A TOPOLOGICAL DEGREE OF SET-VALUED MAPS OF TYPE *(S)*

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La studiul existenţei soluţilor inegalităţilor variaţionale au fost folosite câteva definiţii ale gradului topologic. In cazul operatorilor univalenţi tari monotoni este binecunoscută, de exemplu, metoda lui Szulkin [12]. Deoarece inegalităţiile variationale pot fi rescrise ca incluziuni operatoriale, dupa cum vom arăta mai jos, o abordare mai completă necesită definirea gradelor topologice pentru aplicaţii multivalente. Vom dezvolta o teorie a gradului topologic pentru aplicaţii multivalente de tip (S).

Within the study of the solutions of variational inequalities, some definitions of the topological degree are used. So, Szulkin's method [12] is well-known in the case of strongly monotone operators. Since variational inequalities can be converted into operator inclusions, as we reveal below, a more thorough approach is to define an appropiate degree for set-valued maps. We deal with a theory of the topological degree for mappings of type (S).

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1. Introduction

Roughly speaking, all procedures for solving the operator equations on finite-dimensional spaces are based on Brouwer's degree. One can distinguish between two main directions in the construction of topological degrees on infinite-dimensional spaces. The first method defines a degree as the limit of Brouwer's degrees, in the sense of assuring the strong convergence of the solutions of the restrictited equations to the finite-dimensional subspaces. This Galerkin approach has been used by Browder [3] in the case of type *(S)* operators and, more general, by Skrypnik [10] for operators of type(α), on Banach spaces. The second Hilbertean approach, employing an elliptic super-regularization, takes as starting-point the Leray-Schauder degree in order to define a topological degree for type *(S)* operators on Banach spaces. In this paper, we will pursue the Melnik method [7], which represents a similar construction like that used by Browder-

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Petryshyn for the set-valued degree of A-proper maps. With respect to the second approach with applications to problems involving discontinuous nonlinearities, we refer to the Berkovits-Tienari work [2].

Let X be a real reflexive Banach space, X^* its topological dual and $\langle \cdot,\cdot \rangle : X^* \times X \rightarrowtail R$ the pairing of elements from X^* and X. We denote by 2^{X^*} the totality of all nonempty subset of X^* and consider multivalued or set-valued mapping $A : D(A) \rightarrowtail 2^{X^*}$ with its *effective domain* $D(A) = \{ y \in X \mid A(y) \neq \varnothing \}$.

First, we remind the following equivalence [1]. For a convex, closed subset K of X and element $g \in X^*$, a variational inequality means finding an element $u \in K$ such that

$$< Au - g, x - u > \;\geq\; 0 \quad \text{for all} \;\; x \in K. \tag{1.1}$$

The variational inclusion (1.1) is equivalent to the inclusion

$$u \in K, \qquad g \in A(u) \;+\; N_K(u), \tag{1.2}$$

where $N_K(x) = \{ p \in X^* \mid \sup_{y \in K} < p, x - y > \;\geq\; 0 \}$ is the normal cone to K in $x \in K$.

We notice that N_K is the subdifferential of the indicator function I_K. More general, given a subdifferentiable function $\Psi : X \rightarrowtail R \cup \{+\infty\}$, the problem of finding a solution $u \in dom\ \Psi$ of the (complete) variational inequality:

$$< Au - g, x - u > \;+ \Psi(x) \;- \Psi(u) \;\geq\; 0, \quad \forall\, x \in dom\ \Psi, \tag{1.3}$$

which is equivalent to the inclusion

$$u \in dom\Psi, \qquad g \in A(u) + \partial\Psi(u). \tag{1.4}$$

2. Galerkin's set-valued approximants

Let D be an bounded open subset of $D(A)$, with the boundary ∂D, and $\mathcal{F}(X)$ the class of finite-dimensional subspaces of X so that $D \cap F \neq \emptyset$. We choose a base $\{e_1, e_2, ..., e_n\}$ of every subspace $F \in \mathcal{F}(X)$ and define Galerkin's approximante $A_F : F \rightarrowtail 2^F$ of $A : D(A) \rightarrowtail 2^{X^*}$ relative to F by

$$< A_F(x), y >_{F \times F} = < A(x), y >_{X^* \times X} \quad , \ \forall\, x \in \bar{D} \cap F, \ y \in F.$$

Here the duality on $F \times F$ coincides with the inner product on F. In other words, the set-valued Galerkin restriction A_F of A relative to F have the following structure:

$$A_F(x) = \bigcup_{f \in A(x)} \{\sum_{i=1}^{n} < f, e_i > e_i\} = J_F^*(A(J_F x)), \quad \text{for every} \quad x \in \bar{D}_F = \bar{D} \cap F,$$

where $J_F : F \rightarrowtail X$ is the imedding map and $J_F^* : X^* \rightarrowtail F$ its adjunct. The above structure does not necessarily depend on the base chosen in F.

