MULTIPLE (n,m)-HYBRID LAPLACE TRANSFORM. PART III: APPLICATIONS TO MULTIDIMENSIONAL HYBRID SYSTEMS

Valeriu PREPELIȚĂ[1], Tiberiu VASILACHE[2]

În lucrare sunt prezentate aplicații ale transformării hibride multiple de tip Laplace şi z studiate în partea I şi Partea a II-a. Această transformare este utilizată pentru rezolvarea unor ecuații multidimensionale diferențiale cu diferențe şi integrale. Se deduc soluțiile unor astfel de probleme care apar în teoria aşteptării.

Cele mai importante aplicații ale acestei transformări Laplace multiple se referă la posibilitatea utilizării metodelor frecvențiale la sistemele de comandă multidimensional hibride. Se obțin matricele de transfer ale diferitelor clase de astfel de sisteme, incluzând modele de tip Roesser, Fornasini-Marchesini şi Attasi.

Some applications of the multiple hybrid Laplace and z-type transform studied in Parts I and II are presented. This transform is used to solve multidimensional differential-difference and integral equations. The solutions of such problems which appear in Queueing theory are derived.

The most important applications of this multiple Laplace transform refer to the possibility to apply frequency-domain methods to multidimensional hybrid control systems. Transfer matrices of different classes of such systems are derived, including Roesser, Fornasini-Marchesini and Attasi type models.

Key words: multiple hybrid Laplace and z-type transform, multidimensional differential-difference and integral equations, queueing systems, 2D Roesser models, 2D Attasi models

1. Introduction

A distinct branch of Systems and Control theory is represented by the continuous-discrete multidimensional systems which appear as models in many domains such as image processing, computer tomography, geophysics, in the study of linear repetitive processes [2], [4], [15] or in the iterative learning control synthesis [8]. Such hybrid systems were studied in [5], [6], [7], [10], [11].

In the theory of "classical" 1D systems the frequency domain methods, based on Laplace transform in the continuous case or on z-transform in the

[1] Prof., Mathematics and Informatics Department I, University POLITEHNICA of Bucharest, Romania, e-mail: vprepelita@mathem.pub.ro

[2] Lecturer, Mathematics and Informatics Department I, University POLITEHNICA of Bucharest, Romania

discrete-time case, play a very important role. In order to extend the frequency domain methods to multiple hybrid systems, a multiple (n,m)-Hybrid Laplace Transform was defined in Parts I and II [12], [13] and its main properties were proved, including time-delay theorems, translation, differentiation and difference of the original, differentiation of the image, integration and sum of the original, integration of the image, convolution, product of originals, initial and final values. Some methods and formulas for determining the original of a given (n,m)-Hybrid Laplace transform were given.

The aim of this paper is to present the applications of this multiple hybrid Laplace transform in the study of multidimensional continuous-discrete systems or for solving multiple hybrid differential-difference and integral equations.

In Section 2 the definitions of the original functions and of the (n,m)-Hybrid Laplace Transform are recalled.

In Section 3 it is shown that the hybrid Laplace transform can be used to solve multiple differential-difference and integral equations. This approach is applied to solve such equations which appear in stochastic processes and in the Queueing theory.

Section 4 is devoted to the applications of the hybrid Laplace transform to the study of multidimensional continuous-discrete systems. Transfer matrices are obtained for multiple hybrid Roesser type, Fornasini-Marchesini type and Attasi type models (which generalize the 2D discrete-time models introduced in [1], [3] and [14]), including descriptor systems and systems with delays.

2. Multiple (n,m)-hybrid Laplace transform

We denote by $\langle n\rangle$ the set $\{1,2,\ldots,n\}$.

Definition 2.1. A function $f:\mathbf{R}^n\times\mathbf{Z}^m\to\mathbf{C}$ is said to be a *continuous-discrete original function* (or simply an *original*) if f has the following properties:

(i) $f(t_1,\ldots,t_n;k_1,\ldots,k_m)=0$ if $t_i<0$ or $k_j<0$ for some $i\in\langle n\rangle$ or $j\in\langle m\rangle$.

(ii) $f(\cdot,\ldots,\cdot;k_1,\ldots,k_m)$ is piecewise smooth on $\mathbf{R}_+^n$ for any $(k_1,\ldots,k_m)\in\mathbf{Z}_+^m$.

