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SOLVING UNCERTAIN NONLINEAR COMPLEMENTARITY PROBLEM BY USING A NEW PENALTY METHOD

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In this paper, we develop a new class of uncertain nonlinear complementarity problem (UNCP), that is, the uncertain nonlinear complementarity problem based on uncertainty theory in finite Euclidean spaces. It can be regarded as the generalization of classical nonlinear complementarity problem. In order to find the solution of the UNCP, we firstly convert it into an uncertain mathematical program with equilibrium constraints (UMPECs) by the expected value of uncertain variables. To go one step further, we construct an auxiliary function, which is used to convert the UMPECs into the reformulation of the problem. Then, we present a new penalty method for solving the UNCP. Finally, it is another core of the paper that the rigorous convergence of the penalty method has been proofed.

Keywords: uncertain nonlinear complementarity problem; uncertain mathematical program with equilibrium constraints; convergence analysis; expected value of uncertain variables; optimal solution.

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1. Introduction

Variational inequality problem (VIP) was presented by Hartman et al. [1] as a tool to research partial differential equations, which is also an important discipline of mathematics. Over that last few decades, VIP is one of the most known variational models which can formulate lots of problems arising in engineering, mathematical physics, economics, and other fields [2, 3]. Although there are a lot of derivatives of VIP in infinite dimensional spaces, we are interested in the finite dimensional Euclidean space R^n. Thus, the VIP in finite dimensional Euclidean space can be defined as follows: by finding a point $\bar{y} \in \Upsilon \subseteq R^n$ such that

$$(y - \bar{y})^T G(\bar{y}) \geq 0, \quad \forall y \in \Upsilon, \qquad (\mathbf{1}.1)$$

where Υ is a closed and convex subset of R^n, and $G : \Upsilon \to R^n$ is a mapping. Suppose that there exists a point $\bar{y}$ for $\forall y \in \Upsilon$, then $\bar{y}$ is named a solution to (**1**.1). In fact, we can find that the underlying mapping G and solution $\bar{y}$ in (**1**.1) are all deterministic, i.e., they do not contain fuzziness and uncertainties.

However, in real-word situations, the above problem not only involves deterministic information, but also contains some uncertain and fuzzy factors in those data [4]. In order to characterize the fuzziness and uncertainties, there are two mathematical systems. One is

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probability theory [5], and the other is uncertainty theory [6]. Probability is explained as a frequency that needs enough history data for probabilistic inference, while uncertainty is interpreted as personal belief degree that is from domain experts in the absence of samples [7].

In the probability theory, the stochastic variational inequality problem is to find a vector $\bar{y} \in \Upsilon \subseteq R^n$ satisfying

$$(y - \bar{y})^T G(\bar{y}, \varsigma) \geq 0, \quad \forall y \in \Upsilon, \tag{1.2}$$

where Υ is a closed and convex subset of R^n, $G : \Upsilon \to R^n$ is a mapping, and Ω is the underlying sample space for each ς. In order to get the optimal solution of problem (**1**.2), A new expected residual minimization (ERM) formulation for a class of stochastic VIP was presented in [8]. The progressive hedging algorithm was demonstrated to be applicable to solving multistage stochastic VIPs under monotonicity in [9]. Further, if Υ is the non-negative orthant $R^n_+ = \{y \in R^n | y \geq 0\}$, the problem (**1**.2) is overwritten as the stochastic complementarity problem

$$G(y, \varsigma) \geq 0, \quad y \geq 0, \quad y^T G(y, \varsigma) = 0, \tag{1.3}$$

In general, there is no satisfying (**1**.3) for all $\forall \varsigma \in \Omega$. So, Chen and Fukushima [10] generated observations by the quasi-Monte Carlo methods and proved that every accumulation point of minimizer of discrete approximation problems was a minimum expected residual solution of the stochastic linear complementarity problem. In [9], the stochastic complementarity problems as a special case were explored numerically in a linear two-stage formulation. Lin and Fukushima [11] proposed a smoothed penalty method for solving the stochastic nonlinear complementarity problem and gave a rigorous convergence analysis.

In the uncertainty theory, the uncertain variational inequality problems (UVIP for short) [4]: finding a vector $\bar{y} \in \Upsilon \subseteq R^n$ such that

$$(y - \bar{y})^T G(\bar{y}, \xi) \geq 0, \quad \forall y \in \Upsilon, \tag{1.4}$$

where Υ is a closed and convex subset of R^n, $G : R^n \times B \to R^n$ is a mapping, ξ is an uncertain variable and B is a Borel set. Since there is no solution to problem (**1**.4), Chen and Zhu [4] introduced the expected value of uncertain variables and converted problem (**1**.4) into a classical deterministic variational inequality problem, which can be solved by many algorithms that were developed on the basis of gap functions. In [12], a convex combined expectation regularized gap function with uncertain variable was presented to deal with uncertain nonlinear variational inequality problems. Li et al. [13] established uncertain variational inequality problem as an optimization problem (ERM model) which minimized the expected residual of the so-called regularized gap function. Furthermore, the problem (**1**.4) is also rewritten as the uncertain nonlinear complementarity problem (UNCP)

$$G(y, \xi) \geq 0, \quad y \geq 0, \quad y^T G(y, \xi) = 0, \tag{1.5}$$

However, to the best of our knowledge, up to now, there have not been any papers devoted to solving the UNCP (**1**.5). Motivated by above analyses, we will develop a new penalty method for solving the uncertain nonlinear complementarity problem in this paper.

