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A NOTE ON H_v-LA-SEMIGROUPS

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In this paper, we introduce a generalized class of an H_v-semigroup obtained from an LA-semigroup H. This generalized H_v-structure is called an H_v-LA-semigroup. We provide several examples of H_v-LA-semigroups. Moreover, with the help of an example we obtain that each LA-semigroup endowed with an equivalence relation can induce an H_v-LA-semigroup. We also investigate isomorphism theorems with the help of regular relations. At the end, we introduce the concept of hyperideal and hyperorder in H_v-LA-semigroups and prove some useful results on it.

Keywords: H_v-LA-semigroups, Regular relations, Isomorphism theorems.
MSC2000: 20N20.

1. Introduction

Kazim and Naseeruddin [1] provided the concept of left almost semigroup (abbreviated as LA-semigroup). They generalized some useful results of semigroup theory. Later, Mushtaq [2] and others further investigated the structure and added many useful results to the theory of LA-semigroups; see also [3, 4, 5, 6, 7, 8, 9]. An LA-semigroup is the midway structure between a commutative semigroup and a groupoid. It nevertheless possesses many interesting properties which we usually find in commutative and associative algebraic structures.

Hyperstructure theory was introduced by Marty in 1934, when Marty [10] defined hypergroups, began to analyze their properties, and applied them to groups. Several papers and books have been written on hyperstructure theory; see [11, 12]. Recently a book published on hyperstructures [13] points out on its applications in rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs, and hypergraphs. Recently, Hila and Dine [14] introduced the notion of LA-semihypergroups as a generalization of semigroups, semihypergroups, and LA-semigroups. Yaqoob,

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Corsini and Yousafzai [15] extended the work of Hila and Dine and characterized intra-regular left almost semihypergroups by their hyperideals using pure left identity.

In 1990, Vougiouklis [16] introduced the concept of H_v-structures in Fourth AHA Congress as a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule and so on). After the introduction of the notion of H_v-structures, several authors studied different aspects of H_v-structures. For instance, Vougiouklis [17, 18, 19], Spartalis [20, 21, 22, 23], Spartalis and Vougiouklis [24], Davvaz [25], Nezhad and Davvaz [26] and Hedayati et al. [27, 28].

In this article we introduce a new concept of H_v-LA-semigroups with comprehensive explanation provided in the form of different examples. Moreover we show that every LA-semihypergroup is an H_v-LA-semigroup and each LA-semigroup endowed with an equivalence relation can induce an H_v-LA-semigroup. We also investigate isomorphism theorem with the help of regular relations.

2. Some notions in LA-semigroups and LA-semihypergroups

A groupoid $(S, \cdot)$ is called an LA-semigroup [1], if $(a \cdot b) \cdot c = (c \cdot b) \cdot a$, for all $a, b, c \in S$. The law $(a \cdot b) \cdot c = (c \cdot b) \cdot a$ is called a left invertive law.

Example 2.1. [2] *Let $(\mathbb{Z}, +)$ denote the commutative group of integers under addition. Define a binary operation "$*$" in $\mathbb{Z}$ as follows*:

$$a * b = b - a, \quad \textit{for all } a, b \in \mathbb{Z},$$

where "$-$" denotes the ordinary subtraction of integers. Then $(\mathbb{Z}, *)$ is an LA-semigroup.

By Kazim and Naseerudin [1], in an LA-semigroup S the following law holds $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in S$. This law is known as medial law. In [7], in an LA-semigroup S with left identity, the following law holds $(ab)(cd) = (dc)(ba)$ for all $a, b, c, d \in S$. This law is known as paramedial law. If an LA-semigroup contains a left identity, then by using medial law, we get $a(bc) = b(ac)$, for all $a, b, c, d \in S$.

Definition 2.1. *A map $\circ : S \times S \to \mathcal{P}^*(S)$ is called a hyperoperation or join operation on the set S, where S is a non-empty set and $\mathcal{P}^*(S) = \mathcal{P}(S) \backslash \{\emptyset\}$ denotes the set of all non-empty subsets of S. A hypergroupoid is a set S together with a (binary) hyperoperation.*

Definition 2.2. [14, 15] *A hypergroupoid $(S, \circ)$, which is left invertive (non-associative), that is $(x \circ y) \circ z = (z \circ y) \circ x$, $\forall x, y, z \in S$, is called an LA-semihypergroup.*

Let A and B be two non-empty subsets of S. Then we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad a \circ A = \{a\} \circ A \text{ and } B \circ a = B \circ \{a\}.$$

Example 2.2. [15] *Let $S = \mathbb{Z}$. If we define $x \circ y = y - x + 3\mathbb{Z}$, where $x, y \in \mathbb{Z}$. Then $(S, \circ)$ becomes an LA-semihypergroup.*

3. H$_v$-LA-semigroups

In this section, we will define an H$_v$-LA-semigroup and provide some examples. Throughout the paper H will be considered as an H$_v$-LA-semigroup unless otherwise specified.