(iii) $\exists\, M_j>0$, $\sigma_{fi}\in\mathbf{R}$, $i\in\langle n\rangle$, $R_{fj}>0$, $j\in\langle m\rangle$ such that

$$|f(t_1,\ldots,t_n;k_1,\ldots,k_m)|\le M_f\exp\left(\sum_{i=1}^{n}\sigma_{fi}t_i\right)\prod_{j=1}^{m}R_{fj}^{k_j} \tag{2.1}$$

$\forall\, t_i > 0$, $i \in \langle n \rangle$, $\forall\, k_j \geq 0$, $j \in \langle m \rangle$.

We shall denote by $f(t;k) = f(t_1,\ldots,t_n;k_1,\ldots,k_m)$ the value of f at $t = (t_1,\ldots,t_n)$, $(k_1,\ldots,k_m)$ and similarly $F(s;z) = F(s_1,\ldots,s_n;z_1,\ldots,z_m)$.

Definition 2.2. For any original f, the function

$$F(s_1,\ldots,s_n;z_1,\ldots,z_m) = \int_0^\infty \cdots \int_0^\infty \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} f(t_1,\ldots,t_n;k_1,\ldots,k_m) \cdot \cdot e^{-s_1 t_1} \cdots e^{-s_n t_n} z_1^{-k_1} \cdots z_m^{-k_m} dt_1 \ldots dt_n \tag{2.2}$$

is called the (n,m)-*hybrid Laplace transform* ((n,m)-HLT) or the *image* of f.

We shall use the following notations: for some sets $\alpha = \{i_1,\ldots,i_p\} \subset \langle n \rangle$ and $\beta = \{j_1,\ldots,j_q\} \subset \langle m \rangle$, $|\alpha| = p$ (the cardinality of α), $|\beta| = q$, $E_\alpha = \{\varepsilon \mid \varepsilon \subset \alpha \text{ or } \varepsilon = \varnothing\}$, $E'_\beta = \{\delta \mid \delta \subset \beta \text{ or } \delta = \varnothing\}$; for $\varepsilon \in E_\alpha$ and $\delta \in E'_\beta$, $\tilde{\varepsilon} = \langle n \rangle \setminus \varepsilon$, $\tilde{\delta} = \langle m \rangle \setminus \delta$. If $\varepsilon = \{i\}$ or $\delta = \{j\}$, $\tilde{i}, \tilde{j}$ denote $\tilde{\varepsilon}$ and $\tilde{\delta}$ respectively.

Given $\alpha = \{i_1,\ldots,i_p\} \subset \langle n \rangle$, a p-tuple $(\gamma_{i_1},\ldots,\gamma_{i_p}\} \in \mathbf{N}^p$ is denoted by γ_α or simply by γ and $\dfrac{\partial^\gamma f}{\partial t^\gamma} = \dfrac{\partial^{\gamma_{i_1}+\cdots+\gamma_{i_p}}}{\partial t_{i_1}^{\gamma_{i_1}} \cdots \partial t_{i_p}^{\gamma_{i_p}}}$, $s^\gamma = s_{i_1}^{\gamma_{i_1}} \cdots s_{i_p}^{\gamma_{i_p}}$. The family of all unvoid subsets ε of α is denoted by E_γ^α or E_γ. For $\varepsilon \in E_\gamma^\alpha$, $\tilde{\varepsilon} = \alpha \setminus \varepsilon$, $s_\varepsilon^{\gamma_\varepsilon} = \prod_{i\in\varepsilon} s_i^{\gamma_i}$ and $s_{\tilde{\varepsilon}}^{\gamma_{\tilde{\varepsilon}}} = \prod_{i\in\tilde{\varepsilon}} s_i^{\gamma_i}$ and $s_{\tilde{\varepsilon}}^{\gamma_{\tilde{\varepsilon}}} = 1$ if $\varepsilon = \alpha$; if $\varepsilon = \{\iota_1,\ldots,\iota_p\}$ and $\eta_\varepsilon = (\eta_{\iota_1},\ldots,\eta_{\iota_p}\} \in \mathrm{N}^p$, $\eta_\varepsilon \leq \gamma_\varepsilon$ means $\eta_i \leq \gamma_i$, $\forall i \in \varepsilon$; $f(0_\varepsilon^+;k)$ denotes the limit from the right $f(t_1,\ldots,t_{\iota_1-1},0+,t_{\iota_1+1},\ldots,t_{\iota_p-1},0+,t_{\iota_p+1},\ldots,t_n;k_1,\ldots,k_m)$; if $\varepsilon = \{i\}$ then $f(0_\varepsilon^+;k)$ is denoted by $f(0_i^+;k)$. We denote by $(z-1)^\beta$ the product $\prod_{j\in\beta}(z_j - 1)$.