The specific outline of this paper is as follows. We review some preliminary results about uncertainty theory and other preliminaries in Section 2, which are useful in the remainder of this paper. In section 3, a new penalty method for solving the uncertain nonlinear complementarity problem is developed, and discusses the existence of optimal solutions and stable points of UNCP. Finally, we give some conclusions in Section 4.

2. Preliminaries

In this section, we review some basic concepts of uncertainty theory and other preliminaries required for our study.

2.1. **Uncertainty Theory.** Uncertainty theory is a new branch of axiomatic mathematics, which is introduced by Liu [6] in 2007. It can be defined as follows:

Definition 2.1. [6,14] Let Γ be a nonempty set and Ξ be a σ-algebra on Γ. A set function M is called an uncertain measure if it satisfies the following four axioms:

Axiom 1. (Normality) $M\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2. (Duality) $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event $\Lambda \in \Xi$.

Axiom 3. (Subadditivity) For every sequence$\{\Lambda_i\} \in \Xi$, then $M\left\{\bigcup\limits_{i=1}^{\infty} \Lambda_i\right\} \leq \sum\limits_{i=1}^{\infty} M\{\Lambda_i\}$.

Axiom 4. Let (Γ, Ξ, M) be uncertainty spaces for $i = 1, 2, \ldots$. The product uncertain measure M is an uncertain measure such that $M\left\{\prod\limits_{i=1}^{\infty} \Lambda_i\right\} \leq \prod\limits_{i=1}^{\infty} M(\Lambda_i)$, where $\{\Lambda_i\} \in \Xi$, $i = 1, 2, \ldots$.

Definition 2.2. [6] An uncertain variable ξ is a function from an uncertainty space (Γ, Ξ, M) to the set of real numbers such that for any Borel set B of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is defined an event.

Definition 2.3. [6] Let ξ be an uncertain variable, then the uncertainty distribution of ξ is defined by $\Phi(w) = M\{\xi \leq w\}$ for any real number w.

Definition 2.4. [6] Let ξ be an uncertain variable, then the expected value of ξ is defined by $\Phi(w) = M\{\xi \leq w\}$ for any real number w.

$$E(\xi) = \int_0^{+\infty} M\{\xi \geq w\}\, dw + \int_{-\infty}^0 M\{\xi \leq w\}\, dw, \qquad (\mathbf{2}.1)$$

provided that at least one of the two integrals is finite.

Theorem 2.1. [6] Let ξ be an uncertain variable with uncertainty distribution $\Phi(w)$. If the expected value exists, then

$$E(\xi) = \int_{-\infty}^{+\infty} w d\Phi(w). \qquad (\mathbf{2}.2)$$

Theorem 2.2. [6] Let ξ be an uncertain variable with uncertainty distribution $\Phi(w)$ and $f(w)$ be a strictly monotone function. Then

$$E(\xi) = \int_{-\infty}^{+\infty} f(w) d\Phi(w). \qquad (\mathbf{2}.3)$$

2.2. **Approximation Method.** Because of the subadditivity axiom, there is not density function for uncertain variable, the uncertainty distribution $\Phi(w)$ is usually not differentiable in uncertainty theory. In order to resolve this problem, Li et al. [15] presented an approximation method by using the Stieltjes integral.

Definition 2.5. [15] Let minimum of $\theta^m(x)$ be defined as follows

$$\min \theta^m(x) = \sum_{w_i \in W_\delta} f(x, w_i) \Delta\Phi(w_i),\ x \in R, \qquad (\mathbf{2}.4)$$

where $W_\delta = \{w_i | i = 1, 2, \cdots, m_\delta\}$ is a set satisfying $m_\delta \to +\infty$ as $\delta \to 0$.

Theorem 2.3. [15] For any fixed $x \in R$, there holds that

$$\theta(x) = \lim_{\substack{\delta \to 0 \\ m \to \infty}} \theta^m(x), \qquad (\mathbf{2}.5)$$

where $\theta(x) = E[f(x, \xi)] = \int_{W_\delta} f(x, w) d\Phi(w)$.