Definition 3.1. *Let H be a non-empty set and $*$ be a hyperoperation on H. Then, $(H, *)$ is called an H$_v$-LA-semigroup if it satisfies the weak left invertive law i.e for all $x, y, z \in H$, $(x * y) * z \cap (z * y) * x \neq \emptyset$.*

Example 3.1. *Let $H = (0, \infty)$. We define $x * y = \left\{\frac{y}{x+1}, \frac{y}{x}\right\}$, where $x, y \in H$. Then, for all $x, y, z \in H$, we have*

$$\begin{aligned}(x * y) * z &= \left\{\frac{y}{x+1}, \frac{y}{x}\right\} * z = \left\{\frac{z}{\frac{y}{x+1}+1}, \frac{z}{\frac{y}{x+1}}, \frac{z}{\frac{y}{x}+1}, \frac{z}{\frac{y}{x}}\right\} \\ &= \left\{\frac{z(x+1)}{x+y+1}, \frac{z(x+1)}{y}, \frac{xz}{x+y}, \frac{xz}{y}\right\},\end{aligned}$$

and

$$\begin{aligned}(z * y) * x &= \left\{\frac{y}{z+1}, \frac{y}{z}\right\} * x = \left\{\frac{x}{\frac{y}{z+1}+1}, \frac{x}{\frac{y}{z+1}}, \frac{x}{\frac{y}{z}+1}, \frac{x}{\frac{y}{z}}\right\} \\ &= \left\{\frac{x(z+1)}{y+z+1}, \frac{x(z+1)}{y}, \frac{xz}{y+z}, \frac{xz}{y}\right\},\end{aligned}$$

also

$$\begin{aligned}x * (y * z) &= x * \left\{\frac{z}{y+1}, \frac{z}{y}\right\} = \left\{\frac{\frac{z}{y+1}}{x+1}, \frac{\frac{z}{y+1}}{x}, \frac{\frac{z}{y}}{x+1}, \frac{\frac{z}{y}}{x}\right\} \\ &= \left\{\frac{z}{(x+1)(y+1)}, \frac{z}{x(y+1)}, \frac{z}{y(x+1)}, \frac{z}{xy}\right\}.\end{aligned}$$

Clearly $(H, *)$ is an H$_v$-LA-semigroup because

$$(x * y) * z \cap (z * y) * x = \left\{\frac{xz}{y}\right\} \neq \emptyset.$$

Also it is clear that $(H, *)$ is not an H$_v$-semigroup because

$$(x * y) * z \cap x * (y * z) = \emptyset.$$

Example 3.2. *Consider $H = \{x, y, z\}$ and define a hyperoperation $*$ on H by the following table:*

$*$	x	y	z
x	x	$\{x, z\}$	H
y	$\{x, z\}$	x	x
z	$\{x, y\}$	z	$\{x, z\}$

Then $(H, *)$ is an H_v-LA-semigroup which is not an LA-semihypergroup and not an H_v-semigroup. Indeed, we have

$$\{x, y\} = z * (y * y) \neq (z * y) * y = \{z\}.$$

Thus, $*$ is not associative, and $(z * y) * y \cap z * (y * y) = \emptyset$. Therefore $(H, *)$ is not an H_v-semigroup. Also,

$$\{x, y, z\} = (x * y) * z \neq (z * y) * x = \{x, y\}$$

Thus, $*$ is not left invertive i.e, $(x * y) * z \neq (z * y) * x$. Therefore $(H, *)$ is not an LA-semihypergroup.

Example 3.3. *Let $(H, \cdot)$ be an LA-semigroup with left identity e. We define a hyperoperation $*$ as follows:*

$$\begin{aligned} w * e &= w \cdot e, \; e * w = w, \quad \text{for all } w \text{ in } H. \\ \text{and } x * y &= \{x \cdot y, x, y\}, \quad \text{for all } x, y \text{ in } H \backslash \{e\}. \end{aligned}$$

Then $(H, *)$ becomes an H_v-LA-semigroup which is not an LA-semihypergroup and not an H_v-semigroup. Indeed, we have

$$\{x \cdot e\} = x * (e * e) \neq (x * e) * e = \{x\}.$$

Thus, $*$ is not associative. Therefore $(H, *)$ is not an H_v-semigroup. Also,

$$\begin{aligned} \{x \cdot y, x, y\} &= (e * x) * y \\ &\neq (y * x) * e = \{(y \cdot x) \cdot e, y \cdot e, x \cdot e\} \\ &= \{x \cdot y, y \cdot e, x \cdot e\}. \text{ (by left invertive law)} \end{aligned}$$

Thus, $*$ is not left invertive, and $(e * x) * y \neq (y * x) * e$. Therefore $(H, *)$ is not an LA-semihypergroup.

Note that if $(x * y) * z = (z * y) * x$, then $(H, *)$ becomes an LA-semihypergroup.