In the sequel we shall use the pair (q,r) instead of (n,m)**.**

3. Applications of the multiple (q,r)-hybrid Laplace transformation to multiple differential-difference and integral equations

Let f be an original function ; $j \in \langle r \rangle$, $l \in \mathbf{N}^*$, $\beta = \{j_1,\ldots,j_h\} \subset \langle r \rangle$ and $\theta = \{\theta_{j_1},\ldots,\theta_{j_h}\} \in (\mathbf{N}^*)^h$.

Definition 3.1. The *j-first difference* (*$(j,1)$-difference*) of f is the function

$$\Delta_j f(t;k)=\begin{cases}0 & \text{if } t_i<0 \text{ or } k_l<0 \text{ for some } i\in\langle q\rangle,\, l\in\langle r\rangle \\ f(t_1,\ldots,t_q;k_1,\ldots,k_{j-1},k_j+1,k_{j+1},\ldots,k_r)- & \\ -f(t_1,\ldots,t_q;k_1,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_r) & \text{otherwise.}\end{cases} \tag{3.1}$$

The (j,l)-*difference* of f is defined by induction by

$$\Delta_j^l f(t;k)=\Delta_j(\Delta_j^{l-1} f(t,k)). \tag{3.2}$$

The (β,θ)-*difference* of f $\Delta_\beta^\theta f$ or $\Delta^\theta f$ is defined by induction by

$$\Delta_\beta^\theta f(t;k)=\Delta_{j_h}^{\theta_{j_h}}\cdots\Delta_{j_1}^{\theta_{j_1}} f(t;k). \tag{3.3}$$

Let Γ be a subset of $\bigcup_{i=1}^{q}\mathbf{R}_+^i$ and Θ a subset of $\bigcup_{j=1}^{r}\mathbf{Z}_+^j$. For $\gamma=(\gamma_{i_1},\ldots,\gamma_{i_l})\in\Gamma$ and $\theta=(\theta_{j_1},\ldots,\theta_{j_h})\in\Theta$ we denote a coefficient $a_{\gamma_{i_1},\ldots,\gamma_{i_l};\theta_{j_1},\ldots,\theta_{j_h}}$ by $a_{\gamma\theta}$.

A *multiple differential-difference equation* has the form

$$\sum_{\gamma\in\Gamma}\sum_{\theta\in\Theta} a_{\gamma\theta}\frac{\partial^\gamma}{\partial t^\gamma}\Delta^\theta x(t;k)=f(t;k) \tag{3.4}$$

where $a_{\gamma\theta}\in\mathrm{R}$, $\forall\gamma\in\Gamma$, $\theta\in\Theta$; $x(t;k)$ is un unknown original function and $f(t;k)$ is a given original function.

We consider the boundary conditions

$$\frac{\partial^{\eta_\varepsilon} f}{\partial t^{\eta_\varepsilon}}(0_\varepsilon+;0_\zeta)=g_{\varepsilon,\zeta}(t_{\tilde\varepsilon};k_{\tilde\zeta}),\quad \varepsilon\in E_\gamma,\quad \zeta\in E_\theta \tag{3.5}$$

where t_α, k_β stand for $t_{\alpha_1},\ldots,t_{\alpha_l}$, $k_{\beta_1},\ldots,k_{\beta_h}$ if $\alpha=(\alpha_1,\ldots,\alpha_l)$, $\beta=(\beta_1,\ldots,\beta_h)$; $x(t;k+\theta)$ denotes $x(t;k_1,\ldots,k_{j_1-1},k_{j_1}+\theta_{j_1},k_{j_1+1},\ldots,k_{j_h-1},k_{j_h}+\theta_{j_h},k_{j_h+1},\ldots,k_r)$.

