A NOTE ON THE NUMBER OF PARTITIONS OF n INTO k PARTS

Mircea Cimpoeaş[1]

We prove new formulas and congruences for $p(n,k)$:= the number of partitions of n into k parts and $q(n,k)$:= the number of partitions of n into k distinct parts. Also, we give lower and upper bounds for the density of the set $\{n \in \mathbb{N} \;:\; p(n,k) \equiv i (\bmod\ m)\}$, where $m \geq 2$ and $0 \leq i \leq m-1$.

Keywords: Restricted integer partitions, Restricted partition function.
MSC2010: 11P81, 11P83.

1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum equals n. We define $p(n)$ as the number of partitions of n and for convenience, we define $p(0) = 1$. Let $p(n,k)$ be the number of partitions of n with exactly k summands. Let $q(n,k)$ be the number of partitions of n with k distinct parts and let $q(n)$ total number of partitions of n with distinct parts. For instance, there are 5 partitions of 8 with three summands $1+1+6$, $1+2+5$, $1+3+4$, $2+2+4$, $2+3+3$, hence $p(8,3) = 5$ and $q(8,3) = 2$. Obviously, $p(n,k) = 0$ if and only if $n < k$. Also, $q(n,k) = 0$ if and only if $n < k + \binom{k}{2}$. Moreover, $p(n) = \sum_{k=1}^{n} p(n,k)$ and $q(n) = \sum_{k=1}^{n} q(n,k)$. The function $p(n,k)$ was studied extensively in the literature; see for instance [8]. However, there is no known closed form for $p(n,k)$.

Let $\mathbf{a} = (a_1, \ldots, a_k)$ be a sequence of positive integers and let $p_{\mathbf{a}}(n)$ be the restricted partition function associated to $\mathbf{a}$; see Section 2. In Theorem 3.**2**, we prove new formulas for $p(n,k)$ and $q(n,k)$, using their intrinsic connection with the restricted partition function associated to the sequence $\mathbf{k} := (1, 2, \ldots, k)$. In Proposition 3.**3**, we give another formulas for $p(n,3)$ and $q(n,3)$. In [5] we proved that if a certain determinant is nonzero, then the restricted partition function $p_{\mathbf{a}}(n)$ can be computed by solving a system of linear equations with coefficients which are values of Bernoulli polynomials and Bernoulli Barnes numbers. Using a similar method, we prove that if a certain determinant $\Delta(k)$, which depends only on k, is nonzero, then $p(n,k)$ and $q(n,k)$ can be

[1]Associated Professor, Faculty of Applied Sciences, University "Politehnica" of Bucharest, Romania and, Researcher, Institute of Mathematics of the Romanian Academy, Bucharest, Romania, e-mail: mircea.cimpoeas@upb.ro

expressed in terms of values of Bernoulli polynomials and Bernoulli Barnes numbers; see Theorem 3.**4**.

In Theorem 4.**2**, respectively in Corollary 4.**3**, we provide formulas for $P(n,k)$ = the polynomial part of $p(n,k)$, respectively for $Q(n,k)$ = the polynomial part of $q(n,k)$. In Proposition 5.**2** we prove formulas for the "waves" of $p(n,k)$ and $q(n,k)$, defined analogously as the Sylvester "waves" (see [11],[12]) of the restricted partition function $p_{\mathbf{k}}(n)$.

In Proposition 6.**1** we give new formulas for $p(n,k)$ and $q(n,k)$ in terms of coefficients of a reciprocal polynomial and, as a consequence, in Corollary 6.**2**, we prove some congruence relations for $p(n,k)$ and $q(n,k)$. In a recent preprint [9], K. Grajdzica found lower and upper bounds for the density of the set $\{n \in \mathbb{N} \ : \ p_{\mathbf{a}}(n) \equiv i(\bmod\ m)\}$ for a fixed integer $0 \leq i \leq m-1$. Using this, we prove lower and upper bounds for the density of the set $\{n \in \mathbb{N} \ : \ p(n,k) \equiv i(\bmod\ m)\}$; see Theorem 6.**3**.