Remark 3.1. *Every LA-semihypergroup is an H_v-LA-semigroup but the converse may or may not be true.*

4. Regular relations and isomorphism theorems

In this section we will investigate equivalence relations, regular relations and isomorphism theorems in H_v-LA-semigroups.

In the next example, we will see that each LA-semigroup endowed with an equivalence relation can induce an H_v-LA-semigroup.

Example 4.1. *Let $(H,\cdot)$ be an LA-semigroup, σ an equivalence relation in H and $\sigma(x)$ the equivalence class of the element $x \in H$. On $H/\sigma = \{\sigma(x) : x \in H\}$, we define a hyperoperation $\circledast : H/\sigma \times H/\sigma \longrightarrow \wp^*(H/\sigma)$ by*

$$\sigma(x) \circledast \sigma(y) = \{\sigma(z) : z \in \sigma(x) \cdot \sigma(y)\}, \text{ for all } x, y \in H.$$

Then $(H/\sigma, \circledast)$ is an H_v-LA-semigroup.

The H$_v$-LA-semigroup constructed by Example **4**.1 is some times both an LA-semihypergroup and an H$_v$-LA-semigroup and some times it is only an H$_v$-LA-semigroup. For this consider an LA-semigroup $(H,\cdot)$ defined by the following table:

$\cdot$	x	y	z	w
x	x	z	w	y
y	w	y	x	z
z	y	w	z	x
w	z	x	y	w

Define an equivalence relation as:

$$\sigma = \{(x,x), (y,y), (y,z), (z,y), (z,z), (w,w)\}.$$

The set of equivalence classes related to σ is $H/\sigma = \{\sigma(x), \sigma(y), \sigma(w)\}$, where $\sigma(x) = \{x\}$, $\sigma(y) = \sigma(z) = \{y, z\}$ and $\sigma(w) = \{w\}$. Now, by the hyperoperation $\circledast$ defined in Example **4**.1 we get

$\circledast$	$\sigma(x)$	$\sigma(y)$	$\sigma(w)$
$\sigma(x)$	$\sigma(x)$	$\{\sigma(y), \sigma(w)\}$	$\sigma(y)$
$\sigma(y)$	$\{\sigma(y), \sigma(w)\}$	$\{\sigma(x), \sigma(y), \sigma(w)\}$	$\{\sigma(x), \sigma(y)\}$
$\sigma(w)$	$\sigma(y)$	$\sigma(x)$	$\sigma(w)$

Here $(\sigma(w) \circledast \sigma(x)) \circledast \sigma(x) \cap \sigma(w) \circledast (\sigma(x) \circledast \sigma(x)) = \emptyset$, therefore $(H/\sigma, \circledast)$ is not an H$_v$-semigroup. Also $(H/\sigma, \circledast)$ is not an LA-semihypergroup because

$$\{\sigma(y)\} = (\sigma(x) \circledast \sigma(x)) \circledast \sigma(w) \neq (\sigma(w) \circledast \sigma(x)) \circledast \sigma(x) = \{\sigma(y), \sigma(w)\}.$$

The elements of H/σ satisfies the weak left invertive law, see the following calculations:

$(\sigma(x) \circledast \sigma(x)) \circledast \sigma(x) = \{\sigma(x)\}$
$(\sigma(x) \circledast \sigma(x)) \circledast \sigma(y) = \{\sigma(y), \sigma(w)\}$
$(\sigma(x) \circledast \sigma(x)) \circledast \sigma(w) = \{\sigma(y)\}$
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$(\sigma(w) \circledast \sigma(w)) \circledast \sigma(w) = \{\sigma(w)\}$.
It is clear from the above table that

$$(\sigma(x) \circledast \sigma(y)) \circledast \sigma(w) \cap (\sigma(w) \circledast \sigma(x)) \circledast \sigma(y) \neq \emptyset,$$

for all $x, y, w \in H$. Hence $(H/\sigma, \circledast)$ is an H_v-LA-semigroup which is neither LA-semihypergroup nor H_v-semigroup. But if we define an equivalence relation as:

$$\theta = \{(x,x),(x,w),(y,y),(y,z),(z,y),(z,z),(w,x),(w,w)\}.$$

The set of equivalence classes related to θ is $H/\theta = \{\theta(x), \theta(y)\}$, where $\theta(x) = \theta(w) = \{x, w\}$ and $\theta(y) = \theta(z) = \{y, z\}$. Now, by the hyperoperation $\circledast$ defined in Example **4**.1 we get

$\circledast$	$\theta(x)$	$\theta(y)$
$\theta(x)$	$\{\theta(x), \theta(y)\}$	$\{\theta(x), \theta(y)\}$
$\theta(y)$	$\{\theta(x), \theta(y)\}$	$\{\theta(x), \theta(y)\}$

Clearly, one can see that $(H/\theta, \circledast)$ is an LA-semihypergroup and hence an H_v-LA-semigroup.