By using Definition 3.1, the equation (3.4) can be rewritten as

$$\sum_{\gamma\in\Gamma}\sum_{\theta\in\Theta} b_{\gamma\theta}\frac{\partial^\gamma}{\partial t^\gamma}x(t;k+\theta)=f(t,k). \tag{3.6}$$

By [12, Theorem 2.17] and by applying the (q,r)-HLT to equation (3.6) with boundary conditions (3.5), (3.6) is transformed into an algebraic equation having the solution $X(s;z)=\dfrac{F(s;z)-C(s;z)}{B(s;z)}$, where $B(s;z)$ and $C(s;z)$ are the polynomials

$$B(s;z)=\sum_{\gamma\in\Gamma}\sum_{\theta\in\Theta}b_{\gamma\theta}s^{\gamma}z^{\theta}=\sum_{\gamma\in\Gamma}\sum_{\theta\in\Theta}b_{\gamma_{i_1},\dots,\gamma_{i_l};\theta_{j_1},\dots,\theta_{j_h}}s_{i_1}^{\gamma_{i_1}}\cdots s_{i_l}^{\gamma_{i_l}}z_{j_1}^{\theta_{j_1}}\cdots z_{j_h}^{\theta_{j_h}}$$

and

$$C(s,z)=\sum_{\gamma\in\Gamma}\sum_{\theta\in\Theta}b_{\gamma\theta}z^{\theta}\sum_{\varepsilon\in E_{\gamma}}\sum_{\varsigma\in E_{\theta}}(-1)^{|\varepsilon|+|\zeta|}s_{\tilde{\varepsilon}}^{\gamma_{\tilde{\varepsilon}}}\sum_{\eta_{\varepsilon}\le\gamma_{\varepsilon}-1}s_{\varepsilon}^{\gamma_{\varepsilon}-\eta_{\varepsilon}-1}\cdot$$

$$\cdot\sum_{D\varepsilon,\eta}\mathcal{L}_{q,r}^{\tilde{\varepsilon},\tilde{\zeta}}[g_{\varepsilon,\zeta}(t_{\tilde{\varepsilon}},k_{\tilde{\zeta}})]\left(\prod_{j\in\zeta}z_j^{-k_j}\right)$$

where $\mathcal{L}_{q,r}^{\tilde{\varepsilon},\tilde{\zeta}}$ is the partial hybrid Laplace Transform [12, Definition 2.14].

Application 3.2. A Poisson process $(X_t)_{t\in\mathbf{R}^+}$ is described by the probabilities $P_k(t)=P(X_t=k)$, $k\in\mathbf{N}$, which verify the system of differential equations

$$P_0'(t)=-\lambda P_0(t) \tag{3.7}$$

$$P_{k+1}'(t)=-\lambda P_{k+1}(t)+\lambda P_k(t),\quad k=0,1,\dots \tag{3.8}$$

with the initial conditions $P_0(0)=1$ and $P_k(0)=0$, $k=1,2,\dots$

By using the notation $P_k(t)=x(t,k)$ the system (3.7), (3.8) becomes a differential-difference system which is transformed (by [12, Theorem 2.6]) as above, by applying the (q,r)-HLT, into the algebraic system

$$s\mathcal{L}[x(t,0)]-x(0,0)=-\lambda\mathcal{L}[x(t,0)],\ x(0,0)=P_0(0),$$
$$szX(s,z)-sz\mathcal{L}[x(t,0)]-z\mathcal{Z}[x(0,k)]+zx(0,0)=$$
$$=-\lambda z(X(s,z)-\mathcal{L}[x(t,0)])+\lambda X(s,z).$$

which has the solution $\mathcal{L}[x(t,0)]=\dfrac{1}{s+\lambda}$ and

$$X(s,z)=\frac{z(s+\lambda)}{sz+\lambda z-\lambda}\mathcal{L}[x(t,0)]=\frac{z}{sz+\lambda z-\lambda}$$

whose original is the usual solution $P_k(t)=x(t,k)=e^{-\lambda t}\dfrac{(\lambda t)^k}{k!}$, $k\in\mathbf{N}$.

Application 3.3. In Queueing theory, the transition probabilities $p_{ij}(t)=P(\xi_{\tau+t}=j\,|\,\xi_{\tau}=i)$ of a system $M/M/1$ verify the system of differential equations (see [9])

$$p_{i0}'(t)=-\lambda p_{i0}(t)+\mu p_{i1}(t) \tag{3.9}$$

$$p_{ij}'(t)=\lambda p_{i,j-1}(t)-(\lambda+\mu)p_{ij}(t)+\mu p_{i,j+1}(t) \tag{3.10}$$

with the initial conditions $p_{ij}(0)=\delta_{ij}$, where $0<\lambda<\mu$.