2. **Preliminaries**

Let $\mathbf{a} := (a_1, a_2, \ldots, a_k)$ be a sequence of positive integers, $k \geq 1$. The *restricted partition function* associated to $\mathbf{a}$ is $p_{\mathbf{a}} : \mathbb{N} \to \mathbb{N}$, $p_{\mathbf{a}}(n) :=$ the number of integer solutions $(x_1, \ldots, x_k)$ of $\sum_{i=1}^k a_i x_i = n$ with $x_i \geq 0$. Note that the generating function of $p_{\mathbf{a}}(n)$ is

$$\sum_{n=0}^{\infty} p_{\mathbf{a}}(n) z^n = \frac{1}{(1-z^{a_1}) \cdots (1-z^{a_r})}. \tag{2.1}$$

Let D be a common multiple of a_1, $a_2, \ldots, a_k$. Bell [3] has proved that $p_{\mathbf{a}}(n)$ is a quasi-polynomial of degree $k-1$, with the period D, i.e.

$$p_{\mathbf{a}}(n) = d_{\mathbf{a},k-1}(n) n^{k-1} + \cdots + d_{\mathbf{a},1}(n) n + d_{\mathbf{a},0}(n), \tag{2.2}$$

where $d_{\mathbf{a},m}(n+D) = d_{\mathbf{a},m}(n)$ for $0 \leq m \leq k-1$ and $n \geq 0$, and $d_{\mathbf{a},k-1}(n)$ is not identically zero. Sylvester [11],[12] decomposed the restricted partition in a sum of "waves":

$$p_{\mathbf{a}}(n) = \sum_{j \geq 1} W_j(n, \mathbf{a}), \tag{2.3}$$

where the sum is taken over all distinct divisors j of the components of $\mathbf{a}$ and showed that for each such j, $W_j(n,\mathbf{a})$ is the coefficient of t^{-1} in

$$\sum_{0 \leq \nu < j,\ \gcd(\nu,j)=1} \frac{\rho_j^{-\nu n} e^{nt}}{(1-\rho_j^{\nu a_1} e^{-a_1 t}) \cdots (1-\rho_j^{\nu a_k} e^{-a_k t})},$$

where $\rho_j = e^{\frac{2\pi i}{j}}$ and $\gcd(0,0) = 1$ by convention. Note that $W_j(n,\mathbf{a})$'s are quasi-polynomials of period j. Also, $W_1(n,\mathbf{a})$ is called the *polynomial part* of $p_{\mathbf{a}}(n)$ and it is denoted by $P_{\mathbf{a}}(n)$.

It is well known that $p(n,k)$, the number of partitions of n with exactly k summands, equals to the number of partitions of n whose largest part is k.

It follows that

$$p(n,k) = \begin{cases} p_{(1,2,\ldots,k)}(n-k), \ n \geq k \\ 0, \ n < k \end{cases}. \tag{2.4}$$

There is a 1-to-1 correspondence between the partitions of n with k distinct parts and the partitions of $n - \binom{k}{2}$ with k parts, given by

$$a_1 < a_2 < \ldots < a_k \mapsto a_1 \leq a_2 - 1 \leq \cdots \leq a_k - (k-1).$$

Hence

$$q(n,k) = \begin{cases} p(n - \binom{k}{2}, k), \ n \geq k + \binom{k}{2} \\ 0, \ n < k + \binom{k}{2} \end{cases}. \tag{2.5}$$

From (2.1), (2.4) and (2.5) it follows that

$$\sum_{n=0}^{\infty} p(n,k)z^n = \frac{z^k}{(1-z)(1-z^2)\cdots(1-z^k)}, \ \sum_{n=0}^{\infty} q(n,k)z^n = \frac{z^{k+\binom{k}{2}}}{(1-z)(1-z^2)\cdots(1-z^k)},$$

are the generating functions for $p(n,k)$ and $q(n,k)$ respectively.

3. Main results

Let D_k be the least common multiple of $1, 2, \ldots, k$.

Proposition 3.1. *We have that*

$$p(n,k) = f_{k,k-1}(n)n^{k-1} + \cdots + f_{k,1}(n)n + f_{k,0}(n) \text{ for all } n \geq k,$$

where $f_{k,m}(n) = d_{\mathbf{k},m}(n-k)$, *and* $\mathbf{k} = (1, 2, \ldots, k)$.