Proposition 4.1. *Let $(H, \cdot)$ be an LA-semigroup with left identity and $\emptyset \neq A \subseteq H$. If*

$$(A \cdot (A \cdot x)) \cdot y \cap (A \cdot (A \cdot y)) \cdot x \neq \emptyset, \ \forall x, y \in H,$$

and we define a hyperoperation $A_R^{\otimes}$ on H as $xA_R^{\otimes}y = (x \cdot y) \cdot A$, then $\left(H, A_R^{\otimes}\right)$ becomes an H_v-LA-semigroup.

Proof. Let $x, y, z \in H$, we have

$$\begin{aligned} \left(xA_R^{\otimes}y\right) A_R^{\otimes}z &= ((x \cdot y) \cdot A)\, A_R^{\otimes}z = (((x \cdot y) \cdot A) \cdot z) \cdot A \\ &= ((z \cdot A) \cdot (x \cdot y)) \cdot A = ((y \cdot x) \cdot (A \cdot z)) \cdot A \\ &= (A \cdot (A \cdot z)) \cdot (y \cdot x) = y \cdot ((A \cdot (A \cdot z)) \cdot x), \end{aligned}$$

and on the other hand

$$\begin{aligned}\left(zA_R^{\otimes}y\right)A_R^{\otimes}x &= ((z\cdot y)\cdot A)\,A_R^{\otimes}x = (((z\cdot y)\cdot A)\cdot x)\cdot A\\ &= ((x\cdot A)\cdot(z\cdot y))\cdot A = ((y\cdot z)\cdot(A\cdot x))\cdot A\\ &= (A\cdot(A\cdot x))\cdot(y\cdot z) = y\cdot((A\cdot(A\cdot x))\cdot z).\end{aligned}$$

But, since $(A\cdot(A\cdot x))\cdot y\cap(A\cdot(A\cdot y))\cdot x\neq\emptyset$, $\forall x,y\in H$. It follows that

$$\left(xA_R^{\otimes}y\right)A_R^{\otimes}z\cap\left(zA_R^{\otimes}y\right)A_R^{\otimes}x\neq\emptyset.$$

Hence $\left(H,A_R^{\otimes}\right)$ is an H_v-LA-semigroup. □

Following theorem shows that by any non-empty finite set H with $|H|\geq 3$, we can construct an H_v-LA-semigroup which is some times both an H_v-semigroup and H_v-LA-semigroup.

Theorem 4.1. *Consider a finite set H with $|H|\geq 3$. We define a hyperoperation $*$ on H as follows:*

$$x_i * x_j = \{x_l, x_m\},\ \ \textit{where}\ \begin{array}{l} l\equiv (j+1)-i \pmod{|H|}\\ m\equiv j^2-i \pmod{|H|},\end{array}$$

for all $x_i,x_j\in H$. Then $(H,*)$ becomes an H_v-LA-semigroup.

Proof. For all $x_i,x_j,x_k\in H$, we have

$$\begin{aligned}(x_i*x_j)*x_k &= \left\{x_{j+1-i},x_{j^2-i}\right\}*x_k\\ &= \left\{x_{k+1-j-1+i},x_{k^2-j-1+i},x_{k+1-j^2+i},x_{k^2-j^2+i}\right\},\end{aligned}$$

and

$$\begin{aligned}(x_k*x_j)*x_i &= \left\{x_{j+1-k},x_{j^2-k}\right\}*x_i\\ &= \left\{x_{i+1-j-1+k},x_{i^2-j-1+k},x_{i+1-j^2+k},x_{i^2-j^2+k}\right\}.\end{aligned}$$

This implies that

$$(x_i*x_j)*x_k\cap(x_k*x_j)*x_i=\left\{x_{i-j+k},x_{i+1-j^2+k}\right\}\neq\emptyset.$$

Hence $(H,*)$ is an H_v-LA-semigroup. □

Let H be an H_v-LA-semigroup and θ be an equivalence relation in H. Then we can extend this relation θ to the non-empty subsets A and B of H as follows: $A\overline{\theta}B$ if and only if for all $a\in A$ there exists $b\in B$ such that $a\theta b$ and for all $b\in B$ there exists $a\in A$ such that $b\theta a$. An equivalence relation θ is said to be regular if for all $x,y,z\in H$, $x\theta y\Rightarrow(xz)\,\overline{\theta}\,(yz)$ and $(zx)\,\overline{\theta}\,(zy)$.

Example 4.2. *Let $H=\{x,y,z,w,t\}$ with the binary hyperoperation defined below:*

$*$	x	y	z	w	t
x	x	$\{y,z\}$	y	$\{w,t\}$	t
y	$\{y,z\}$	$\{x,y,z\}$	$\{x,y\}$	$\{w,t\}$	t
z	y	x	z	$\{w,t\}$	t
w	$\{w,t\}$	$\{w,t\}$	$\{w,t\}$	w	t
t	t	t	t	t	t

Here $(H,*)$ *is an* H_v*-LA-semigroup. We define a regular relation on* H *as:*

$$\begin{aligned}\theta &= \{(x,x),(x,y),(x,z),(y,x),(y,y),(y,z),(z,x),(z,y),\\ &\quad (z,z),(w,w),(w,t),(t,w),(t,t)\}.\end{aligned}$$

Here the θ*-regular classes are the subsets* $\{x,y,z\}$ *and* $\{w,t\}$.

