitală de mari dimensiuni este descrisă cu sisteme de referinţă care au originea în centrul atractiv (Pamantul), problema integrarii ecuaţiilor de mişcare prezintă unele dificultăţi, deoarece unele coordonate (ca razele vectoare) au valori foarte mari, iar altele (ca distanţele între corpuri) au valori foarte mici. Unele dificultăţi pot fi evitate, dacă mişcarea relativă a sistemului este studiată în raport cu un sistem de referintă cu mişcare cunoscută. Studiul mişcării relative nu este impus de considerente de integrare, acesta este impus de aspecte practice.

Modelele şi metoda elaborată permit rezolvarea unui numă mare de probleme de dinamică sistemelor în camp gravitaţional.

Some problems of dynamics in non-inertial frames are presented in this paper. Motion of particles in gravitational field are studied. Lagrange equations for relative motion with respect to a reference frame with known trajectory are written and a Binet type equation for relative motion is established.

When the motion of a system of bodies which compos a large orbital station is described within reference frames having the origin in the center of the attractive body (Earth), the problem of integration of motion equations presents some difficulties, because some coordinates (like the vector radii) have very great values, and others (like distances between bodies) have very small values. Some difficulties can be avoided if relative motion of the system is studied with respect to a reference frame with a known motion. Relative motion study isn't imposed by integration considerations; it is imposed by practical aspects. The models and the elaborated method allow solving a large number of problems of systems dynamics in gravitational field.

Keywords: dynamics, relative motion, non-inertial frames.

1. Introduction

The non-inertial reference frame Oxyz is moving with respect to the inertial reference frame $O_1x_1y_1z_1$ (Fig. 1), so that the Oxy plane coincides with the $O_1x_1y_1$ plane, during the motion.

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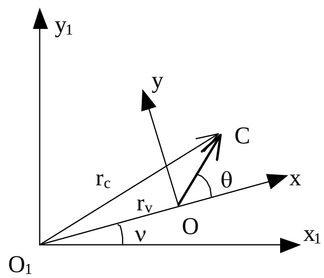

Fig.1. References frames

The polar coordinates r_v, ν are used in the inertial frame and the cylindrical coordinates ρ, θ, z,- in the moving frame.

Motion equations can be obtained by the theorem of linear momentum with respect to the moving frame, considering the relative acceleration, the acceleration of transport and the Coriolis acceleration or by specific equations of the analytical mechanics. The second method is preferred. It leads to a compact form of the equations and prime integrals can be found.

2. Motion Equations

Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_k}\right) - \frac{\partial E}{\partial q_k} = Q_k \quad k = 1,2,\ldots,n \tag{1.1}$$

are used.

The coordinates of the mass center C in an inertial reference frame are:

$$x_{1c} = r_v \cos\nu + \rho\cos(\nu+\theta), \quad y_{1c} = r_v \sin\nu + \rho\sin(\nu+\theta), \quad z_{1c} = z \tag{1.2}$$

The kinetic energy is

$$E = \frac{m}{2}\left(\dot{x}_{1c}^2 + \dot{y}_{1c}^2 + \dot{z}_{1c}^2\right) == \frac{m}{2}[\dot{r}_v^2 + r_v^2\dot{\nu}^2 + \dot{\rho}^2 + \rho^2\left(\dot{\nu}+\dot{\theta}\right)^2 + \dot{z}^2$$
$$+ 2\dot{\rho}\left(\dot{r}_v\cos\theta + r_v\dot{\nu}\sin\theta\right) + 2\rho\left(\dot{\nu}+\dot{\theta}\right)\left(-\dot{r}_v\sin\theta + r_v\dot{\nu}\cos\theta\right)] \tag{1.3}$$

Using the function

$$\Phi = \dot{r}_v\cos\theta + r_v\dot{\nu}\sin\theta = \Phi\left(r_v,\theta,\dot{r}_v,\dot{\nu}\right), \tag{1.4}$$

which represents the projection of O, the origin of the moving frame, on the radius OC**,** the kinetic energy becomes

$$E = \frac{m}{2}[\dot{r}_v^2 + r_v^2\dot{\nu}^2 + \dot{\rho}^2 + \rho^2\left(\dot{\nu}+\dot{\theta}\right)^2 + \dot{z}^2 + 2\dot{\rho}\Phi + 2\rho\left(\dot{\nu}+\dot{\theta}\right)\frac{\partial\Phi}{\partial\theta}] \tag{1.3'}$$