The main elements of the Brouwer and Leray-Schauder degree theory for set-valued maps are presented in the chapter VI of Lloyd's monography [6].

With a view to simplification, let $C(X^*)$ be the family of the non-empty, convex, closed subsets of X^*, and we consider the following definition:

The map $A : D \rightarrowtail C(X^*)$ is said to be *of type* $S(D)$, if any sequence $y_n \in D$ with $y_n \to y$ in X and $f_n \in A(y_n)$ with $f_n \to f$ in X^*, such that

$$\limsup_{n\to\infty} < A(y_n), y_n - y > \ \leq 0 \tag{2.1}$$

implies the strong convergence $y_n \to y$ în X.

The condition (2.1) can be re-written as $\limsup_{n\to\infty} < A(y_n), y_n > \ \leq \ < f, y >$ and maps of type $S(D)$ coincide with those of type (α).

3.Topological degree for set-valued maps of type *S*.

With a view of a general definition to the (S) degree, let us consider that the following hypotheses hold:

(i) $A : D \rightarrowtail C(X^*)$

(ii) $A \in S(D)$;

(iii) $0 \notin A(y)$ for every $y \in \partial D$.

Like in the univalent case (Skrypnik [10], p.39), we can establish the existence of a finite-dimensional space $F_o \in \mathcal{F}(X)$ with the following features:

(A) $0 \notin A_F(y)$ for every $y \in \partial D$ and

(B) $deg(A_F, D, 0) = deg(A_{F_o}, D, 0)$, for any space $F \in \mathcal{F}(X)$ with $F_o \subseteq F$.

The word *deg* refers here to the Brouwer degree.

Under the hypotheses (i)-(iii). we take by definition

$$d_S(A, D, 0) = deg(A_{F_o}, D, 0),$$

as the degree *(S)* of the set-valued map $A : D \rightarrowtail C(X^*)$ on the D subset with respect to $0 \in X^*$. Moreover, due to the invariance of Brouwer's degree under translations, for any $f \in X^* \setminus A(\partial D)$, we have

$$d_S(A, D, f) = deg(A - f, D, 0).$$

The above-defined degree verifies all the axioms of the classic topological degree (Pascali [9]). We say that $A_t : [0,1] \times \bar{D} \rightarrowtail C(X^*)$ is a homotopy *of type* $S(D)$ if every sequence $\{y_n\} \subset D$ with $y_n \to y_0$ in *X*, $\{t_n\} \subset [0,1]$ with $t_n \to t$ and $f_n \in A(t_n, y_n)$ with $f_n \to f$ în X^* and the condition

$$\limsup_{n\to\infty} \langle A_{t_n}(y_n), y_n \rangle \leq \langle f, y \rangle$$

is in fact the strongly convergent $y_n \to y_0$ în X and $A_{t_n}(y_n) \to A_t(y)$ results in X^*.

Maps $A_0, A_1 : \overline{D} \subset X \rightarrowtail C(X^*)$ belonging to the class $S(D)$ and satisfying condition $0 \notin A_i(y)$, for *i =1,2*, for each $y \in \partial D$, are called *homotopic* on $\overline{D}$ if there is a bounded map $A : [0,1] \times \overline{D} \rightarrowtail 2^{X^*}$ satisfying the following conditions:

(i) $A(0, \cdot) = A_0$, $A(1, \cdot) = A_1$;

(ii) A satisfies condition $S(D)$;

(iii) $0 \notin A(t, y)$ for any $t \in [0,1]$ and any $y \in \partial D$;

(iv) A is demiclosed; i.e., if $t_n \to t_0$ and $y_n \to y_0$ strongly in *X,* and $d_n \to d_0$ in X^*, for any $d_n \in A(t_n, y_n)$, then $d_0 \in A(t_0, y_0)$.

If A_0 and A_1 are homotopic on $\overline{D}$, then

$$\deg\left(A_0,\overline{D},0\right)=\deg\left(A_1,\overline{D},0\right).$$

If $A:\overline{D}\subset X \rightarrowtail 2^{X^*}$ is a map of class $S(D)$ and $0\notin A(y)$ for any $y\in\overline{D}$, then $\deg\left(A,\overline{D},0\right)=0$.