Proof. It follows from (2.2) and (2.4). □

Theorem 3.2. (1) *For $n \geq k$ we have that:*

$$p(n,k) = \frac{1}{(k-1)!} \sum_{\substack{0 \leq j_1 \leq \frac{D_k}{1}-1, \ \ldots, 0 \leq j_k \leq \frac{D_k}{k}-1 \\ j_1+2j_2+\cdots+kj_k \equiv (n-k) \bmod D_k}} \prod_{\ell=1}^{k-1} \left(\frac{n-k-j_1-2j_2-\ldots-kj_k}{D_k} + \ell \right).$$

(2) *For $n \geq k + \binom{k}{2}$ we have that:*

$$q(n,k) = \frac{1}{(k-1)!} \sum_{\substack{0 \leq j_1 \leq \frac{D_k}{1}-1, \ \ldots, 0 \leq j_k \leq \frac{D_k}{k}-1 \\ j_1+2j_2+\cdots+kj_k \equiv (n-k-\binom{k}{2}) \bmod D_k}} \prod_{\ell=1}^{k-1} \left(\frac{n-k-\binom{k}{2}-j_1-2j_2-\ldots-kj_k}{D_k} + \ell \right).$$

Proof. (1) The result follows from [4, Corollary 2.10] and (2.4).
(2) It follows from (1) and (2.5). □

The *unsigned Stirling numbers* $\left[{n \atop k}\right]$ are defined by the identity

$$(x)^n := x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} \left[{n \atop k}\right] x^k.$$

The *Bernoulli numbers* B_ℓ's are defined by the identity

$$\frac{t}{e^t-1}=\sum_{\ell=0}^{\infty}\frac{t^\ell}{\ell!}B_\ell.$$

$B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$ and $B_n=0$ is n is odd and $n\geq 1$.

Proposition 3.3. (1) *For $n\geq 3$, we have that:*

$$p(n,3)=\sum_{m=1}^{3}\frac{(-1)^{m-1}}{6(m-1)!}\sum_{i_1+i_2+i_3=2-m}\frac{B_{i_1}B_{i_2}B_{i_3}}{i_1!i_2!i_3!}2^{i_2}3^{i_3}(n-3)^{m-1}+$$
$$+\frac{1}{12}\sum_{j=2}^{3}\sum_{\ell=1}^{j}\rho_j^\ell\sum_{k=0}^{2}\frac{1}{6^k}\begin{bmatrix}3\\k+1\end{bmatrix}\sum_{\substack{0\leq j_1\leq 5,\ 0\leq j_2\leq 2,\ 0\leq j_3\leq 1\\ j_1+2j_2+3j_3\equiv\ell(\bmod\ j)}}(j_1+2j_2+3j_3)^k,$$

where $\rho_j=e^{\frac{2\pi i}{j}}$.

(2) *For $n\geq 6$, we have that:*

$$q(n,3)=\sum_{m=1}^{3}\frac{(-1)^{m-1}}{6(m-1)!}\sum_{i_1+i_2+i_3=2-m}\frac{B_{i_1}B_{i_2}B_{i_3}}{i_1!i_2!i_3!}2^{i_2}3^{i_3}(n-6)^{m-1}+$$
$$+\frac{1}{12}\sum_{j=2}^{3}\sum_{\ell=1}^{j}\rho_j^\ell\sum_{k=0}^{2}\frac{1}{6^k}\begin{bmatrix}3\\k+1\end{bmatrix}\sum_{\substack{0\leq j_1\leq 5,\ 0\leq j_2\leq 2,\ 0\leq j_3\leq 1\\ j_1+2j_2+3j_3\equiv\ell(\bmod\ j)}}(j_1+2j_2+3j_3)^k,$$

Proof. (1) Since $1,2,3$ are coprime, the conclusion follows from [6, Proposition 4.3] and the fact that $p(n,3)=p_{(1,2,3)}(n-3)$ for $n\geq 3$.

(2) Follows from (1) and the fact that $q(n,3)=p(n-3,3)$ for $n\geq 6$. □

The *Bernoulli polynomials* are defined by

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k.$$

For $\mathbf{a}=(a_1,\dots,a_k)$, the Bernoulli-Barnes numbers (see [1]) are

$$B_j(\mathbf{a})=\sum_{i_1+\cdots+i_j=j}\binom{j}{i_1,\dots,i_k}B_{i_1}\cdots B_{i_k}a_1^{i_1}\cdots a_k^{i_k}.$$

