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ISOPERIMETRIC INEQUALITIES IN MINKOWSKI SPACE M_2

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*In prima parte a acestei lucrări sunt prezentate două formule variaționale în spațiul **M_2**, obținute de O. Biberstein în ([1]). Acestea conduc la problema izoperimetrică rezolvată de H. Busemann în ([2]), ([3]).*

*Autorul obține inegalitatea, $a^2 L_\Gamma^2 \geq 4 A_\Gamma A_{\Gamma'}$, unde Γ este o curbă convexă, închisă de clasa C^1 în spațiul **M_2**, cu lungimea L_Γ şi aria A_Γ. Curba Γ' este un anti-cerc în spațiul **M_2**, cu raza $a > 0$ şi aria $A_{\Gamma'}$.*

*Dacă $a = 1$ şi $\Gamma' = T$, unde T este izoperimetrica spațiului **M_2**, se obține inegalitatea izoperimetrică, $L^2 \geq 4 A_\Gamma A_T$, unde A_T este aria izoperimetricei. În partea a doua a lucrării, această inegalitate este generalizată pentru curbe închise din spațiul **M_2**, care nu sunt convexe.*

*In the first part of this paper are presented two variational formulas in the space **M_2**, obtained by O. Biberstein in ([1]). These lead to the isoperimetric problem solved by H. Busemann in ([2]), ([3]).*

*The author obtain the inequality, $a^2 L_\Gamma^2 \geq 4 A_\Gamma A_{\Gamma'}$, where Γ is a convex, closed curve of class C^1 in **M_2** space, with the length L_Γ and the area A_Γ. The curve Γ' is a anti-cercle in **M_2** space, with the radius $a > 0$ and area $A_{\Gamma'}$.*

*If $a = 1$ and $\Gamma' = T$, where T is the isoperimetric in **M_2** space, we obtain the isoperimetric inequality, $L^2 \geq 4 A_\Gamma A_T$, where A_T is the area of isoperimetric T.*

*In the second part of the paper, this inequality is generalized for closed curves in **M_2**, which are not convex.*

1. Introduction

Let $\mathbf{V}_n$ be a differentiable manifold of class C^1 and let **F** be a family of submanifolds of dimension $p < n$, which depend on r real parameters.

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An arbitrary submanifold $\boldsymbol{S_p} \in \mathbf{F}$ is given by a submersion which is represented with respect to two local maps $\left(H_1; x^1,..., x^n\right)$ on $\mathbf{V}_{\boldsymbol{n}}$ and $\left(H_2; u^1,..., u^p\right)$ on $\boldsymbol{S_p}$ as follows:

$$x^i = x^i\left(u^1,..., u^p, \alpha^1,..., \alpha^r\right), \quad i = \overline{1, n} \tag{1}$$

where the functions x^i ar of class C[1] relative to $u^j, \alpha^k, j = \overline{1, p}, k = \overline{1, r}$.

Let ω be a differentiable p - form of class C[1] on $\mathbf{V}_{\boldsymbol{n}}$ an $\omega \bullet \mathbf{v}$ the inner product of the vector field **v** defined on $\mathbf{V}_{\boldsymbol{n}}$ with the differentiable form ω.

For each $\boldsymbol{S_p} \in \mathbf{F}$ we consider functional

$$\mathrm{I} = \int_{\boldsymbol{S_p}} \omega. \tag{2}$$

Throughout this paper, the manifold $\mathbf{V}_{\boldsymbol{n}}$ will be replaced with Minkowski space $\mathbf{M_2}$, and instead of relation (1) we shall use the vectorial notation,

$$\overline{\mathbf{M}} = \overline{\mathbf{M}}\left(u^1,..., u^p, \alpha^1,..., \alpha^r\right). \tag{3}$$

We denote

$$\delta I = \sum_{k=1}^{r} \frac{\partial I}{\partial \alpha^k} \delta\alpha^k, \quad \delta\overline{\mathbf{M}} = \sum_{k=1}^{r} \frac{\partial \overline{\mathbf{M}}}{\partial \alpha^k} \delta\alpha^k .$$

The variation of the functional (2) will be:

$$\delta I = \int_{\boldsymbol{S_p}} \mathrm{d}\,\omega \bullet \delta\overline{\mathbf{M}} + \int_{\partial \boldsymbol{S_p}} \omega \bullet \delta\overline{\mathbf{M}}, \tag{4}$$

where $\partial \boldsymbol{S_p}$ is the boundary of the manifold $\boldsymbol{S_p}$ and $\mathrm{d}\,\omega$ is the exterior differential of the form ω.