Let $A:\overline{D}\subset X \rightarrowtail 2^{X^*}$ be a map of class $S(D)$ and $f\in X^*$ satisfying condition $f\notin A(y)$ for any $y\in\partial D$. In order that the inclusion $A(y)\ni f$ to have a solution in *D*, it is sufficient that $\deg\left(A,\overline{D},f\right)\neq 0$.

Moreover, let $A:\overline{D}\subset X \rightarrowtail C\left(X^*\right)$ is a map of class $S(D)$ coercive with respect to $f\in X^*$, i.e., $\langle A(y)-f,y\rangle\geq 0$, for any $y\in\partial D$. In this case, we have $\deg\left(A,\overline{D},f\right)\neq 0$ and so the inclusion $A(y)\ni f$ has solutions in D.

Furthermore, the next result of the theory of the topological degree can be generalised. Suppose that *D* is a symmetric bounded neighborhood of zero, $A:\overline{D}\subset X \rightarrowtail C\left(X^*\right)$ is a map of class $S(D)$ and $0\notin A\left(\partial D\right)$. Suppose, in addition, that

$$A(y)\cap\lambda A(-y)=\varnothing \quad \text{for } y\in\partial D \text{ and } \lambda\in[0,1],$$

then $\deg\left(A,\overline{D},0\right)$ is an odd number.

The above-mentioned approach is also valid in the case of variational inequalities of Solonoukha type [11] if we consider the next definition:

The $A:D\rightarrowtail C\left(X^*\right)$ is a map of type $S_-(D)$, if for any sequence $\{y_n\}$ in D onverging weakly to some $y\in X$ and for any sequence $f_n\in A(y_n)$ converging weakly to some $f\in X^*$ such that

$$\limsup_{n\to\infty}\left[A(y_n),y_n-y\right]_-\leq 0$$

it follows that $y_n\to y$ in *X*.

4. Degree for pseudomonotone maps.

We will now take the line traced by the definition presented at the end of the previous chapter and we define

A mapping A from D into $C\left(X^*\right)$ is *weakly pseudomonotone* or $PM_{-}(D)$ if for any sequence $\{y_n\}$ in D converging weakly to some $y \in X$ and for any sequence $f_n \in A(y_n)$ converging weakly to some $f \in X^*$ such that

$$\limsup_{n\to\infty}\left[A\left(y_n\right), y_n - y\right]_{-} \le 0$$

it follows that $f \in A(y)$ and $< f_n, y_n > \ \to \ < f, y >$.

We remark that if A is weakly pseudomonotone, then

$$\limsup_{n\to\infty} < Ay_n - Ay, y_n - y > \ = \ 0.$$

We extend a basic relation due to B Calvert and J.R.L. Webb (see [9}) between the pseudomonotones operators and those of type S.

Theorem 4.1. *Let* $D \subset X$ *be a open, bounded subset and* $A_0 : \overline{D} \subset X \rightarrowtail C\left(X^*\right)$ *satisfies condition* $S_{-}\left(D\right)$. *Then demicontinuous operator* $A : \overline{D} \subset X \rightarrowtail C\left(X^*\right)$ *is pseudomonoton and* $0 \notin \overline{A\left(\partial D\right)}$ *if and only if* $A_\varepsilon = \varepsilon A_0 + A : \overline{D} \subset X \rightarrowtail C\left(X^*\right)$ *satisfies condition* $S_{-}\left(D\right)$, *for each* $\varepsilon > 0$.

Proof. The "if" part. Let A be a weakly pseudomonotone operator and assume that

$$\limsup_{n\to\infty}\left[\left(\varepsilon A_0 + A\right)x_n - \left(\varepsilon A_0 + A\right)x, x_n - x\right]_{-} \le 0$$

whenever $x_n \to x$ in X. Since $A \in PM_{-}(D)$, we have $\limsup_{n\to\infty}\left[A_0 x_n - A_0 x, x_n - x\right]_{-} \le 0$, we infer that $x_n \to x$ in X, that is, $\varepsilon A_0 + A$ is of type $S_{-}\left(D\right)$.