3. Penalty method formation of uncertain

In this section, we mainly study how to solve the UNCP. In fact, it would be difficult to find a vector in (**1**.5) satisfying the complementarity conditions for (almost) all $\xi \in B$. In order to find an optimal solution of problem (**1**.5), we can adopt a recourse function $s(\xi) \geq 0$ to the inequality $G(y, \xi) \geq 0$ and try to solve a vector $y \geq 0$ in (**1**.5) that minimizes the expected recourse. Therefore, we can get the following uncertain optimization problem

$$\begin{aligned} &\min \ E_\xi \left[c^T s\left(\xi\right)\right] \\ &s.t. \ y \geq 0, \ G(y, \xi) + s(\xi) \geq 0 \\ &\quad y^T \left[G(y, \xi) + s(\xi)\right] = 0 \\ &\quad s(\xi) \geq 0, \xi \in B, \end{aligned} \tag{3.1}$$

where E_ξ is the expectation in respect of the uncertain variable $\xi \in B$ and c is a constant vector with positive elements. Through it all, we suppose that the function G is continuously differentiable with respect to y, and if the uncertain variable ξ is a continuous uncertain variable, the function G is also continuous with respect to ξ and monotonous. We still name problem (**3**.1) UNCP, although it is in effect an uncertain mathematical program with equilibrium constraints (UMPECs).

Mathematical programs with equilibrium constraints (MPECs) play a major role in many fields such as engineering design, economic equilibrium, multilevel games, and so forth [16]. Some of the methods have been developed to solve the stochastic mathematical program with equilibrium constraints (SMPEC) [17]. However, to the best of our knowledge, there have not been any papers devoted to solving the UMPECs. In such cases, we will broaden the approach in [17] to the uncertain nonlinear case and propose a new method for solving the UMPECs by converting to equivalence problem in this section.

3.1. **Construction of auxiliary function and discussion its properties.** In order to present a mathematical method for solving the UMPECs, we firstly construct the following auxiliary function.

Definition 3.1. Let ξ be an uncertain variable and $\forall y \in R^n$, a function $P : R^n \times B \to [0, +\infty]$ is defined by

$$P(y, \xi) = Sup \left\{ -(v + q(\xi)y)^T G(y, \xi) | v + q(\xi)y \leq c, v \geq 0, q(\xi) \leq 0 \right\}, \tag{3.2}$$

where v is a vector function, $q(\xi)$ is an uncertain variable and c is a nonnegative constant vector.

According to the duality theorem of nonlinear programming, we can get that, for $y \in R^n$ and uncertain variable ξ, $P(y, \xi) < +\infty$ if and only if

$$S(y, \xi) = \left\{ s(\xi) | y^T \left[G(y, \xi) + s(\xi)\right] \leq 0, G(y, \xi) + s(\xi) \geq 0, s(\xi) \geq 0 \right\} \tag{3.3}$$

is nonempty, then

$$P(y, \xi) = \inf \left\{ c^T s(\xi) | s(\xi) \in S(y, \xi) \right\} \tag{3.4}$$

holds.

Theorem 3.1. Let $y \in R^n$ and ξ be an uncertain variable, then $P(y, \xi) < +\infty$ if and only if $y[j]G_j(y, \xi) \leq 0$ for every j.

Proof: In the following, we will show the proof in two steps.

(I) First of all, prove that $P(y, \xi) < +\infty \Rightarrow y[j]G_j(y, \xi) \leq 0$ for every j. We will prove this conclusion by using reduction to absurdity. Suppose that there is an index j satisfying $y[j] > 0$ and $G_j(y, \xi) > 0$, and let $v(q)$ be assigned by $v(q) = q(\xi)y[j]e_j - q(\xi)y$. Then, for $q(\xi) \leq 0$, we can get $v(q) \geq 0$ and $v(q) + q(\xi)y \leq c$. In accordance with the definition of $P(y,\xi)$, we have $P(y,\xi) \geq Sup\left\{-(v+q(\xi)y)^T G(y,\xi)|q(\xi)\leq 0\right\} = Sup\left\{q(\xi)y[j]G_j(y,\xi)|q(\xi)\leq 0\right\} = +\infty$, which is a contradiction. Therefore, the conclusion holds.

(II) secondly, prove that, for every j, $y[j]G_j(y,\xi) \leq 0 \Rightarrow P(y,\xi) < +\infty$. Since $y[j]G_j(y,\xi) \leq 0$ for $\forall y \in R^n$, then let $N_1 = [j|y\,[j] = 0, G_j(y,\xi) \in R]$, $N_2 = [j|y\,[j] > 0, G_j(y,\ \xi) = 0]$ and $N_3 = [j|y\,[j] > 0, G_j(y,\xi) < 0]$. It is easy to get $N_1 \cup N_2 \cup N_3 = \{1,2,\ldots,n\}$. According to the definition of $P(y,\xi)$, then

$$P(y,\xi) = Sup\left\{-\sum_{j=1}^{n}(v+q(\xi)y)\,[j]G_j(y,\xi)|\,(v+q(\xi)y)\,[j] \leq c, v[j] \geq 0, q(\xi) \leq 0\right\}. \qquad (\mathbf{3}.5)$$

holds.