We consider the determinant:

$$\Delta(k):=\begin{vmatrix}\frac{B_1(\frac{1}{D_k})}{1} & \cdots & \frac{B_1(1)}{1} & \cdots & \frac{B_k(\frac{1}{D_k})}{k} & \cdots & \frac{B_k(1)}{k}\\ \frac{B_2(\frac{1}{D_k})}{2} & \cdots & \frac{B_1(1)}{1} & \cdots & \frac{B_{k+1}(\frac{1}{D_k})}{k+1} & \cdots & \frac{B_{k+1}(1)}{k+1}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \frac{B_{kD_k}(\frac{1}{D_k})}{kD_k} & \cdots & \frac{B_{kD_k}(1)}{kD_k} & \cdots & \frac{B_{kD_k+k-1}(\frac{1}{D_k})}{kD_k+k-1} & \cdots & \frac{B_{kD_k+k-1}(1)}{kD_k+k-1}\end{vmatrix}.$$

Theorem 3.4. *If $\Delta(k) \neq 0$, then $p(n,k)$ can be computed in terms of $B_j(\frac{v}{D_k})$, $1 \leq v \leq k$, $1 \leq j \leq kD_k$ and $B_j(\mathbf{k})$, $0 \leq j \leq kD_k$, where $\mathbf{k} = (1,2,\dots,k)$.*

Proof. According to [5, Formula (1.8)], we have that:

$$\sum_{m=0}^{k-1}\sum_{v=1}^{D_k} d_{\mathbf{k},m}(v) D^{n+m} \frac{B_{n+m+1}(\frac{v}{D})}{n+m+1} = \frac{(-1)^{n-1}k!}{(n+k)!} B_{n+k}(\mathbf{k}) - \delta_{0n},\ (\forall) n \geq 0, \quad (3.1)$$

where δ_{0n} is the *Kronecker's symbol.* Giving values $0 \leq n \leq kD_k - 1$ in (3.1), and seeing $d_{\mathbf{k},m}(v)$'s as variables, we obtain a system of kD_k linear equations, with the determinant equal to $\pm D_k^N \Delta(k)$ for some integer $N \geq 1$. By hypothesis, $\Delta(k) \neq 0$, hence we can determine $d_{\mathbf{k},m}(v)$ by solving the system. From (2.2), one has

$$p_{\mathbf{k}}(n) = d_{\mathbf{k},k-1}(n) n^{k-1} + \cdots + d_{\mathbf{k},1}(n) n + d_{\mathbf{a},0}(n).$$

Hence, the conclusion follows from (2.4) and (2.5). □

4. The polynomial part of $p(n,k)$ and $q(n,k)$

We recall the following basic facts on quasi-polynomials [10, Proposition 4.4.1]:

Proposition 4.1. *The following conditions on a function $f : \mathbb{N} \to \mathbb{C}$ and integer $D > 0$ are equivalent.*

(i) *$f(n)$ is a quasi-polynomial of period D.*

(ii) *$\sum_{n=0}^{\infty} f(n) z^n = \frac{L(z)}{M(z)}$, where $L(z), M(z) \in \mathbb{C}[z]$, every zero λ of $M(z)$ satisfies $\lambda^D = 1$ (provided $\frac{L(z)}{M(z)}$ has been reduced to lowest terms), and* $\deg L(z) < \deg M(z)$.

(iii) *For all $n \geq 0$, $f(n) = \sum_{\lambda^D=1} F_\lambda(n) \lambda^{-n}$, where each $F_\lambda(n)$ is a polynomial function. Moreover, $\deg F_\lambda(n) \leq m(\lambda) - 1$, where $m(\lambda)$ = multiplicity of λ as a root of $M(z)$.*

We define the *polynomial part* of $f(n)$ to be the polynomial function $F(n) = F_1(n)$, with the notation of Proposition 4.**1**. The polynomial part $F(n)$ of a quasi-polynomial $f(n)$ gives a rough approximation of $f(n)$, which is useful for studying the asymptotic behaviour of $f(n)$, when $n \gg 0$. If $\mathbf{a} = (a_1, \dots, a_k)$ is a sequence of positive integers and $p_{\mathbf{a}}(n)$ is the restricted partition function associated to $\mathbf{a}$, we denote $P_{\mathbf{a}}(n)$, the polynomial part of $p_{\mathbf{a}}(n)$. Several formulas of $P_{\mathbf{a}}(n)$ were proved in [2], [7] and [4].

We consider the following functions:

$$P(n,k) = P_{(1,2,\dots,k)}(n-k),\ n \geq k,$$
$$Q(n,k) = P_{(1,2,\dots,k)}\left(n - k - \binom{k}{2}\right),\ n \geq k + \binom{k}{2},$$

and we called them, the *polynomial part* of $p(n,k)$ and $q(n,k)$, respectively.