The „only if" part. Assume that $\varepsilon A_0 + A$ is of type $S_{-}\left(D\right)$ for each $\varepsilon > 0$. If A is not pseudomonotone, then there exists a sequence $\{x_n\}$ such that $x_n \to x$ and $\limsup_{n\to\infty}\left[Ax_n - Ax, x_n - x\right]_{-} = -\delta$ with $\delta > 0$. Then, since A is demicontinuous $\{x_n\}$ can not be strongly convergent to x. On the other side, because $\{x_n\}$ is bounded, there is an $M > 0$ such that $\|x_n\| \le M$ and $\left|\varepsilon\left[A_0 x_n - A_0 x, x_n - x\right]_{-}\right| < 4\varepsilon M^2$. Take $\varepsilon < \dfrac{\delta}{8M^2}$ and have

$$\limsup_{n\to\infty}\left[\left(\varepsilon A_0 + A\right)x_n - \left(\varepsilon A_0 + A\right)x, x_n - x\right]_{-} \le -\frac{1}{2}\delta < 0$$

and $\{x_n\}$ doesn't converge strongly to *x*, which contradict our initial assumption that $\varepsilon A_0 + A$ is of type $S_-(D)$. □

Moreover, $0 \notin A_\varepsilon(y)$ for any $y \in \partial D$. Thus, $\deg\left(A_\varepsilon, \overline{D}, 0\right)$ is defined if $0 < \varepsilon < \delta_0 M^{-1}$. Let us show that the degree defined in this way does not depend on ε. Suppose that $0 < \varepsilon_i < \delta_0 M^{-1}$ for $i = 1,2$ and consider the corresponding A_{ε_i}. Put $A(t,y) = \left(y\varepsilon_2 + (1-t)\varepsilon_1\right)A_0(y) + A(y)$. We have $A(0,y) = A_{\varepsilon_1}(y)$, $A(1,y) = A_{\varepsilon_2}(y)$, and $0 \notin A(t,y)$ for any $t \in [0,1]$ and any $y \in \partial D$. Consequently,

$$\deg\left(A_{\varepsilon_1}, \overline{D}, 0\right) = \deg\left(A_{\varepsilon_2}, \overline{D}, 0\right).$$

Hence, the limit $\lim_{\varepsilon \to \infty} \deg\left(A_\varepsilon, \overline{D}, 0\right)$ exists, we call it the degree of the pseudo-monotone map *A* on the domain $\overline{D}$ with respect to $0 \in X^*$ and denote it by $\deg\left(A, \overline{D}, 0\right)$.

We notice that in the above construction the limit does not depend on map A_0 and therefore the degree of the weak pseudomonotones maps is well-defined.

A similar degree theory of the $S(D)$ mappings was elaborated in the book of D. O'Regan, Y. J. Cho, Y.-Q. Chen [8] in 2006. Applications of the (S)-degree for perturbations of maximal monotone operators in the theory of variational inequalities have been tackled in [8] as well.

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R E F E R E N C E S

[1] *P. Anghel*, Monotone variational inequalities revisited, Math. Reports, 2007. (to appear)
[2] *J. Berkovits – M. Tienari*, Topological degree theory for some classes of multis with applications to hyperbolic and elliptic problems involving discontinuous nonlinearities, Dynam. Systems Appl., **5** (1996), no. 1, 1-18; MR 96m:47112.
[3] *F. E. Browder*, Nonlinear eigenvalue problems and Galerkin approximations, Bull. Amer. Math. Soc., **74** (1968), 651-656.
[4] *F.E. Browder – W. V. Petryshyn*, Approximation methods and the generalized topological degree for nonlinear mappings in Banach spaces, J. Funct. Anal., **3** (1969), 217-274.

[5] *S. Fulina*, On pseudomonotone variational inequalities, An. St. Univ. Ovidius Constantza, Ser. Mat., **14,** *1*, (2006), 83-90.
[6] *N. G. Lloyd*, Degree theory, Cambridge University Press , 1978.
[7] *V. S. Mel'nik*, Multivariational inequalities and operator inclusions in Banach spaces with mappings of the class $(S)_+$, Ukrainian Mathematical Journal, **52** (2000), no. 11, MR 2002f:47135.
[8] *D. O'Regan - Y. J. Cho - Y. Q. Chen*, Topological degree theory and applications, Series Math. Anal. Appl.10, Chapman & Hall CRC, Bocca Raton,2006; MR 2007b; 47168
[9] *D. Pascali*, Topological Methods in Nonlinear Analysis: Topological degree for monotone mappings, Ovidius Univ. Constanța and Courant Institute, New York University, 2001.
[10] *I.V. Skrypnik,* Methods of Analysis of Nonlinear Elliptic Boundary Value Problems, (translated from the 1990 Russian original by Dan Pascali), Translations of Math. Monographs **139,** Amer. Math. Soc., Providence, RI, 1994, MR 95i:35109.
[11] *O. V. Solonoukha,* On the stationary variational inequalities with the generalized pseudomonotone operators, Methods of Functional Analysis and Topology, **3** (1997), no. 3, 81-95; MR 2002d:47091.
[12] *A. Szulkin,* Positive solutions of variational inequalities: A degree-theoretic approach, J. Diff. Equations, **57** (1985), 90-111: MR 86i:47072.