It is known ([1], [5]) that fixing in space $\mathbf{R}^2$ a closed, convexe curve **U** of class C[3], without stationary points, with area π and central simmetric with respect to the origin **O**, it can be defined minkowskian norm in $\mathbf{M_2}$.

The curve **U** named indicatrix has the parametric equation:

$$t = t(\varphi), 0 \le \varphi \le 2\pi,$$

where φ represents the double of the area of the sector $\left(\mathbf{O}, \boldsymbol{t_0}, \boldsymbol{t}\right)$, named the amplitude of ***t***.

The vector $\boldsymbol{t_0}$ corresponds to a fixed point on **U**.

The curve **T**, $x=-n(\varphi)$, where $n(\varphi)=\dfrac{\mathrm{d}t}{\mathrm{d}\varphi}$ is named the isoperimetric of the space $\mathbf{M_2}$ ([3]).

Minskowian frames will be those affine frames with respect to whom the area of the indicatrix **U** is π.

The geometrical interpretation of the parameter φ leads to the relation:

$$[t,n]=1,$$

where by $[x,y]$ we denote the determinant constructed on the vectors $x,y\in\mathbf{M_2}$, relative to a minkowskian basis.

Minkowskian curvature of the curve **T** will be denoted by $\lambda(\varphi)$ and satisfies the relation:

$$\frac{\mathrm{d}n}{\mathrm{d}\varphi}=-\lambda^{-1}(\varphi)t, \tag{5}$$

where $\lambda(\varphi)>0$.

Let $\{\mathrm{M},e_1,e_2\}$ be a minkowskian frame. Then the motion equations are ([1]):

$$\mathrm{d}\bar{\mathbf{M}}=\sigma_1e_1+\sigma_2e_2;\quad \mathrm{d}e_1=e_2\,\mathrm{d}\varphi, \mathrm{d}e_2=-\lambda^{-1}(\varphi)e_1\,\mathrm{d}\varphi,$$

where φ is the amplitude of the vector e_1 on the indicatrix **U**.

Structure equations will be:

$$\mathrm{d}\sigma_1=\lambda^{-1}(\varphi)\sigma_2\wedge\mathrm{d}\varphi,\ \mathrm{d}\sigma_2=\sigma_1\wedge\mathrm{d}\varphi.$$

2. Isoperimetric problem in the space $\mathbf{M_2}$

Let $\boldsymbol{\Gamma}_\alpha$ be a family of closed curves in the space $\mathbf{M_2}$, depending on the differentiable parameter $\alpha\in\mathbf{R}^2$, given by the vectorial equation:

$$\bar{\mathbf{M}}=\bar{\mathbf{M}}(s,\alpha),$$

where $\bar{\mathbf{M}}$ is a function of class C^1 relative to minkowskian arc s.

We consider the functionals:

$$L(\alpha)=\int_{\Gamma_\alpha}\omega_1,$$

that represent the length of a curve from the family Γ_α, where $\omega_1=\sigma_1=\mathrm{d}s$ and

$$A(\alpha)=\int_{\boldsymbol{\Gamma}_\alpha}\boldsymbol{\omega}_2,$$

that represent the area of a curve from the family Γ_α,

$$\omega_2 = \frac{1}{2}\left[\overline{\mathbf{M}}, \mathrm{d}\,\overline{\mathbf{M}}\right].$$

Using relation (4) for closed curves we obtain

$$\frac{\mathrm{d}\,L}{\mathrm{d}\,\alpha} = \int_{\Gamma_\alpha} \mathrm{d}\,\boldsymbol{\omega}_1 \bullet \frac{\partial\overline{\mathbf{M}}}{\partial\alpha}, \quad \frac{\mathrm{d}\,A}{\mathrm{d}\,\alpha} = -\int_{\Gamma_\alpha} \mathrm{d}\,\boldsymbol{\omega}_2 \bullet \frac{\partial\overline{\mathbf{M}}}{\partial\alpha},$$

because both integrals in (4) extended to $\partial\boldsymbol{\Gamma}_\alpha$ will be zero.