Theorem 4.2. *For $n \geq k$, we have that:*

(1) $$P(n,k) = \frac{1}{D_k(k-1)!} \sum_{0\leq j_1 \leq \frac{D_k}{1}-1,\ ,\dots,0\leq j_k\leq \frac{D_k}{k}-1} \prod_{\ell=1}^{k-1}\left(\frac{n-k-j_1-2j_2-\dots-kj_k}{D_k}+\ell\right).$$

(2) $$P(n,k) = \frac{1}{k!}\sum_{u=0}^{k-1}\frac{(-1)^u}{(k-1-u)!}\sum_{i_1+\cdots+i_k=u}\frac{B_{i_1}\cdots B_{i_k}}{i_1!\cdots i_k!}1^{i_1}\cdots k^{i_k}(n-k)^{k-1-u}.$$

Proof. (1) It follows from [4, Corollary 3.6] and (2.4). See also [7, Theorem 1.1].

(2) It follows from [4, Corollary 3.11] and (2.4). See also [2, p.2]. □

Corollary 4.3. *For $n \geq k + \binom{k}{2}$, we have that:*

(1) $$Q(n,k) = \frac{1}{D_k(k-1)!} \sum_{0\leq j_1 \leq \frac{D_k}{1}-1,\ ,\dots,0\leq j_k\leq \frac{D_k}{k}-1} \prod_{\ell=1}^{k-1}\left(\frac{n-k-\binom{k}{2}-j_1-2j_2-\dots-kj_k}{D_k}+\ell\right).$$

(2) $$Q(n,k) = \frac{1}{k!}\sum_{u=0}^{k-1}\frac{(-1)^u}{(k-1-u)!}\sum_{i_1+\cdots+i_k=u}\frac{B_{i_1}\cdots B_{i_k}}{i_1!\cdots i_k!}1^{i_1}\cdots k^{i_k}\left(n-k-\binom{k}{2}\right)^{k-1-u}.$$

Proof. It follows from Theorem 4.**2** and (2.5). □

5. The Sylvester waves of $p(n,k)$ and $q(n,k)$

Let $\mathbf{k} := (1,2,\dots,k)$. According to (2.3), the restricted partition function $p_{\mathbf{k}}(n)$ can be writen as a sum of "waves", $p_{\mathbf{k}}(n) = \sum_{j=1}^{k} W_j(n,\mathbf{k})$. We define the functions

$$W_j(n,k) = W_j(n-k,\mathbf{k}),\ n\geq k, \tag{5.1}$$

$$\widetilde{W}_j(n,k) := W_j(n-k-\binom{k}{2},\mathbf{k}),\ n\geq k+\binom{k}{2}, \tag{5.2}$$

and we call them the "waves" of $p(n,k)$ and $q(n,k)$, respectively.

Remark 5.1. Note that $P(n,k) = W_1(n,k)$ and $Q(n,k) = \widetilde{W}_1(n,k)$ are the polynomial parts of $p(n,k)$ and $q(n,k)$.

Proposition 5.2. (1) *For any positive integers $1 \leq j \leq k \leq n$, we have that:*

$$W_j(n,k) = \frac{1}{D_k(k-1)!}\sum_{m=1}^{k}\sum_{\ell=1}^{j}\rho_j^{\ell}\sum_{t=m-1}^{k-1}\begin{bmatrix}k\\ t+1\end{bmatrix}(-1)^{t-m+1}\binom{t}{m-1}\cdot$$
$$\cdot \sum_{\substack{0\leq j_1\leq D_k-1,\dots,0\leq j_k\leq \frac{D_k}{k}-1\\ j_1+\cdots+kj_k\equiv \ell(\bmod j)}} D_k^{-t}(j_1+\cdots+kj_k)^{t-m+1}(n-k)^{m-1}.$$

(2) *For any positive integers* $1 \leq j \leq k$ *and* $n \geq k + \binom{k}{2}$, *we have that:*

$$\widetilde{W}_j(n,k) = \frac{1}{D_k(k-1)!} \sum_{m=1}^{k} \sum_{\ell=1}^{j} \rho_j^{\ell} \sum_{t=m-1}^{k-1} \begin{bmatrix} k \\ t+1 \end{bmatrix} (-1)^{t-m+1} \binom{t}{m-1} \cdot$$

$$\cdot \sum_{\substack{0 \leq j_1 \leq D_k - 1, \ldots, 0 \leq j_k \leq \frac{D_k}{k} - 1 \\ j_1 + \cdots + kj_k \equiv \ell (\bmod j)}} D_k^{-t} (j_1 + \cdots + kj_k)^{t-m+1} (n - k - \binom{k}{2})^{m-1}.$$

Proof. (1) It follows from [6, Proposition 4.2] and (5.1).