In what follows, we will prove that the function $P(y,\xi)$ is bounded.

(II$_1$) When$\forall j \in N_1$, then $y\,[j] = 0$ and $G_j(y,\xi) \in R$, we can get $(v+q(\xi)y)\,[j] = v[j]$. By (**3**.5), we have $0 \leq v[j] \leq c[j]$, and

1)when $G_j(y,\xi) < 0$, then $-\sum\limits_{j\in N_1}(v+q(\xi)y)\,[j]G_j(y,\xi) =$
$-\sum\limits_{j\in N_1}v[j]G_j(y,\xi) = \sum\limits_{j\in N_1}v[j]G_j(y,\xi) \leq \sum\limits_{j\in N_1}c[j]G_j(y,\xi)$;

2)when $G_j(y,\xi) = 0$, then $-\sum\limits_{j\in N_1}(v+q(\xi)y)\,[j]G_j(y,\xi) = 0$;

3)when$G_j(y,\xi) > 0$, then $-\sum\limits_{j\in N_1}(v+q(\xi)y)\,[j]G_j(y,\xi) = -\sum\limits_{j\in N_1}v[j]G_j(y,\xi) \leq \sum\limits_{j\in N_1}c[j]G_j(y,\xi)$.

It is easy to get that the function $P(y,\xi)$ is bounded from 1), 2) and 3).

(II$_2$)When $\forall j \in N_2$, then $y\,[j] > 0$ and $G_j(y,\xi) = 0$, we have
$-\sum\limits_{j\in N_2}(v+q(\xi)y)\,[j]G_j(y,\xi) = 0$, which shows that the function $P(y,\xi)$ is bounded.

(II$_3$)When $\forall j \in N_3$, then $y\,[j] > 0$ and $G_j(y,\xi) < 0$, we can obtain

$$-\sum_{j\in N_3}(v+q(\xi)y)\,[j]G_j(y,\xi) \leq \sum_{j\in N_3}c[j]G_j(y,\xi),$$

which explains that the function $P(y,\xi)$ is bounded.

Based on the above proof, we can get $P(y,\xi) < +\infty$. Therefore, the conclusion holds. In order to be convenient for computation, we can present the following corollary from Theorem **3**.1.

Corollary 3.1. Let $y \in R^n$ and ξ be an uncertain variable. If $P(y,\xi) < +\infty$, then

$$P(y,\xi) = c^T s(y,\xi), \qquad (\mathbf{3}.6)$$

where $s(y,\xi) = \max\{-G(y,\xi), 0\}$.

In order to find a solution to the problem (**3**.1) numerically, it will be proved that the problem (**3**.1) is equivalent to the following one in the discrete case

$$\min_{y\geq 0}\ E_\xi\left[P(y,\xi)\right]. \qquad (\mathbf{3}.7)$$

3.2. **Discrete case.** Let $y \in R^n$ and ξ be an uncertain variable with uncertainty distribution $\Phi(w)$. In order to get the solution of problems (**3**.1) and (**3**.7), we try to discretize the problem by the (**2**.4) and (**2**.5). In other words,

$$\theta(y) = E\left[P(y,\xi)\right] = \int_{W_\delta} P(y,w)d\Phi(w) = \lim_{\substack{\delta\to 0\\ m\to\infty}}\theta^m(y) = \lim_{\substack{\delta\to 0\\ m\to\infty}}\sum_{w_i\in W_\delta}P(y,w_i)\Delta\Phi(w_i). \qquad (\mathbf{3}.8)$$

For easy figures, let $G(y,\xi)$ and $P(y,\xi)$ be defined as $G_i(y)$ and $P_i(y)$. Thus, problems (**3**.1) and (**3**.7) will become the following problems, respectively

$$\begin{aligned} \min\ & \lim_{\substack{\delta\to 0\\ m\to\infty}} \sum_{w_i\in W_\delta} c^T s(w_i)\Delta\Phi(w_i) \\ s.t.\ & y\geq 0,\ G_i(y)+s(w_i)\geq 0 \\ & y^T\left(G_i(y)+s(w_i)\right)=0 \\ & s(w_i)\geq 0,\ i=1,2,\ldots,m_\delta, \end{aligned} \tag{3.9}$$

and

$$\min \rho(y)=\lim_{\substack{\delta\to 0\\ m\to\infty}} \sum_{w_i\in W_\delta} P_i(y)\Delta\Phi(w_i). \tag{3.10}$$

3.3. **A new penalty method for discrete problems.** In this subsection, we will mainly discuss the discrete problem (**3**.10). In fact, according to the Theorem **3**.1, it is easy to get that problem (**3**.10) can be equivalent to the following problem:

$$\begin{aligned} \min\ & \rho(y)=\lim_{\substack{\delta\to 0\\ m\to\infty}} \sum_{w_i\in W_\delta} P_i(y)\Delta\Phi(w_i) \\ s.t.\ & y\geq 0,\ y[j]G_{ji}(y)\leq 0 \\ & j=1,2,\cdots,n, i=1,2,\ldots,m_\delta. \end{aligned} \tag{3.11}$$

Further, from Corollary **3**.1, we have

$$\min \rho(y)=\lim_{\substack{\delta\to 0\\ m\to\infty}} \sum_{w_i\in W_\delta} c^T \max\left\{-G_i(y),0\right\}\Delta\Phi(w_i), \tag{3.12}$$

for $\forall y\in Y$, where Y is the feasible region of problem (**3**.11).