Along the curve $\boldsymbol{\Gamma}_\alpha$ we have,

$$\frac{\partial\overline{\mathbf{M}}}{\partial\alpha} = a_1(s)e_1 + a_2(s)e_2,$$

where $\{\mathrm{M}, e_1, e_2\}$ is Frenet frame associated to the curve $\boldsymbol{\Gamma}_\alpha$ in the point $\overline{\mathbf{M}}(s,\alpha)$.

We obtain,

$$\mathrm{d}\,\sigma_1 \bullet \frac{\partial\overline{\mathbf{M}}}{\partial\alpha} = \lambda^{-1}(\varphi)(\sigma_2 \wedge \mathrm{d}\,\varphi) \bullet \frac{\partial\overline{\mathbf{M}}}{\partial\alpha} = -\lambda^{-1}(\varphi)\left(\sigma_2 \bullet \frac{\partial\overline{\mathbf{M}}}{\partial\alpha}\right)\mathrm{d}\,\varphi +$$

$$+\lambda^{-1}(\varphi)\left(\mathrm{d}\,\varphi \bullet \frac{\partial\overline{\mathbf{M}}}{\partial\alpha}\right)\sigma_2.$$

Along the curve $\boldsymbol{\Gamma}_\alpha$, we have $\sigma_2 = 0$ and when we pass from a curve to another curve from the family $\boldsymbol{\Gamma}_\alpha$, $\alpha \in \mathbf{R}$, the value of the form σ_2 is

$$\sigma_2 \bullet \frac{\partial\overline{\mathbf{M}}}{\partial\alpha} = a_2(s).$$

It results the variational formula:

$$\frac{\mathrm{d}\,L}{\mathrm{d}\,\alpha} = -\int_{\Gamma_\alpha} \lambda^{-1} a_2(s) \frac{\mathrm{d}\,\varphi}{\mathrm{d}\,s}\,\mathrm{d}\,s = -\int_{\Gamma_\alpha} \tilde{k}(s) a_2(s)\,\mathrm{d}\,s, \tag{6}$$

where $\tilde{k} = \lambda^{-1}(\varphi)\dfrac{\mathrm{d}\,\varphi}{\mathrm{d}\,s} = \lambda^{-1}(\varphi)k$.

The curvature of the curve $\boldsymbol{\Gamma}_\alpha$ in the point $\overline{\mathbf{M}}(s,\alpha)$ is $k = \dfrac{\mathrm{d}\,\varphi}{\mathrm{d}\,s}$, and $\tilde{k}$ is its anti-curvature in the same point.

In the same way, we obtain the second variational formula,

$$\frac{\mathrm{d}\,A}{\mathrm{d}\,\alpha} = -\int_{\Gamma_\alpha} a_2(s)\,\mathrm{d}\,s. \tag{7}$$

Theorem 1. Let be a curve from the family Γ_α which varies such that $\frac{\mathrm{d}A}{\mathrm{d}\alpha}=0$. In order that such a curve satisfies the condition $\frac{\mathrm{d}L}{\mathrm{d}\alpha}=0$ it is necessary and sufficient that along this curve its anti-curvature $\tilde{k}$ should be constant, i.e. Γ_α is anti-circle.

The anti-circle with the center in the origin of the space $\mathbf{M_2}$ and radius 1 is the isoperimetric **T**.

3. Isoperimetric inequality in the space $\mathbf{M_2}$

Let $\{0,t,n\}$ be minkowskian frame with respect to whom we consider the family of straight lines $\Delta\varphi : x=-H(\varphi)n+\rho t, \quad \rho\in\mathbf{R}$.