Lemma 4.1. *Let* θ *be a regular relation on* H_v*-LA-semigroup, then*

$$\{\theta(z) : z \in \theta(x)\theta(y)\} = \{\theta(z) : z \in xy\} \;\; \forall\, x,y \in H.$$

Proof. Proof is straightforward. □

Next we show that each H$_v$-LA-semigroup can induce a new H$_v$-LA-semigroup through a regular relation.

Theorem 4.2. *Let* θ *be a regular relation on* H_v*-LA-semigroup* H, *then* ($H/\theta, \circledast$) *is an* H_v*-LA-semigroup with the mapping* $\circledast : H/\theta\times\, H/\theta \to \wp^*(H/\theta)$ *defined by* $\theta(x) \circledast \theta(y) = \{\theta(z) : z \in \theta(x)\theta(y)\}$, *for all* $\theta(x)$, $\theta(y) \in H/\theta$.

Proof. It follows from Lemma **4**.1. □

A mapping $\phi : H_1 \to H_2$, where both H_1 and H_2 are H$_v$-LA-semigroups is said to be homomorphism if $\phi(xy) = \phi(x)\phi(y)$ $\forall x,y \in H_1$. If it is 1-1 and onto then it is called isomorphism, and in that case two H$_v$-LA-semigroups H_1 and H_2 are said to be isomorphic and it is denoted by $H_1 \cong H_2$.

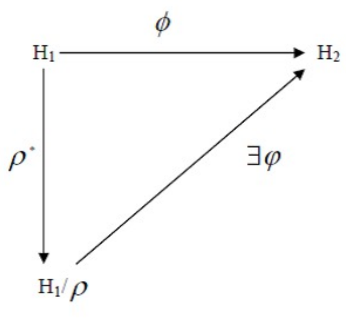
Let $\phi : H_1 \to H_2$ be a homomorphism of H$_v$-LA-semigroups and we define the relation $\rho = \phi^{-1} * \phi = \{(x,y) \in H_1 \times H_1 : \phi(x) = \phi(y)\}$.

Lemma 4.2. *The relation* $\rho = \phi^{-1} * \phi = \{(x,y) \in H_1 \times H_1 : \phi(x) = \phi(y)\}$ *is regular on* H_1.

Proof. The relation ρ is obviously an equivalence relation. For regularity let $x,y,z \in H_1$ such that $x\rho y \Rightarrow \phi(x) = \phi(y) \Rightarrow \phi(xz) = \phi(yz)$ and $\phi(zx) = \phi(zy)$. So $(xz)\,\overline{\rho}\,(yz)$ and $(zx)\,\overline{\rho}\,(zy)$. Thus $x\rho y \Rightarrow (xz)\,\overline{\rho}\,(yz)$ and $(zx)\,\overline{\rho}\,(zy)$. Hence $\rho = \phi^{-1} * \phi = \{(x,y) \in H_1 \times H_1 : \phi(x) = \phi(y)\}$ is regular on H_1. □

Remark 4.1. *Since* $\rho = \phi^{-1} * \phi = \{(x,y) \in H_1 \times H_1 : \phi(x) = \phi(y)\}$ *is regular on* H_1, *by Theorem* **4**.*2, it follows that* H_1/ρ *is an* H_v*-LA-semigroup.*

Theorem 4.3. *Let* $\phi : H_1 \to H_2$ *be a homomorphism of* H_v*-LA-semigroups. Then there exist a monomorphism* $\varphi : H_1/\rho \to H_2$ *such that Im*ϕ *=Im*φ *and the diagram*



commutes i.e. $\varphi * \rho^* = \phi$, *where the mapping* $\rho^* : H_1 \to H_1/\rho$ *is defined by* $\rho^*(x) = \rho(x)\ \forall x \in H_1$.

Proof. Let us define $\varphi : H_1/\rho \to H_2$ by $\varphi(\rho(x)) = \phi(x)\ \forall\ x \in H_1$. Then φ is obviously well defined and $1-1$. Now, for all $x, y \in H_1$ we have

$$\begin{aligned}\varphi(\rho(x) \circledast \rho(y)) &= \{\varphi(\rho(z)) : z \in xy\} = \{\phi(z) : z \in xy\} \\ &= \phi(xy) = \phi(x)\phi(y) = \varphi(\rho(x)) \circledast \varphi(\rho(y)).\end{aligned}$$

Hence φ is homomorphism and it is easy to prove that Imϕ =Imφ.

Now for all $x \in H_1$, we have

$$(\varphi * \rho^*)(x) = \varphi(\rho^*(x)) = \varphi(\rho(x)) = \phi(x).$$

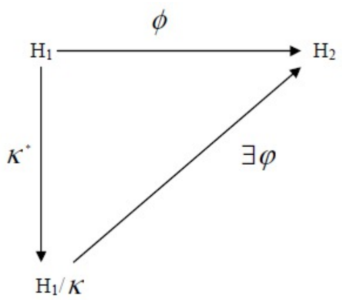
Hence diagram commutes. This completes the proof. □

Now with the help of regular relation ρ, we state the first isomorphism theorem.