However, problem (**3**.11) is not any more an SMPECs and not easy to solve. On the one hand, the objective function of problem (**3**.11) is not differentiable everywhere, on the other hand, when $m\to\infty$, the calculation of objective function becomes more and more complex due to more and more constraints in problem (**3**.11). In order to solve these complex difficulties, we will next develop a new penalty method for solving problem (**3**.11).

Suppose γ is a nonnegative real number and the penalty function $\psi: R\to[0,+\infty)$ can be defined by

$$\psi_\gamma(z)=\eta z+\sqrt{\eta^2 z^2+\gamma^2}, \tag{3.13}$$

where $\eta\in[0,1]$.

From (**3**.13), it is easy to see that is differentiable everywhere for every $\gamma>0$. Then, with the help of the penalty method, we can construct the following penalty approximation of the problem (**3**.11)

$$\min_{y\geq 0} \mu_\gamma(y)+\kappa\varphi_\gamma(y), \tag{3.14}$$

where $\kappa>0$ is a penalty parameter and let

$$\mu_\gamma(y)=\lim_{\substack{\delta\to 0\\ m\to\infty}} \tilde{\mu}_\gamma(y), \tag{3.15}$$

$$\varphi_\gamma(y)=\lim_{\substack{\delta\to 0\\ m\to\infty}} \tilde{\varphi}_\gamma(y), \tag{3.16}$$

where $\tilde{\mu}_\gamma(y)=\sum_{w_i\in W_\delta}\sum_{j=1}^{n} c[j]\psi_\gamma\left(-G_{i,j}(y)\right)\Delta\Phi(w_i)$ and $\tilde{\varphi}_\gamma(y)=\sum_{w_i\in W_\delta}\sum_{j=1}^{n}\psi_\gamma\left(y[j]G_{i,j}(y)\right)$.

Suppose that $\bar{\kappa}>0$ is a sufficiently large constant. When $\gamma\to 0$ and $\kappa=\bar{\kappa}$ in (**3**.15) and (**3**.16), then problem (**3**.14) degenerates to the following problem

$$\min_{y\geq 0} \rho(y)+\bar{\kappa}\varphi_0(y), \tag{3.17}$$

where $\varphi_0(y)=\lim_{\substack{\delta\to 0\\ m\to\infty}} \sum_{w_i\in W_\delta}\sum_{j=1}^{n} \max\left\{y[j]G_{i,j}(y),0\right\}$.

In order to find the relationship between problem (**3**.11) and problem (**3**.17), we firstly give the following definitions.

Definition 3.2. If there are Lagrange multiplier vectors $\bar{\omega}$ and $\bar{\tau}_i$ $(i = 1, 2, \cdots, m_\delta)$ such that

$$0 \in \partial\tilde{\rho}(\bar{y}) - \bar{\omega} + \sum_{w_i \in W_\delta} \sum_{j=1}^{n} \bar{\tau}_i[j] \left(G_{i,j}(\bar{y})e_j + \bar{y}[j]\nabla G_{i,j}(\bar{y})\right), \quad (\mathbf{3}.18)$$

$$\bar{\omega} \perp \bar{y} = 0, \quad (\mathbf{3}.19)$$

$$\bar{\tau}_i[j] \perp (-\bar{y}[j]G_{i,j}(\bar{y})) = 0, \ \forall i, \ \forall j, \quad (\mathbf{3}.20)$$

then $\bar{y} \in Y$ is an optimal solution of problem (**3**.11).

For $\forall j$ and $\forall i$, let $\tilde{\tilde{\rho}}_{i,j}(y) = \max\{-G_{i,j}(y), 0\}$, then

$$\tilde{\partial}\rho_{i,j}(\bar{y}) = \begin{cases} co\{-\nabla G_{i,j}(\bar{y}), 0\}, G_{i,j}(\bar{y}) = 0, \\ \{-\nabla G_{i,j}(\bar{y})\}, \qquad G_{i,j}(\bar{y}) < 0, \\ \{0\}, \qquad\qquad\qquad G_{i,j}(\bar{y}) > 0, \end{cases} \quad (\mathbf{3}.21)$$

and

$$\partial\tilde{\rho}(\bar{y}) = \sum_{w_i \in W_\delta} \sum_{j=1}^{n} c[j]\tilde{\partial}\rho_{i,j}(\bar{y})\Delta\Phi(w_i), \quad (\mathbf{3}.22)$$

where co expresses the convex hull.