By denoting $p_{ij}(t) = x_i(t, j)$ and $\mathcal{L}[x_i(t, j)] = X_i(s, z)$ and by applying the (q,r)-HLT and [12, Theorem 2.6] as in Application 3.2, the differential-difference system is transformed directly into a set of algebraic equations having the solutions

$$X_i(s,z) = \frac{z^{-i+1} + \mathcal{L}[x_i(t,0)]\mu z(1-z)}{(s+\lambda+\mu)z - \lambda - \mu z^2}.$$

Definition 3.4. A multiple *continuous-discrete convolution integral equation* has the form

$$Ax(t_1,\dots,t_q;k_1,\dots,k_r) + \int_0^{t_1}\dots\int_0^{t_q}\sum_{l_1=0}^{k_1}\dots\sum_{l_r=0}^{k_r} x(u_1,\dots,u_q;l_1,\dots,l_r)\cdot$$
$$\cdot g(t_1-u_1,\dots,t_q-u_q;k_1-l_1,\dots,k_r-l_r)du_1\dots du_q = f(t_1,\dots,t_q;k_1,\dots,k_r), \tag{3.11}$$

where $A \in \mathrm{R}$ and f and g are original functions.

By applying the (q,r)-HLT, due to [15, Theorem 2.8], equation (3.11) is transformed into the algebraic equation

$$AX(s;z) + X(s;z)G(s;z) = F(s;z).$$

4. State space representations and transfer matrices of (q,r)-D hybrid systems

We shall use the multiple hybrid Laplace transformation to obtain the transfer matrices of different classes of hybrid systems. We shall denote the operator $\mathcal{L}_{n,m}$ by $\mathcal{L}_{q,r}$, $q,r \in \mathbf{N}^*$.

The (q,r)-D *hybrid Roesser type model* has the state space representation

$$\begin{bmatrix} \frac{\partial}{\partial t} x^c(t;k) \\ \sigma x^d(t;k) \end{bmatrix} = A\begin{bmatrix} x^c(t;k) \\ x^d(t;k) \end{bmatrix} + Bu(t;k) \tag{4.1}$$

$$y(t;k) = C\begin{bmatrix} x^c(t;k) \\ x^d(t;k) \end{bmatrix} + Du(t;k) \tag{4.2}$$

where $t = (t_1,\dots,t_q) \in \mathbf{R}_+^q$, $k = (k_1,\dots,k_r) \in \mathbf{Z}_+^r$,

$$x^c(t;k) = \begin{bmatrix} x_1^c(t;k) \\ \dots \\ x_q^c(t;k) \end{bmatrix}, \quad x^d(t;k) = \begin{bmatrix} x_1^d(t;k) \\ \dots \\ x_r^d(t;k) \end{bmatrix}, \quad x_i^c(t;k) \in \mathbf{R}^{n_{c_i}}, \quad n_{c_i} \in \mathbf{N}^*, \quad i = \overline{1,q},$$

$x_j^d(t;k) \in \mathbf{R}^{n_{d_j}}$, $n_{d_j} \in \mathbf{N}^*$, $j \in \langle r \rangle$, $u(t;k) \in \mathbf{R}^m$, $y(t;k) \in \mathbf{R}^p$, $m, p \in \mathbf{N}^*$; the operators $\frac{\partial}{\partial t}$ and σ are defined by

$$\frac{\partial}{\partial t} x^c(t;k) \begin{bmatrix} \frac{\partial}{\partial t_1} x_1^c(t;k) \\ \frac{\partial}{\partial t_2} x_2^c(t;k) \\ \cdots \\ \frac{\partial}{\partial t_q} x_q^c(t;k) \end{bmatrix}, \quad \sigma x^d(t;k) = \begin{bmatrix} x_1^d(t;k_1+1,k_2,\ldots,k_r) \\ x_2^d(t;k_1,k_2+1,\ldots,k_r) \\ \cdots \\ x_r^d(t;k_1,k_2,\ldots,k_r+1) \end{bmatrix}.$$