Theorem 4.4. *Let* $\phi : H_1 \to H_2$ *be a homomorphism of* H_v*-LA-semigroups. Then* $H_1/\rho \cong$*Im*ϕ.

Proof. It follows from Theorem **4**.3. □

Theorem 4.5. *Let* $\phi : H_1 \to H_2$ *be a homomorphism of* H_v*-LA-semigroups. If* κ *is regular relation on* H_1 *such that* $\kappa \subseteq \rho$, *then there exists an unique monomorphism* $\varphi : H_1/\kappa \to H_2$ *such that Im*ϕ =*Im*φ *and the diagram*



commutes i.e. $\varphi * \kappa^* = \phi$, *where the mapping* $\kappa^* : H_1 \to H_1/\kappa$ *is defined by* $\kappa^*(x) = \kappa(x)\ \forall\ x \in H_1$.

Proof. Proof is straightforward. □

Lemma 4.3. *Let* θ *and* σ *be two regular relations on an* H_v*-LA-semigroup* H *such that* $\theta \subseteq \sigma$. *Then* σ/θ *is a regular relation on* H/θ.

Proof. Let us define $\sigma/\theta : H/\theta \times H/\theta \to \wp^*(H/\theta)$ by $\sigma/\theta(\theta(x)) = \theta(x)$ for all $\theta(x) \in H/\theta$. This mapping is well-defined as consider $\theta(x) = \theta(y) \Rightarrow (x, y) \in \theta \subseteq \sigma \Rightarrow (\theta(x), \theta(y)) \in \sigma/\theta$ and so $\sigma/\theta(\theta(x)) = \sigma/\theta(\theta(y))$. Next we will show that σ/θ is an equivalence relation. Let $x \in H$, then $(x, x) \in \sigma \Rightarrow (\theta(x), \theta(x)) \in \sigma/\theta$, thus σ/θ is reflexive. Also let $x, y \in H$, such that $(\theta(x), \theta(y)) \in \sigma/\theta$. As $(x, y) \in \sigma \Rightarrow (y, x) \in \sigma$ due to the symmetry of σ.

Which implies that $(\theta(y), \theta(x)) \in \sigma/\theta$. Hence σ/θ is symmetric. Again let $x, y, z \in H$, such that $(\theta(x), \theta(y)), (\theta(y), \theta(z)) \in \sigma/\theta$ and $(x, y), (y, z) \in \sigma \Rightarrow (x, z) \in \sigma$ due to the transitivity of σ. Which implies that $(\theta(x), \theta(z)) \in \sigma/\theta$. Hence σ/θ is transitive. Thus σ/θ is an equivalence relation. Now we will show that it is regular. For it let $x, y, z \in H$, such that

$$\begin{aligned}(\theta(x))\,\sigma/\theta\,(\theta(y)) &\Rightarrow (x, y) \in \sigma \Rightarrow x\sigma y \Rightarrow (xz)\,\overline{\sigma}\,(yz) \\ &\Rightarrow \{\theta(\mu) : \mu \in xz\}\,\overline{\sigma/\theta}\,\{\theta(\nu) : \nu \in yz\}.\end{aligned}$$

Which implies that $(\theta(x) \circledast \theta(z))\,\overline{\sigma/\theta}\,(\theta(y) \circledast \theta(z))$ and similarly we can show that

$$(\theta(x))\,\sigma/\theta\,(\theta(y)) \Rightarrow (z\theta x \circledast \theta(x))\,\overline{\sigma/\theta}\,(\theta(z) \circledast \theta(z)).$$

Hence σ/θ is a regular relation on H/θ. □

Remark 4.2. *Since σ/θ is a regular relation on H/θ, it implies that $(H/\theta)/(\sigma/\theta)$ is an H_v-LA-semigroup.*

Theorem 4.6. (*3rd isomorphism theorem*) *Let θ and σ be two regular relations on an H_v-LA-semigroup H such that $\theta \subseteq \sigma$. Then $(H/\theta)/(\sigma/\theta) \cong H/\sigma$.*

Proof. Let us define $\varphi : (H/\theta)/(\sigma/\theta) \to H/\sigma$ by $\varphi(\sigma/\theta(\theta(x))) = \sigma(x)\ \forall\ x \in H$. It is easy to show that this map is bijective. We will only show that it is homomorphism. For that suppose $x, y \in H$, then

$$\begin{aligned}\varphi(\sigma/\theta(\theta(x)) \circledast \sigma/\theta(\theta(y))) &= \varphi(\{\sigma/\theta(\theta(z)) : \theta(z) \in \theta(x) \circledast \theta(y)\}) \\ &= \varphi(\{\sigma/\theta(\theta(z)) : z \in xy\}) \\ &= \{\varphi(\sigma/\theta(\theta(z))) : z \in xy\} \\ &= \{\sigma(z) : z \in xy\} = \sigma(x)\sigma(y) \\ &= \varphi(\sigma/\theta(\theta(x))) \circledast \varphi(\sigma/\theta(\theta(x))).\end{aligned}$$