Definition 3.3. If there is a Lagrange multiplier vector ω such that

$$0 \in \partial\tilde{\rho}(\bar{y}) - \omega + \bar{\kappa}\partial\tilde{\varphi}_0(\bar{y}), \quad (\mathbf{3}.23)$$

$$\omega \perp \bar{y} = 0, \quad (\mathbf{3}.24)$$

then $\bar{y} \geq 0$ is an optimal solution of problem (**3**.17).

For $\forall j$ and $\forall i$, let $\tilde{\varphi}_{i,j}(y) = \max\{y[j]G_{i,j}(y), 0\}$, then

$$\partial\tilde{\varphi}_{i,j}(y) = \begin{cases} co\{G_{i,j}(y)e_j + y[j]\nabla G_{i,j}(y), 0\}, \ y[j]G_{i,j}(y) = 0, \\ \{G_{i,j}(y)e_j + y[j]\nabla G_{i,j}(y)\}, \qquad y[j]G_{i,j}(y) > 0, \\ \{0\}, \qquad\qquad\qquad\qquad\qquad y[j]G_{i,j}(y) < 0, \end{cases} \quad (\mathbf{3}.25)$$

and

$$\tilde{\varphi}_0(\bar{y}) = \sum_{w_i \in W_\delta} \sum_{j=1}^{n} \partial\tilde{\varphi}_{i,j}(\bar{y}). \quad (\mathbf{3}.26)$$

Theorem 3.2. Let $\bar{y}$ is an optimal solution of problem (**3**.17), and $\varphi_0(\bar{y}) = 0$ for $\bar{y} \in Y$, then $\bar{y}$ is an optimal solution of problem (**3**.11). Conversely, if $\bar{y}$ is an optimal solution of problem (**3**.11), then $\bar{y}$ is an optimal solution of problem (**3**.17) for any $\bar{\kappa}$ large enough.

Based on the Theorem **3**.2, we can give a new algorithm for solving the problem (**3**.14), which is called the **Algorithm UNCP** and the concrete steps are shown as follows:

Step 1. Initialize the parameters of the algorithm UNCP. Set $l = 0$ and $\eta = 1/2$. Select $\gamma^l > 0$ and $\kappa^l > 0$.

Step 2. Set γ=γ^l and κ=κ^l.

Step 3. Solve the problem (**3**.14) to obtain a stable point y=y^l.

Step 4. If the algorithm UNCP is converged or y is smaller than $y_{\min}$, then go to step 6, otherwise go to step 5.

Step 5. Select $\gamma^{l+1} < \gamma^l$ and $\kappa^{l+1} > \kappa^l$, and set $l = l + 1$. Go to step 2.

Step 6. The algorithm UNCP terminates and outputs the optimal solution.

Now, let $\{\gamma^l\}$ and $\{\kappa^l\}$ be two sequences satisfying

$$\lim_{l\to\infty} \gamma^l = 0, \ \lim_{l\to\infty} \kappa^l = \bar{\kappa}. \quad (\mathbf{3}.27)$$

In what follows, we will study the limiting form of the sequence $\{y^l\}$ generated by the Algorithm UNCP. The convergence result is described concretely as below.

Theorem 3.3. Let Algorithm UNCP generate a sequence $\{y^l\}$ of stable points of problem (**3**.14) with $\gamma=\gamma^l$ and $\kappa=\kappa^l$, then any generated point $\bar{y}$ of the sequence $\{y^l\}$ is a stable point of problem (**3**.17). Further, if $\varphi_0(\bar{y})=0$ for $\bar{y}\in Y$, then $\bar{y}$ is a stable point of problem (**3**.11).

Proof: Since y_l is a stable point for problem (**3**.14) with $\gamma=\gamma^l$ and $\kappa=\kappa^l$, then there must exit some Lagrange multiplier vectors ω^l satisfying

$$\nabla\tilde{\mu}_{\gamma^l}(y^l)+\kappa_l\nabla\tilde{\varphi}_{\gamma^l}(y^l)-\omega^l=0, \tag{3.28}$$

$$\omega^l\bot y^l=0, \tag{3.29}$$

and

$$\bar{y}=\lim_{l\to\infty}y_l, \tag{3.30}$$

where

$$\nabla\tilde{\mu}_{\gamma^l}(y^l)=-\sum_{w_i\in W_\delta}\sum_{j=1}^{n}c[j]\psi'{}_{\gamma}\left(-G_{i,j}(y^l)\right)\Delta\Phi(w_i)\nabla G_{i,j}(y^l), \tag{3.31}$$

and

$$\nabla\tilde{\varphi}_{\gamma^l}(y^l)=\sum_{w_i\in W_\delta}\sum_{j=1}^{n}\psi'{}_{\gamma^l}\left(y^l[j]G_{i,j}(y^l)\right)\left[G_{i,j}(y^l)e_j+y^l[j]\nabla G_{i,j}(y^l)\right]. \tag{3.32}$$