The real matrices A,B,C,D have the dimensions $n \times n$, $n \times m$, $p \times n$, $p \times m$ respectively, where $n = \sum_{i=1}^{q} n_{c_i} + \sum_{j=1}^{r} n_{d_j}$. The state vector is $x(t;k) \in \mathbf{R}^n$, $x(t,k) = [x^c(t,k)^T \, x^d(t;k)^T]^T$. For $\tau = \{i_1,\ldots,i_l\} \subset \langle q \rangle$ and $\delta = \{j_1,\ldots,j_h\} \subset \langle r \rangle$, the operators $\frac{\partial}{\partial \tau}$ and σ_δ are defined by

$$\frac{\partial}{\partial \tau} x(t;k) = \frac{\partial^l}{\partial t_{i_1} \cdots \partial t_{i_l}} x(t;k), \quad \sigma_\delta x(t;k) = x(t;k+e_\delta),$$

where $e_\delta = e_{j_1} + \cdots + e_{j_h}$, $e_j = (0,\ldots,0,1,0,\ldots,0) \in \mathbf{R}^r$; $\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}$, $\sigma_\delta = \sigma$ when $\tau = \langle q \rangle$, $\delta = \langle r \rangle$; $\frac{\partial}{\partial \tau} x(t;k) = x(t,k)$ and $\sigma_\delta x(t;k) = x(t;k)$ if $\tau = \varnothing$ and $\delta = \varnothing$.

Now we shall denote by $A_{\varnothing\varnothing}, A_{\tau,\varnothing}, A_{\varnothing,\delta}, A_{\tau,\delta}$ the matrices $A_0, A_{i_1,\ldots,i_l;0}, A_{0;j_1,\ldots,j_h}, A_{i_1,\ldots,i_l;j_1,\ldots,j_h}$.

The notation $(\tau,\delta) \subset (\langle q \rangle, \langle r \rangle)$ means that τ and δ are subsets of $\langle q \rangle$ and $\langle r \rangle$ respectively, and $(\tau,\delta) \neq (\langle q \rangle, \langle r \rangle)$.

The (q,r)-D *hybrid Fornasini-Marchesini type model* has the state-space representation

$$\frac{\partial}{\partial t} \sigma x(t;k) = \sum_{(\tau,\delta) \subset (\langle q \rangle, \langle r \rangle)} A_{\tau,\delta} \frac{\partial}{\partial \tau} \sigma_\delta x(t;k) + \sum_{(\tau,\delta) \subset (\langle q \rangle, \langle r \rangle)} B_{\tau,\delta} \frac{\partial}{\partial \tau} \sigma_\delta u(t;k) \tag{4.3}$$

where $A_{\tau,\delta}$, $B_{\tau,\delta}$, C and D are $n \times n$, $n \times m$, $p \times n$ and $p \times m$ real matrices.

Particularly, given $q+r$ commuting $n\times n$ matrices A_{c_i}, $i\in\langle q\rangle$, A_{d_j}, $j\in\langle r\rangle$, one obtains the (q,r)-D *hybrid Attasi type model*, with the state equation

$$\frac{\partial}{\partial t}\sigma x(t;k)=\sum_{(\tau,\delta)\subset(\langle q\rangle,\langle r\rangle)}(-1)^{q+r-|\tau|-|\delta|-1}\left(\prod_{i\in\widetilde{\tau}}A_{c_i}\right)\left(\prod_{j\in\widetilde{\delta}}A_{d_j}\right)\frac{\partial}{\partial\tau}\sigma_\delta x(t;k)+Bu(t;k) \quad (4.5)$$

where $\widetilde{\tau}=\langle q\rangle\setminus\tau$, $\widetilde{\delta}=\langle r\rangle\setminus\delta$.

The descriptor models for the three types of systems have the state-equations similar to (4.1), (4.3) and (4.5), but with the lefthand member replaced respectively by

$$E\begin{bmatrix}\frac{\partial}{\partial t}x^c(t,k)\\ \sigma x^d(t,k)\end{bmatrix},\quad E\frac{\partial}{\partial t}\sigma x(t;k) \quad (4.6)$$

where E are square singular matrices of appropriate dimensions.

Now, let us consider the constant delay times $a=(a_1,\dots,a_q)\in\mathbf{R}_+^q$ and $b=(b_1,\dots,b_r)\in\mathbf{Z}_+^r$. We denote by $x(t-a;k-b)$ the vector

$$x(t_1-a_1,\dots,t_q-a_q;k_1-b_1,\dots,k_r-b_r).$$

The (q,r)-D hybrid *Roesser type model with time delay* has the state space representation

$$E\begin{bmatrix}\frac{\partial}{\partial t}x^c(t;k)\\ \sigma x^d(t;k)\end{bmatrix}=A_0\begin{bmatrix}x^c(t;k)\\ x^d(t;k)\end{bmatrix}+A_1\begin{bmatrix}x^c(t-a;k-b)\\ x^d(t-a;k-b)\end{bmatrix}+ \quad (4.7)$$
$$+B_0u(t;k)+B_1u(t-a;k-b),$$

$$y(t;k)=C_0\begin{bmatrix}x^c(t;k)\\ x^d(t;k)\end{bmatrix}+C_1\begin{bmatrix}x^c(t-a;k-b)\\ x^d(t-a;k-b)\end{bmatrix}+ \quad (4.8)$$
$$+D_0u(t;k)+D_1u(t-a;k-b).$$