With the above expressions and function Φ given by (1.4), the motion equations (1.1) become:

$$\ddot{\rho}-\rho\left(\dot{\nu}+\dot{\theta}\right)^2+\frac{d\,\Phi}{dt}-\left(\dot{\nu}+\dot{\theta}\right)\frac{\partial\Phi}{\partial\theta}=\frac{1}{m}Q_{\rho},$$
$$\frac{d}{dt}\left[\rho^2\left(\dot{\nu}+\dot{\theta}\right)\right]+\rho\frac{d}{dt}\left(\frac{\partial\Phi}{\partial\theta}\right)+\rho\left(\dot{\nu}+\dot{\theta}\right)\Phi=\frac{1}{m}\,Q_{\theta},\ \ddot{z}=\frac{1}{m}Q_z \tag{1.5}$$

2.1. Generalized Forces

Force function which corresponds to the mass center M and to an orbital station of mass m_v is

$$U=\frac{f\,M\,m}{\left(r_v^2+\rho^2+2r_v\rho\cos\theta+z^2\right)^{\frac{1}{2}}}+\frac{f\,m_v\,m}{\rho} \tag{1.6}$$

Generalized forces are:

$$Q_{\rho}=-\frac{\left(\rho+r_v\cos\theta\right)f\,M\,m}{\left(r_v^2+\rho^2+2r_v\cos\theta+z^2\right)^{3/2}}-\frac{f\,m_v\,m}{\rho^2},\quad Q_{\theta}=\frac{f\,M\,m r_v\,\rho\sin\theta}{\left(r_v^2+\rho^2+2r_v\,\rho\cos\theta+z^2\right)^{3/2}},$$
$$Q_z=-\frac{z\,f\,M\,m}{\left(r_v^2+\rho^2+2r_v\,\rho\cos\theta+z^2\right)^{3/2}}. \tag{1.7}$$

For

$$\rho << r_v \quad and \quad z << r_v \tag{1.8}$$

relations (1.7) may be approximated by:

$$Q_{\rho}\approx-\frac{f\,M\,m\left(r_v\cos\theta+\rho\right)}{r_v^3}-\frac{f\,m_v\,m}{\rho^2},\quad Q_{\theta}\approx\frac{f\,M\,m r_v\,\rho\sin\theta}{r_v^3},\quad Q_z\approx-\frac{z\,f\,M\,m}{r_v^3}. \tag{1.7'}$$

2.2. Particular Motions of the Moving Frame

Equations (1.5) can be used for any plane curve $r_v=r_v(\nu)$, which is described by the origin O of the moving system. In some cases, the relative motion within frames with a particular motion, which corresponds to practical problems, are analyzed.

2.2.1. Relative Motion with Respect to a Frame in Rotation

In this case the origin O of the moving reference frame coincides with the origin O_1 of the fix**ed** (inertial) frame and the relations from bellow can be written:

$$r_v=0,\quad \dot{r}_v=0 \tag{1.9}$$

Function Φ, given by (1.4) becomes

$$\Phi=0 \tag{1.4'}$$

With the above relations, from (1.5) it follows:

$$\ddot{\rho} - \rho\left(\dot{\nu} + \dot{\theta}\right)^2 = \frac{1}{m}Q_\rho, \quad \frac{d}{dt}\left[\rho^2\left(\dot{\nu} + \dot{\theta}\right)\right] = \frac{1}{m}Q_\theta, \quad \ddot{z} = \frac{1}{m}Q_z \tag{1.10}$$

Equations from [3, pg. 289], which are used in the study of the relative motion with respect to the Earth-Moon system, performing a rotation with respect to the mass center of the system are found again.

2.2.2. Relative motion with respect to a frame with the origin on a circle

If the origin O of the moving frame is moving on a circle of radius r_v, with the angular velocity $\dot{\nu} = n$, the function (1.4) becomes

$$\Phi = r_v n \cos\theta = \Phi(\theta), \tag{1.11}$$

and the equations (1.5) are written:

$$\ddot{\rho} - \rho\left(n + \dot{\theta}\right)^2 - r_v n^2 \cos\theta = \frac{1}{m}Q_\rho, \quad \frac{d}{dt}\left[\rho^2\left(n + \dot{\theta}\right)\right] + \rho r_v n^2 \sin\theta = \frac{1}{m}Q_\theta, \quad \ddot{z} = \frac{1}{m}Q_z \tag{1.12}$$

The above equations can be used in the study of the body motion with respect to a large vehicle which is not influenced by the presence of the first body.