From (**3**.13), we have

$$\psi'_{\gamma^l}(z)=\eta+z\eta^2\Big/\sqrt{\eta^2z^2+\gamma^2}, z\in R. \tag{3.33}$$

Taking (**3**.31) and(**3**.32) into (**3**.28), we can obtain

$$\omega^l=\sum_{w_i\in W_\delta}\sum_{j=1}^{n}\tau_i^l[j]\left[G_{i,j}(y^l)e_j+y^l[j]\nabla G_{i,j}(y^l)\right]-\sum_{w_i\in W_\delta}\sum_{j=1}^{n}c[j]\pi_i^l[j]\Delta\Phi(w_i)\nabla G_{i,j}(y^l) \tag{3.34}$$

where, for $\forall j$ and $\forall i$

$$\tau_i^l[j]=\kappa^l\psi'_{\gamma^l}\left(y^l[j]G_{i,j}(y^l)\right), \tag{3.35}$$

$$\pi_i^l[j]=\psi'_{\gamma}\left(-G_{i,j}(y^l)\right). \tag{3.36}$$

In the light of (**3**.27) and (**3**.30), we can get that sequences $\{\gamma^l\}$ and $\{\kappa^l\}$ are bounded. Further, for $\forall i$, $\{\tau_i^l\}$ and $\{\pi_i^l\}$ are also bounded. It can be seen that the sequence $\{\omega^l\}$ is also bounded. Thus, the following limits exist $\tau_i=\lim_{l\to\infty}\tau_i^l$, $\pi_i=\lim_{l\to\infty}\pi_i^l$, $\omega=\lim_{l\to\infty}\omega^l$, $\forall i$. Seeking the limit of (**3**.29) and (**3**.34), respectively, we can get $\omega\bot\bar{y}=0$, and

$$\omega=\sum_{w_i\in W_\delta}\sum_{j=1}^{n}\tau_i[j]\left[G_{i,j}(\bar{y})e_j+\bar{y}[j]\nabla G_{i,j}(\bar{y})\right]-\sum_{w_i\in W_\delta}\sum_{j=1}^{n}c[j]\pi_i[j]\Delta\Phi(w_i)\nabla G_{i,j}(\bar{y}) \tag{3.37}$$

In what follows, in order to explain that $\bar{y}$ is an optimal solution of problem (**3**.17), then comparing (**3**.23) with (**3**.37), we just need to prove

$$\left[\sum_{w_i\in W_\delta}\sum_{j=1}^{n}\tau_i[j]\left[G_{i,j}(\bar{y})e_j+\bar{y}[j]\nabla G_{i,j}(\bar{y})\right]-\sum_{w_i\in W_\delta}\sum_{j=1}^{n}c[j]\pi_i[j]\Delta\Phi(w_i)\nabla G_{i,j}(\bar{y})\right]\in\partial\tilde{\rho}(\bar{y})+\bar{\kappa}\partial\tilde{\varphi}_0(\bar{y}). \tag{3.38}$$

Namely,

$$-\sum_{w_i\in W_\delta}\sum_{j=1}^{n}c[j]\pi_i[j]\Delta\Phi(t_i)\nabla G_{i,j}(\bar{y})\in\partial\tilde{\rho}(\bar{y}), \tag{3.39}$$

and

$$\sum_{w_i \in W_\delta} \sum_{j=1}^{n} \tau_i[j] \left[G_{i,j}(\bar{y})e_j + \bar{y}[j]\nabla G_{i,j}(\bar{y})\right] \in \bar{\kappa}\partial\tilde{\varphi}_0(\bar{y}). \qquad (\mathbf{3}.40)$$

We next prove that (**3**.39) and (**3**.40) hold, respectively.

(I)First, we prove that (**3**.39) is true. According to the (**3**.22), it is easy to see that we only need to prove

$$-\pi_i[j]\nabla G_{i,j}(\bar{y}) \in \tilde{\partial}\rho_{i,j}(\bar{y})\forall i, \forall j \qquad (\mathbf{3}.41)$$

Next, we will prove (**3**.41) in three cases.

(I_1) If $G_{i,j}(\bar{y}) > 0$, by taking (**3**.36) into (**3**.33), then we can get

$$\pi_i^l[j] = \psi_\gamma'\left(-G_{i,j}(y^l)\right) = \eta - G_{i,j}(y^l)\eta^2 \Big/ \sqrt{\eta^2 (G_{i,j}(y^l))^2 + \gamma_l^2}. \qquad (\mathbf{3}.42)$$

Getting the limit of $\pi_i[j] = \lim\limits_{l\to\infty} \pi_i^l[j] = \lim\limits_{l\to\infty}\left[\eta - G_{i,j}(y^l)\eta^2 \Big/ \sqrt{\eta^2 (G_{i,j}(y^l))^2 + \gamma_l^2}\right]$ $= \eta - \eta = 0$. So, $-\pi_i[j]\nabla G_{i,j}(\bar{y}) = 0 \in \tilde{\partial}\rho_{i,j}(\bar{y})$ is true.