The (q,r)-D hybrid Fornasini-Marchesini type model with time delays has the representation

$$E\frac{\partial}{\partial t}\sigma x(t,k)=\sum_{(\tau,\delta)\subset(\langle q\rangle,\langle r\rangle)}A_{0,\tau,\delta}\frac{\partial}{\partial\tau}\sigma_\delta x(t;k)+$$
$$+\sum_{(\tau,\delta)\subset(\langle q\rangle,\langle r\rangle)}A_{1,\tau,\delta}\frac{\partial}{\partial\tau}\sigma_\delta x(t-a;k-b)+\sum_{(\tau,\delta)\subset(\langle q\rangle,\langle r\rangle)}B_{0,\tau,\delta}\frac{\partial}{\partial\tau}\sigma_\delta u(t;k)+ \quad (4.9)$$
$$+\sum_{(\tau,\delta)\subset(\langle q\rangle,\langle r\rangle)}B_{1,\tau,\delta}\frac{\partial}{\partial\tau}\sigma_\delta u(t-a;k-b),$$

$$y(t;k) = C_0 x(t;k) + C_1 x(t-a;k-b) + D_0 u(t;k) + D_1 u(t-a;k-b). \quad (4.10)$$

The (q,r)-D hybrid Attasi type model with time delays can be defined similarly.

By applying the multiple (q,r)-hybrid Laplace transformation to the systems described above and taking into account [12, Theorems 2.11, 2.13 and 2.18] we can get the input-output maps of these systems in the frequency domain.

In order to simplify the formulae, we shall consider null boundary conditions, i.e. $x(t_1,\ldots,t_q;k_1,\ldots,k_r) = 0$ if $t_i = 0$ for some $i \in \langle q \rangle$ or $k_j = 0$ for some $j \in \langle r \rangle$. We use the notations $X(s;z) = \mathcal{L}_{q,r}[x(t;k)]$, $X^c(s;z) = \mathcal{L}_{q,r}[x^c(t;k)]$ etc. By [12, (2.12i) and (2.14i)], the state equation (4.1), modified as in (4.6), becomes

$$E\begin{bmatrix} s_1 X_1^c(s;z) \\ \cdots \\ s_q X_q^c(s;z) \\ z_1 X_1^d(s;z) \\ \cdots \\ z_r X_r^d(s;z) \end{bmatrix} = A\begin{bmatrix} X_1^c(s;z) \\ \cdots \\ X_q^c(s;z) \\ X_1^d(s;z) \\ \cdots \\ X_r^d(s;z) \end{bmatrix} + Bu(s;z), \quad (4.11)$$

hence the left-hand member can be written as

$$E\left(\left(\bigoplus_{i=1}^{q} s_i I_{n_{c_i}}\right) \oplus \left(\bigoplus_{j=1}^{r} z_j I_{n_{d_j}}\right)\right) X(s;z);$$

the output equation (4.2) is transformed into

$$Y(s;z) = C\begin{bmatrix} X^c(s,z) \\ X^d(s,z) \end{bmatrix} + DU(s;z). \quad (4.12)$$

By solving (4.11) with respect to the state vectors $X_i^c(s;z)$, $X_j^d(s;z)$ and by replacing them in (4.12) one obtains the input-output map $Y(s;z) = H(s;z)U(s;z)$, where $H(s;z)$ is the transfer matrix of the (q,r)-D hybrid Roesser type system:

$$H(s;z) = C\left(E\left(\left(\bigoplus_{i=1}^{q} s_i I_{n_{c_i}}\right) \oplus \left(\bigoplus_{j=1}^{r} z_j I_{n_{d_j}}\right)\right) - A\right)^{-1} B + D. \quad (4.13)$$

Here $\oplus$ indicates the direct sum and I_n the unit matrix of order n. For $E = I_n$ (4.13) gives the transfer matrix for the standard system.