With the above conditions, kinetic energy given by (1.3’) becomes

$$E = \frac{m}{2}[r_v^2 n^2 + \dot{\rho}^2 + \rho^2\left(n + \dot{\theta}\right)^2 + \dot{z}^2 + 2\dot{\rho} r_v n \sin\theta + 2\rho\left(n + \dot{\theta}\right) r_v n \cos\theta] \tag{1.13}$$

If a U force function exists, so that:

$$Q_\rho = \frac{\partial U}{\partial \rho}, \quad Q_\theta = \frac{\partial U}{\partial \theta}, \quad Q_z = \frac{\partial U}{\partial z}, \tag{1.14}$$

Jacobi’s prime integral

$$E_2 - E_0 - U = h \tag{1.15}$$

becomes, for (1.12) system

$$\frac{m}{2}\left[\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2 - n^2\left(r_v^2 + \rho^2 + 2\rho r_v \cos\theta\right)\right] - U = h. \tag{1.16}$$

For $r_v = 0$, prime integrals which correspond to the motion with respect to the system in rotation, from [3, pag.289], are obtained.

For some problems, the independent variable t is replaced with the independent variable ν. Derivative of a function f is

$$\frac{df}{dt} = \frac{df}{d\nu}\frac{d\nu}{df} = n\frac{df}{d\nu} \tag{1.17}$$

and (1.12) equations become

$$\frac{d^2\rho}{d\nu^2}-\rho\left(1+\frac{d\theta}{d\nu}\right)^2-r_\nu\cos\theta=\frac{1}{mn^2}Q_\rho,\quad \frac{d}{d\nu}\left[\rho^2\left(1+\frac{d\theta}{d\nu}\right)\right]+\rho r_\nu\sin\theta=\frac{1}{mn^2}Q_\theta,$$
$$\frac{d^2z}{d\nu^2}=\frac{1}{mn^2}Q_z \tag{1.18}$$

3. Binet's Equation in Relative Motion

If generalized forces (1.7') are replaced in (1.18) equations, it follows:

$$\frac{d^2\rho}{d\nu^2}-\rho\left(1+\frac{d\theta}{d\nu}\right)^2-r_\nu\cos\theta=-\frac{1}{n^2}\left(\frac{fM(r_\nu\cos\theta+\rho)}{r_\nu^3}+\frac{fm_\nu}{\rho^2}\right),$$
$$\frac{d}{d\nu}\left[\rho^2\left(1+\frac{d\theta}{d\nu}\right)\right]+\rho r_\nu\sin\theta=\frac{1}{n^2}\frac{fMr_\nu\rho\sin\theta}{r_\nu^3},\quad \frac{d^2z}{d\nu^2}=-\frac{1}{n^2}\frac{fMz}{r_\nu^3} \tag{1.19}$$

Solution of the last (1.19) equation is

$$z=C_z\cos\left(\frac{1}{n}\sqrt{\frac{fM}{r_\nu^3}}\nu-\nu_0\right). \tag{1.20}$$

For circular orbits,

$$\frac{1}{n}\sqrt{\frac{fM}{r_\nu^3}}=1 \tag{1.21}$$

and (1.20) becomes

$$z=C_z\cos(\nu-\nu_0). \tag{1.20'}$$

With (1.21) condition, the first two (1.19) equations become:

$$\frac{d^2\rho}{d\nu^2}-\rho\left(1+\frac{d\theta}{d\nu}\right)^2=-\frac{1}{n^2}\left(\frac{fM\rho}{r_\nu^3}+\frac{fm_\nu}{\rho^2}\right),\quad \frac{d}{d\nu}\left[\rho^2\left(1+\frac{d\theta}{d\nu}\right)\right]=0. \tag{1.22}$$

From (1.22) it follows

$$\rho^2\left(1+\frac{d\theta}{d\nu}\right)=C_\rho. \tag{1.23}$$

Using the first (1.22) relation and (1.23) relation, the path of relative motion is searched, in the form

$$\rho=\rho(\theta) \tag{1.24}$$

With (1.23) relation, the derivative of a function f with respect to the variable ν is written

$$\frac{df}{d\nu}=\frac{df}{d\theta}\frac{d\theta}{d\nu}=\left(\frac{C_\rho}{\rho^2}-1\right)\frac{df}{d\theta}\,. \tag{1.25}$$