(1) It follows from [6, Proposition 4.2] and (5.2). □

6. New formulas and congruences for $p(n,k)$ and $q(n,k)$

We consider the function:

$$f(n,k) := \#\{(j_1, \ldots, j_k) \ : \ j_1 + 2j_2 + \cdots + kj_k = n, \ 0 \leq j_i \leq \frac{D_k}{i} - 1, \ 1 \leq i \leq k\},$$

where D_k is the least common multiple of $1, 2, \ldots, k$. Let $d_k := kD_k - \binom{k+1}{2}$. Note that $f(n,k) = f(d-n,k)$, for $0 \leq n \leq d$, and $f(n,k) = 0$ for $n \geq d+1$. It follows that $F(n,k) = \sum_{\ell=0}^{d_k} f(n,k)x^{\ell}$ is a reciprocal polynomial. With the above notations we have:

Proposition 6.1. (1) *For* $n \geq k$ *we have that:*

$$p(n,k) = \sum_{j=\left\lceil \frac{n+\binom{k}{2}}{D_k} \right\rceil - k}^{\left\lfloor \frac{n-k}{D_k} \right\rfloor} \binom{k+j-1}{j} f(n-k-jD_k).$$

(2) *For* $n \geq k + \binom{k}{2}$ *we have that:*

$$q(n,k) = \sum_{j=\left\lceil \frac{n}{D_k} \right\rceil - k}^{\left\lfloor \frac{n-k-\binom{k}{2}}{D_k} \right\rfloor} \binom{k+j-1}{j} f(n-k-\binom{k}{2} - jD_k).$$

Proof. (1) It follows from [6, Proposition 2.2], [6, Corollary 2.3] and (2.4).

(2) It follows from (1) and (2.5). □

Corollary 6.2. (1) *For* $n \geq k$ *we have that:*

$$(k-1)!p(n,k) \equiv 0 \mod (j+\ell+1)(j+\ell+2)\cdots(j+k-1),$$

where $\ell = \left\lfloor \frac{n-k}{D_k} \right\rfloor - \left\lceil \frac{n+\binom{k}{2}}{D_k} \right\rceil + k$.

(2) *For* $n \geq k + \binom{k}{2}$ *and* $\ell' = \left\lfloor \frac{n-k-\binom{k}{2}}{D_k} \right\rfloor - \left\lceil \frac{n}{D_k} \right\rceil + k$ *we have that:*

$$(k-1)!q(n,k) \equiv 0 \mod (j+\ell'+1)(j+\ell'+2)\cdots(j+k-1),$$

Theorem 6.3. (1) *The inequality* $\lim_{N\to\infty} \frac{\#\{n\leq N\,:\,p(n,k)\equiv 1(\bmod\ 2)\}}{N} \leq \frac{2}{3}$, *holds for infinitely many positive integers* k. *Moreover, if the above inequality is not satisfied for some positive integers* k, *then it holds for* $k+1$.

(2) *Let* $m>1$ *be a positive integer. For each positive integer* k *we have*

$$\lim_{N\to\infty} \frac{\#\{n\leq N\ :\ p(n,k)\not\equiv 0(\bmod m)\}}{N} \geq \frac{1}{\binom{k+1}{2}}.$$

The above results hold if we replace $p(n,k)$ *with* $q(n,k)$.

Proof. (1) Since $p(n,k)=p_{\mathbf{k}}(n-k)$ for $n\geq k$, where $\mathbf{k}=(1,2,\ldots,k)$, the result follows from [9, Theorem 4.2]. (2) Follows from [9, Theorem 5.2]. The asymptotic behavior is the same if we replace $p_{\mathbf{k}}(n)$ with $p(n,k)$ or $q(n,k)$. □

7. Conclusions

We proved new formulas and congruences for $p(n,k)$ (and $q(n,k)$), the number of partitions of n into k parts (into k distinct parts), and their polynomial parts. Also, we gave lower and upper bounds for the density of the set $\{n\in\mathbb{N}\ :\ p(n,k)\equiv i(\bmod m)\}$. Our method can be adapted for other partition functions.

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