The function $H(\varphi)$ is a positive, periodic function with the period 2π, of class C^2 and is called the support function ([1], [2]) for the convex envelope Γ of the family of straight lines. This will have the tangential equation Γ:

$$x=-H(\varphi)n+H^{(1)}(\varphi)t .$$

The length and the area determined by the convex and closed curve Γ will be,

$$L_\Gamma=\int_0^{2\pi}\lambda^{-1}(\varphi)H(\varphi)\,\mathrm{d}\varphi ,$$

$$A_\Gamma=\frac{1}{2}\int_0^{2\pi}\left[x,\frac{\mathrm{d}x}{\mathrm{d}\varphi}\right]\mathrm{d}\varphi=\frac{1}{2}\int_0^{2\pi}\left(\lambda^{-1}(\varphi)H^2(\varphi)-H^{(1)^2}(\varphi)\right)\mathrm{d}\varphi . \tag{8}$$

Using the notion of mixed area ([2]) for two closed, convex curves Γ_1 and Γ_2, of class C^1, we obtain,

$$A(\Gamma_1,\Gamma_2)=\frac{1}{2}\int_0^{2\pi}\left[x_1,\frac{\mathrm{d}x_2}{\mathrm{d}\varphi}\right]\mathrm{d}\varphi ,$$

where Γ_i has the equation $x_i=-H_i(\varphi)n+H_i^{(1)}t,\ i=1,2$, H_i being the support functions for the curves Γ_i.

Because the indicatrix **U** has the area equal with π, Lebesgue measure on $\mathbf{R}^2$ coincides with minkowskian bidimensional measure on $\mathbf{M_2}$.

Therefore, by Brunn - Minkowski inequality ([3], [4]) we obtain,

$$A^2(\Gamma_1,\Gamma_2)\geq A_{\Gamma_1}A_{\Gamma_2} , \tag{9}$$

where A_{Γ_i}, $i = 1,2$ is the area determined by the curve Γ_i.

For $\Gamma_1 = \Gamma$ and $\Gamma_2 = \Gamma'$, where Γ' are the equation $x = -an$, $a > 0$, we have

$$\begin{aligned} A(\Gamma, \Gamma') &= \frac{1}{2}\int_0^{2\pi}\left[-H(\varphi)n + H^{(1)}(\varphi)t, -a\frac{\mathrm{d}\,n}{\mathrm{d}\,\varphi}\right]\mathrm{d}\,\varphi = \\ &= \frac{a}{2}\int_0^{2\pi} H(\varphi)\lambda^{-1}(\varphi)\,\mathrm{d}\,\varphi = \frac{a}{2}L_\Gamma. \end{aligned} \tag{10}$$

Replacing (10) in (9), we obtain the inequality,

$$a^2 L_\Gamma^2 \geq 4A_\Gamma A_{\Gamma'}.$$

For $a = 1$ and $\Gamma' = T$, we obtain following theorem:

Theorem 2. For every convex, closed curve $\Gamma \subset \mathbf{M_2}$, of class C^1, with area A_Γ and the length L_Γ, the following isoperimetric inequality holds:

$$L_\Gamma^2 \geq 4A_\Gamma A_{\mathbf{T}}, \tag{11}$$

where **T** is the isoperimetric of the space $\mathbf{M_2}$.

In (11) the equality holds only for $\Gamma = \mathbf{T}$ or for every curve homothetic with **T**.

If the space $\mathbf{M_2}$ is the euclidian space, then $A_{\mathbf{T}} = \pi$ and from (11) we obtain classic inequality $L_\Gamma^2 \geq 4\pi A_\Gamma$.

4. The generalization of isoperimetric inequality in the space $\mathbf{M_2}$

Let $\mathbf{B} \subset \mathbf{M_2}$ be a bounded and connected set and let $\mathbf{B}^*$ be the convex cover of **B**.

Theorem 3. The set of straight lines $X = \{g \mid g \cap \mathbf{B} \neq \varnothing\}$ and $X^* = \{g \mid g \cap \mathbf{B}^* \neq \varnothing\}$ coincide.

The proof is mode by double inclusion.