Hence φ is homomorphism. Thus $(H/\theta)/(\sigma/\theta) \cong H/\sigma$. □

Proposition 4.2. *Let $(H, *)$ be an H_v-LA-semigroup and $\emptyset \neq N \subseteq H$. If we define a well defined hyperoperation $\circledcirc$ on $H/N = \{N * a | a \in H\}$ as $(N * a) \circledcirc (N * b) = \{N * n | n \in a * b\}, \forall\, a, b \in H$, then $(H/N, \circledcirc)$ is an H_v-LA-semigroup.*

Proof. Let $(N * a), (N * b), (N * c) \in H/N$, $\forall\ a, b, c \in H$. Consider

$$\begin{aligned}((N * a) \circledcirc (N * b)) \circledcirc (N * c) &= (\{N * n | n \in a * b\}) \circledcirc (N * c) \\ &= \{N * m | m \in n * c\} \\ &= \{N * m | m \in (a * b) * c\}.\end{aligned}$$

On the other hand

$$\begin{aligned}((N * c) \circledcirc (N * b)) \circledcirc (N * a) &= (\{N * n_1 | n_1 \in c * b\}) \circledcirc (N * a) \\ &= \{N * m_1 | m_1 \in n_1 * a\} \\ &= \{N * m_1 | m_1 \in (c * b) * a\}.\end{aligned}$$

Now using the fact that $(H, *)$ is an H_v-LA-semigroup i.e,

$$(a * b) * c \cap (c * b) * a \neq \emptyset.$$

Thus

$$((N * a) \circledcirc (N * b)) \circledcirc (N * c) \cap ((N * c) \circledcirc (N * b)) \circledcirc (N * a) \neq \emptyset.$$

Hence $(H/N, \circledcirc)$ is an H_v-LA-semigroup. □

5. H_v-LA-subsemigroups and ideals

A non-empty subset K of $(H, *)$ is said to be an H_v-LA-subsemigroup if it is itself an H_v-LA-semigroup or $a * b \in K$, $\forall\ a, b \in K$. K is called proper H_v-LA-subsemigroup if $K \neq H$.

Proposition 5.1. *Intersection of two H_v-LA-subsemigroups is again an H_v-LA-subsemigroup if it is non-empty.*

Proof. Proof is straightforward. □

On the other hand union of two H_v-LA-subsemigroups may be or may be not an H_v-LA-subsemigroup. From Example **4**.2, it is easy to observe that $\{x\}$ and $\{z\}$ are H_v-LA-subsemigroups but $\{x\} \cup \{z\}$ is not an H_v-LA-subsemigroup.

Definition 5.1. *A non-empty subset K of $(H, *)$ is said to be an ideal of H if $a * K \subseteq K$, for all $a \in H$.*

Every ideal of H is an H_v-LA-subsemigroup but converse is not true. From Example **4**.2, it is easy to observe that $\{x, y, z\}$ is an H_v-LA-subsemigroup but not an ideal.

The following results are obviously true individually and we state here without proof because it is straightforward.

Theorem 5.1. *Let $(H, *)$ be an H_v-LA-semigroup, we have the following:*

- If K is an ideal and L is an H_v-LA-subsemigroup then $K \cap L$ is an ideal of L.
- If K and L are ideals of an H_v-LA-semigroup H, then $K \cap L$ is an ideal of H and $K \cap L = K * L$.

Proposition 5.2. *Let $\phi : H_1 \to H_2$ be a homomorphism of H_v-LA-semigroups. Then,*

(i) If K is an H_v-LA-subsemigroup (resp., ideal) of H_1 then $f(K)$ is an H_v-LA-subsemigroup (resp., ideal) of H_2.

(ii) If ϕ is surjective and L is an H_v-LA-subsemigroup (resp., ideal) of H_2, then $\phi^{-1}(L) = \{a \in H_1 | \phi(a) \in L\}$ is an H_v-LA-subsemigroup (resp., ideal) of H_2.

Proof. Proof is straightforward. □

6. Hyperorder on H$_v$-LA-semigroups

Definition 6.1. *Let $(H, *)$ be an H_v-LA-semigroup, and let $a, b \in H$. We write $a \rhd b$ if $a * c \subseteq b * c, \forall\ c \in H$, and we call $\rhd$ a hyperorder on H.*

Definition 6.2. *Let $(H, *)$ be an H_v-LA-semigroup, and let $a, b \in H$. If $a \rhd b$ and $b \rhd a$ then we say a is hyperequal to b, and it is denoted by $a \trianglerighteq b$.*

The relation " $\trianglerighteq$ " is an equivalence relation on H.