(I_2) If $G_{i,j}(\bar{y}) = 0$, by taking (**3**.36) into (**3**.33), then we have

$$\pi_i^l[j] = \psi_\gamma'\left(-G_{i,j}(y^l)\right) = \eta + \left(-G_{i,j}(y^l)\right)\eta^2 \Big/ \sqrt{\eta^2 (-G_{i,j}(y^l))^2 + \gamma_l^2} = \eta, \qquad (\mathbf{3}.43)$$

which shows

$$0 \le \pi_i^l[j] \le 1, \ \forall l. \qquad (\mathbf{3}.44)$$

Finding the limit of (**3**.44), then $0 \le \pi_i[j] = \lim\limits_{l\to\infty}\pi_i^l[j] \le 1$. Thus, from (**3**.21), we can obtain that $-\pi_i[j]\nabla G_{i,j}(\bar{y}) \in co\{-\nabla G_{i,j}(\bar{y}), 0\} = \tilde{\partial}\rho_{i,j}(\bar{y})$ holds.

(I_3) Let $G_{i,j}(\bar{y}) < 0$, by taking (**3**.36) into (**3**.33), then we can obtain

$$\pi_i^l[j] = \psi_\gamma'\left(-G_{i,j}(y^l)\right) = \eta + G_{i,j}(y^l)\eta^2 \Big/ \sqrt{\eta^2 (G_{i,j}(y^l))^2 + \gamma_l^2}. \qquad (\mathbf{3}.45)$$

Let $l \to \infty$ in (**3**.45), then

$$\pi_i[j] = \lim_{l\to\infty}\pi_i^l[j] = \lim_{l\to\infty}\left[\eta + G_{i,j}(y^l)\eta^2 \Big/ \sqrt{\eta^2 (G_{i,j}(y^l))^2 + \gamma_l^2}\right] = \eta + \eta = 2\eta. \qquad (\mathbf{3}.46)$$

When $\eta = 1/2$ in (**3**.46), then $\pi_i[j] = 2\eta = 2*1/2 = 1$. Thus, $-\pi_i[j]\nabla G_{i,j}(\bar{y}) = -\nabla G_{i,j}(\bar{y}) \in \tilde{\partial}\rho_{i,j}(\bar{y})$. The conclusion is true.

(II)According to the (**3**.26), it is easy to see that we only need to prove

$$\tau_i[j]\left[G_{i,j}(\bar{y})e_j + \bar{y}[j]\nabla G_{i,j}(\bar{y})\right] \in \bar{\kappa}\partial\tilde{\varphi}_0(\bar{y}), \ \forall i, \forall j. \qquad (\mathbf{3}.47)$$

Taking (**3**.35) into (**3**.33), we have

$$\tau_i^l[j] = \kappa^l\left(\eta + y^l[j]G_{i,j}(y^l)\eta^2 \Big/ \sqrt{\eta^2 [y^l[j]G_{i,j}(y^l)]^2 + \gamma^2}\right) \qquad (\mathbf{3}.48)$$

Let $\eta = 1/2$,from (**3**.25), (**3**.27) and (**3**.30), it is easy to get

$$\forall i, \forall j, \begin{cases} \tau_i[j] \in [0, \bar{\kappa}], & \bar{y}[j]G_{i,j}(\bar{y}) = 0, \\ \tau_i[j] = \bar{\kappa}, & \bar{y}[j]G_{i,j}(\bar{y}) > 0, \\ \tau_i[j] = 0, & \bar{y}[j]G_{i,j}(\bar{y}) < 0, \end{cases} \qquad (\mathbf{3}.49)$$

which shows that (**3**.48) is true.

Though (I) and (II), we can get that any generated point $\bar{y}$ of the sequence $\{y^l\}$ is a stable point of (**3**.17). In addition, since $\varphi_0(\bar{y}) = 0$ for $\bar{y} \in Y$, then, according to the Theorem **3**.2, we must obtain that $\bar{y}$ is a stable point of (**3**.11). This completes the proof of the theorem.

4. Conclusions

In this paper, we study a new class of nonlinear complementarity problem, which is named the uncertain nonlinear complementarity problem in view of the uncertainty theory. It can be seen as another generalization of the classical nonlinear complementarity problems in addition to the stochastic nonlinear complementarity problems. In order to present some efficient methods for solving the UNCP, we firstly converted the UNCP into an uncertain mathematical program with equilibrium constraints by introducing the expected value model of uncertain variables. Further, some equivalent problems were given, and discussed the existence of solutions of the problems. Finally, a new penalty method for solving the UNCP is developed based on the equivalent problems and the rigorous proof of its convergence has also been given.

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