In the same manner, by using [12, Theorem 2.13] one can obtain the transfer matrix of the time delay system (4.7), (4.8):

$$H(s;z) = \left[C_0 + C_1\left(\exp\left(\sum_{i=1}^{q} a_i s_i\right)\right)\left(\sum_{j=1}^{r} z_j^{-b_j}\right)\right]\times$$
$$\times\left[E\left(\left(\bigoplus_{i=1}^{q} s_i I_{n_{c_i}}\right)\oplus\left(\bigoplus_{j=1}^{r} z_j I_{n_{d_j}}\right)\right) - A_0 -\right.$$
$$\left. - A_1\left(\exp\left(-\sum_{i=1}^{q} a_i s_i\right)\right)\left(\sum_{j=1}^{r} z_j^{-b_j}\right)\right]^{-1}\times \tag{4.14}$$
$$\times\left[B_0 + B_1\left(\exp\left(-\sum_{i=1}^{q} a_i s_i\right)\right)\left(\sum_{j=1}^{r} z_j^{-b_j}\right)\right]+$$
$$+ D_0 + D_1\left(\exp\left(-\sum_{i=1}^{q} a_i s_i\right)\right)\left(\sum_{j=1}^{r} z_j^{-b_j}\right).$$

Now, let us consider the hybrid Fornasini-Marchesini type systems. By [12, (2.14 ii)], for null boundary conditions, equation (4.9) becomes, with the notations $s_\tau = \prod_{i\in\tau} s_i$, $s_\delta = \prod_{i\in\delta} z_j$ for $\tau \subset \langle q\rangle$, $\delta \subset \langle r\rangle$:

$$Es_{\langle q\rangle} z_{\langle r\rangle} X(s,z) = \sum_{(\tau,\delta)\subset(\langle q\rangle,\langle r\rangle)} A_{\tau,\delta} s_\tau z_\delta X(s;z) + \sum_{(\tau,\delta)\subset(\langle q\rangle,\langle r\rangle)} B_{\tau,\delta} s_\tau z_\delta U(s;z)$$

By solving this equation with respect to $X(s;z)$ and by replacing $X(s;z)$ in the transformed equation (4.4), one obtains the transfer matrix of the system (4.3), (4.4) (with (4.6))

$$H(s;z) = C\left(Es_{\langle q\rangle} z_{\langle r\rangle} - \sum_{(\tau,\delta)\subset(\langle q\rangle,\langle r\rangle)} A_{\tau,\delta} s_\tau z_\delta\right)^{-1}\left(\sum_{(\tau,\delta)\subset(\langle q\rangle,\langle r\rangle)} B_{\tau,\delta} s_\tau z_\delta\right).$$

Particularly, for the (q,r)-D hybrid Attasi type system (4.5), (4.4), we get the separable transfer matrix

$$H(s,z) = C\left(\prod_{i=1}^{q} (s_i I - A_{c_i})\right)^{-1}\left(\prod_{j=1}^{r} (z_j I - A_{d_j})\right)^{-1} B + D.$$

Similarly, we can show that the corresponding time-delay system (4.9), (4.10) has the transfer matrix

$$H(s;z)=(C_0+C_1e^{-as}z^{-b})\cdot$$
$$\cdot\left(Es^{\langle q\rangle}z^{\langle r\rangle}-\sum_{(\tau,\delta)\subset(\bar{q},\bar{r})}A_{0,\tau,\delta}s_\tau z_\delta-\sum_{(\tau,\delta)\subset(\bar{q},\bar{r})}A_{1,\tau,\delta}s_\tau z_\delta e^{-as}z^{-b}\right)^{-1}\cdot \tag{4.16}$$
$$\cdot\left(\sum_{(\tau,\delta)\subset(\bar{q},\bar{r})}B_{0,\tau,\delta}s_\tau z_\delta+\sum_{(\tau,\delta)\subset(\bar{q},\bar{r})}B_{1,\tau,\delta}s_\tau z_\delta e^{-as}z^{-b}\right)+D_0+D_1e^{-as}z^{-b},$$

were e^{-as} denotes $\exp\left(-\sum_{i=1}^{q}a_is_i\right)$ and z^{-b} denotes $\prod_{j=1}^{r}z_j^{-b_j}$.

6. Conclusion

In this paper the applications of the multiple (n,m)-Hybrid Laplace Transformation have been emphasized, including differential-difference and integral equations, as well as the frequency-domain representation of multidimensional hybrid control systems. This approach can be continued by some topics such as the realization theory of these systems or the frequency-domain compensation.

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