Proposition 6.1. *Let $(H, *)$ be an H_v-LA-semigroup, we define class $[a] = \{b \in H | a \trianglerighteq b\}$ and let $C(H) = \{[a] | a \in H\}$ denotes the set of all classes and if we define the hyperoperation on $C(H)$ as $[a] \otimes [b] = \{[n] | n \in a * b\}$, then $(C(H), \otimes)$ is an H_v-LA-semigroup.*

Proof. Let $[a], [b], [c] \in C(H)$. Consider

$$\begin{aligned} ([a] \otimes [b]) \otimes [c] &= (\{[n] | n \in a * b\}) \otimes [c] \\ &= \{[m] | m \in n * c\} = \{[m] | m \in (a * b) * c\}, \end{aligned}$$

and

$$\begin{aligned} ([c] \otimes [b]) \otimes [a] &= (\{[n_1] | n_1 \in c * b\}) \otimes [a] \\ &= \{[m_1] | m_1 \in n_1 * a\} = \{[m_1] | m_1 \in (c * b) * a\}. \end{aligned}$$

Since $(H, *)$ is an H$_v$-LA-semigroup i.e,

$$(a * b) * c \cap (c * b) * a \neq \emptyset.$$

Hence

$$([a] \otimes [b]) \otimes [c] \cap ([c] \otimes [b]) \otimes [a] \neq \emptyset, \text{ for all } [a], [b], [c] \in C(H).$$

Hence $(C(H), \otimes)$ is an H$_v$-LA-semigroup. □

7. Direct products of H$_v$-LA-semigroups

Let $(H_1, *)$ and $(H_2, \bullet)$ be two H$_v$-LA-semigroups. Given $(H_1 \times H_2, \otimes)$, $\otimes$ is a hyperoperation on $H_1 \times H_2$, such that

$$(a_1, b_1) \otimes (a_2, b_2) = \{(c, d) | c \in a_1 * a_2,\ d \in b_1 \bullet b_2\},$$

for all $(a_1, b_1), (a_2, b_2) \in H_1 \times H_2$. Then we say $(H_1 \times H_2, \otimes)$ is the direct product of H$_v$-LA-semigroups $(H_1, *)$ and $(H_2, \bullet)$.

Proposition 7.1. *The direct product of two H_v-LA-semigroups is again an H_v-LA-semigroup.*

Proof. Let $(H_1, *)$ and $(H_2, \bullet)$ be two H$_v$-LA-semigroups. We will show that their direct product $(H_1 \times H_2, \otimes)$ is also an H$_v$-LA-semigroup. Let (a_1, b_1), (a_2, b_2), $(a_3, b_3) \in H_1 \otimes H_2$, then

$$\begin{aligned} ((a_1, b_1) \otimes (a_2, b_2)) \otimes (a_3, b_3) &= (\{(c, d) | c \in a_1 * a_2,\ d \in b_1 \bullet b_2\}) \otimes (a_3, b_3) \\ &= \{(e, f) | e \in c * a_3,\ f \in d \bullet b_3\} \\ &= \{(e, f) | e \in (a_1 * a_2) * a_3,\ f \in (b_1 \bullet b_2) \bullet b_3\}. \end{aligned}$$

On the other hand

$$\begin{aligned}((a_3,b_3)\otimes(a_2,b_2))\otimes(a_1,b_1) &= (\{(c_1,d_1)\,|c_1\in a_3*a_2,\ d_1\in b_3\bullet b_2\})\otimes(a_1,b_1)\\ &= \{(e_1,f_1)\,|e_1\in c_1*a_1,\ f_1\in d_1\bullet b_1\}\\ &= \{(e_1,f_1)\,|e_1\in(a_3*a_2)*a_1,\ f_1\in(b_3\bullet b_2)\bullet b_1\}\,.\end{aligned}$$

Now since $(H_1,*)$ and $(H_2,\bullet)$ are H$_v$-LA-semigroups so

$$(a_1*a_2)*a_3\cap(a_3*a_2)*a_1\neq\emptyset,$$

and

$$(b_1\bullet b_2)\bullet b_3\cap(b_3\bullet b_2)\bullet b_1\neq\emptyset.$$

Hence by using this we get

$$((a_1,b_1)\otimes(a_2,b_2))\otimes(a_3,b_3)\cap((a_3,b_3)\otimes(a_2,b_2))\otimes(a_1,b_1)\neq\emptyset.$$

Which proves that $(H_1\times H_2,\otimes)$ is also an H$_v$-LA-semigroup, i.e direct product of two H$_v$-LA-semigroups is again an H$_v$-LA-semigroup. □

Proposition 7.2. *If $(K,*)$ and $(L,\bullet)$ are two H_v-LA-subsemigroups (resp., ideals) of $(H_1,*)$ and $(H_2,\bullet)$, respectively, then the direct product $K\times L$ is also an H_v-LA-subsemigroup (resp., ideal) of $(H_1\times H_2,\otimes)$.*

Proof. Proof is straightforward. □

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