The first (1.16) equation becomes

$$\frac{1}{\rho^2}\left(C_\rho-\rho^2\right)^2\frac{d^2\left(\frac{1}{\rho}\right)}{d\theta^2}+2\rho\left(C_\rho-\rho^2\right)\left(\frac{d\left(\frac{1}{\rho}\right)}{d\theta}\right)^2+\frac{C_\rho^2}{\rho^3}=\rho+\frac{1}{M^2}\frac{f m_\nu}{\rho^2} \tag{1.26}$$

With notation

$$u=\frac{1}{\rho} \tag{1.27}$$

The (1.26) equation becomes:

$$u^2\left(C_\rho-\frac{1}{u^2}\right)^2\frac{d^2u}{d\theta}+\frac{2}{u}\left(C_\rho-\frac{1}{u^2}\right)\left(\frac{du}{d\theta}\right)^2+C_\rho^3u^3=\frac{1}{u}+\frac{f m_\nu}{n^2}u^2 \tag{1.28}$$

The trajectory of relative motion in the form (1.24) is obtained by integration of the above equation.

4. Motion of the System of Two Particles

In the case of a system formed by two material points which are connected through a negligible mass tether, force function is given by

$$U=\frac{\mu m_1}{r_1}+\frac{\mu m_2}{r_2}\,, \tag{1.29}$$

and

$$U_2=\frac{\mu m}{r_\nu}-\frac{\mu m^* S^2}{2r_\nu^3}\left(1-3\cos^2\theta\cos^2\varphi\right) \tag{1.30}$$

is obtained,
where $S=S_1+S_2$ is the distance between the two points of the system, $m=m_1+m_2$ is the system mass, and

$$m^*=\frac{m_1m_2}{m_1+m_2} \tag{1.31}$$

Using the relation for the force function U_2, motion equations for the studied system in S, θ, and φ coordinates can be obtained. In the case of a circular orbit the following equations can be written:

$$S'' - S\left[\phi'^2 + (1+\theta')^2\cos^2\phi - 1 + 3\cos^2\theta\cos^2\phi\right] = \frac{1}{m^* n^2} Q_\rho^*$$

$$\frac{d}{d\nu}\left[S^2(1+\theta')\cos^2\phi\right] + 3S^2\sin\theta\cos\theta\cos^2\phi = \frac{1}{m^* n^2} Q_\theta^* \quad (1.32)$$

$$\frac{d}{d\nu}\left(S^2\phi'\right) + S^2(1+\theta')^2\sin\phi\cos\phi + 3S^2\cos^2\theta\sin\phi\cos\phi = \frac{1}{m^* n^2} Q_\phi^*$$

In the fig. 2 variation of the ratio between length of the tether and initialy length of the tether with respect to number of mass center rotations in the orbit are presented . Comparation between results obtained in this case and results obtained from the second equation (1.32), on the basis of a linear model, show a more rapid deployement of the tether than in a case of a linear model. Diference betweent these two cases is espected because linearization of equations leads to loosing of the inertial forces which favorise a rapid deployement of the tether. In the figure 3 deployement with a constant strenght in the tether is presented.

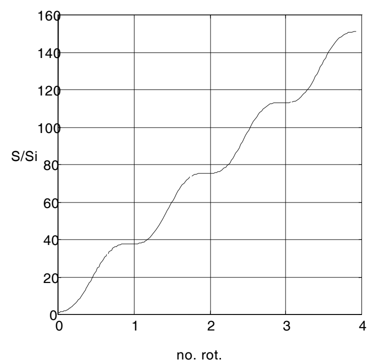

Fig. 2 Relative length variation of the tether (without strength)

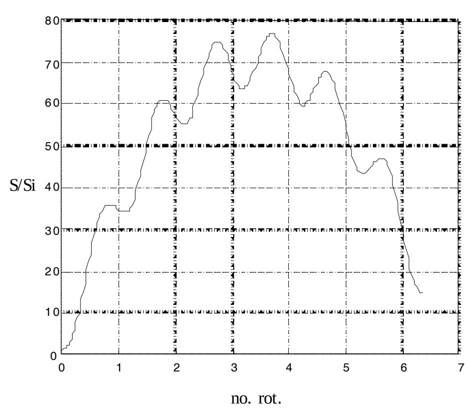

Fig. 3 Relative length variation of the tether (constant strength

5. Conclusions

Relative motion of material points in non-inertial frames with known path is studied in this paper and interesting results are obtained.
Equations and prime integrals from celestial mechanics ([3]) are found again.

Well known Binet's equation is written in inertial reference frame. Similar equation for the motion in non-inertial frame is established in this paper.

Results presented in figs. 2 and 3 can be useful in dynamics of tethered satellites.

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