Theorem 4. Let $\mathbf{B} \subset \mathbf{M_2}$ be a convex and bounded set and $\xi_g(\mathbf{B})$ be minkowskian length of the linear set determined by the intersection $g \cap \mathbf{B}$.

Then, relativ to the elementary measure of the set of straight lines in the space $\mathbf{M_2}$, $\mathrm{d}\,g = \lambda^{-1}(\varphi)|\mathrm{d}\,H \wedge \mathrm{d}\,\varphi|$, we have ([5]):

$$\int_X \xi_{\mathbf{B}}(g)\,\mathrm{d}g = A_{\mathbf{T}} A_{\mathbf{B}} \qquad (12)$$

where $A_{\mathbf{T}}$ is the area of isoperimetric **T** and $A_{\mathbf{B}}$ is the area of the set **B**.

In space $\mathbf{M_2}$ holds Crofton's integral formula ([5]):

$$\int_X \nu\,\mathrm{d}g = 2L_{\mathbf{B}}, \qquad (13)$$

where ν is the number of intersections of with the boundary $\partial\mathbf{B}$ of the set **B** and $L_{\mathbf{B}}$ is the length of the boundary $\partial\mathbf{B}$, supposed of class C^1.

If **B** is not a convex set, then in (12) $\nu \geq 2$ and we obtain,

$$L_{\mathbf{B}} \geq \int_X \mathrm{d}g. \qquad (14)$$

When **B** is convex, $L_{\mathbf{B}} = \int_X \mathrm{d}g$, because $\nu = 2$.

The following theorem extend inequality (11).

Theorem 5. For every bounded an connected set $\mathbf{B} \subset \mathbf{M_2}$ whose boundary $\partial\mathbf{B}$ is a closed curve of class C^1, with the length $L_{\mathbf{B}}$, the inequality holds,

$$L_{\mathbf{B}}^2 \geq 4\int_X \xi_{\mathbf{B}}(g)\,\mathrm{d}g. \qquad (15)$$

Proof. Let $\mathbf{B}^*$ be the convex cover of the set **B** and let $L_{\mathbf{B}^*}$ be minkowskian length of its boundary.

Then, by (14) and theorem 3, we have,

$$L_{\mathbf{B}}^2 \geq \left(\int_X \mathrm{d}g\right)^2 = \left(\int_{X^*} \mathrm{d}g\right)^2. \qquad (16)$$

By (11) and (12), for the convex set $\mathbf{B}^*$ we have,

$$L_{\mathbf{B}^*}^2 \geq A_{\mathbf{B}^*} A_{\mathbf{T}} \text{ and } \int_{X^*} \xi_{\mathbf{B}^*}(g)\,\mathrm{d}g = A_{\mathbf{T}} A_{\mathbf{B}^*},$$

hence

$$L_{\mathbf{B}^*}^2 \geq 4\int_{X^*} \xi_{\mathbf{B}^*}(g)\,\mathrm{d}g. \qquad (17)$$

Because $L_{\mathbf{B}^*} = \int_{X^*} \mathrm{d}g$, by relations (16) and (17) we obtain,

$$L_{\mathbf{B}}^2 \geq \left(\int_X \mathrm{d}\, g \right)^2 = \left(\int_{X^*} \mathrm{d}\, g \right)^2 = L_{\mathbf{B}^*}^2 \geq 4 \int_{X^*} \xi_{\mathbf{B}^*}(g) \mathrm{d}\, g \qquad (18)$$

The inclusion $\mathbf{B} \subset \mathbf{B}^*$ implies that $g \cap \mathbf{B} \subset g \cap \mathbf{B}^*$, for all g. Hence $\|g \cap \mathbf{B}\| \leq \|g \cap \mathbf{B}^*\|$, that is $\xi_{\mathbf{B}^*}(g) \geq \xi_{\mathbf{B}}(g)$.

So relation (18) becomes (15), that is

$$L_{\mathbf{B}}^2 \geq 4 \int_{X^*} \xi_{\mathbf{B}}(g) \mathrm{d}\, g = 4 \int_X \xi_{\mathbf{B}}(g) \mathrm{d}\, g\,. \qquad (19)$$

In case that **B** is convex, from (15) and (12) it can be obtained relation (11).

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