NUMERICAL RECKONING COMMON FIXED POINT IN CAT(0) SPACES FOR A GENERAL CLASS OF OPERATORS

Abdul Latif[1], Mihai Postolache[2], Monairah Omar Alansari[3]

In this work, we propose a numerical algorithm for the computation of a common fixed point of two mappings endowed with the L_2 *property, in the setting of* CAT(0) *spaces. Its* Δ*-convergence is proved, and necessary and sufficient conditions for its strong convergence are given. These results generalize fixed point results in literature.*

Keywords: CAT(0) spaces, common fixed point, Δ-convergence.
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1. Introduction

Due to its importance both from theoretical and applied point of view, fixed point theory has encountered a wide development in recent years. The directions of study mainly have focused on three parts. The first one is that of finding various generalizations to the Banach contraction principle; please, see [4, 8, 23]. The second one refers to the setting in which fixed point results are obtained, including here ordered metric spaces, dislocated metric spaces, quasi metric spaces, b-metric spaces, extended b-metric spaces, CAT(0) spaces, hyperbolic spaces and so on; please, see [3, 19, 21, 25]. The third direction comprises designing numerical methods for the approximation of fixed points or common fixed points associated with appropriate generalized contractive operators. The development of numerical algorithms for the reckoning of fixed points, which started with the Picard sequence, has been the object of study of many scientists. Mann [17] proposed the next numerical scheme, for a selfmapping T defined on a convex subset K on a Banach space, by the use of an auxiliary sequence $\{\alpha_n\}$ from the interval $(0,1)$.

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \ \ n \geq 0,$$

where x_0 is a point in K.

In the same regard, Ishikawa [10] introduced a two step algorithm, by means of two auxiliary sequences $\{\alpha_n\}$, and $\{\beta_n\}$ in $(0,1)$, as follows

$$\begin{aligned} y_n &= (1-\alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} &= (1-\beta_n)x_n + \beta_n T y_n, \ \ n \geq 0, \end{aligned}$$

[1] Corresponding author. King Abdulaziz University, Department of Mathematics, POB 80203, Jeddah 21589, Saudi Arabia, email alatif@kau.edu.sa

[2] Corresponding author. Business School, Sichuan University, Chengdu 610064, China; and Romanian Academy, Gh. Mihoc-C. Iacob Institute of Mathematical Statistics and Applied Mathematics, Bucharest 050711, Romania and University "Politehnica" of Bucharest, Department of Mathematics and Informatics, Bucharest 060042, Romania, email mihai.postolache@upb.ro

[3] King Abdulaziz University, Department of Mathematics, POB 80203, Jeddah 21589, Saudi Arabia, email malansari@kau.edu.sa

where $x_0 \in K$, and T is a selfmapping on the convex subset K.

Agarwal *et al.* [2] proposed a two step iterative method, as follows

$$\begin{aligned} y_n &= (1-\alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} &= (1-\beta_n)T x_n + \beta_n T y_n, \;\; n \geq 0, \end{aligned}$$

with the starting point x_0 in the convex subset K, and the control sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $(0,1)$. This scheme was extended to a common fixed point algorithm, by Sahu *et al.* [20].

Noor [18] introduced a three step iterative method for the determination of a fixed point for a mapping T on a convex subset K of a Banach space, by the use of three control parameter sequences from $(0,1)$, as in the next lines

$$\begin{aligned} z_n &= (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n, \\ x_{n+1} &= (1-\gamma_n)x_n + \gamma_n T y_n, \;\; n \geq 0, \end{aligned}$$

where $x_0 \in K$.

In 2016, Thakur *et al.* [24] introduced the following scheme for the reckoning of fixed points of nonexpansive mappings

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n, \\ z_n &= (1-\gamma_n)y_n + \gamma_n T y_n, \\ x_{n+1} &= (1-\alpha_n)T y_n + \alpha_n T z_n, \;\; n \geq 0, \end{aligned}$$

$\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ in $(0,1)$, with convergence analysis in Banach spaces; see also [9].

Abbas and Nazir [1] constructed, starting with a point x_0 in K, the next three-step algorithm

$$\begin{aligned} z_n &= (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n &= (1-\beta_n)T x_n + \beta_n T z_n, \\ x_{n+1} &= (1-\gamma_n)T y_n + \gamma_n T z_n, \;\; n \geq 0, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are in sequences from $(0,1)$.

In [22], Sintunawarat and Pitea introduced the next numerical algorithm for the determination of a fixed point of a mapping $T\colon K \to K$, of Berinde type, as follows

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n y_n, \\ x_{n+1} &= (1-\alpha_n)T z_n + \alpha_n T y_n, \;\; n \geq 0, \end{aligned} \tag{1}$$

where $x_0 \in K$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$, and K is a closed, convex subset of a Banach space.

The purpose of this work is to develop a numerical scheme for the reckoning of a common fixed point of two mappings which are endowed with an adequate property defined by means of an inequality condition imposed on some superior limits associated to it. The paper is organized as follows. Section 2 is dedicated to preliminary concepts and properties needed in the sequel. Section 3 contains the main results on Δ-convergence and strong convergence of an algorithm for the determination of a common fixed points of two operators which satisfy property (L_2). The last section concludes the work.

2. Preliminaries

We start by introducing some facts on CAT(0) spaces, the framework chosen for our results.

Consider x and y points in a metric space (X, d). A continuous mapping $\gamma\colon [0, a] \to X$, with $\gamma(0) = x$, $\gamma(a) = y$, for which $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$, for any t_1, $t_2 \in [0, a]$, is called a geodesic path which joins x and y, while the image of this mapping is the geodesic segment from x and y. (X, d) is a geodesic space if any two points of it are linked by a geodesic. Moreover, if these geodesics are unique, then the space is called uniquely geodesic.

A geodesic triangle in a geodesic space (X, d) is formed by three distinct points and the geodesic segments between them. A comparison triangle is a triangle in the Euclidian plane such the lengths of its edges are the same as those of the initial triangle. (X, d) is a CAT(0) space if the inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}),$$

holds for any x and y in X, where $\bar{x}$, $\bar{y}$ are the corresponding comparison points.

A characterization of CAT(0) spaces was provided in [4], by the use of a condition of Bruhat and Tits [5]. They stated that a space (X, d) is a CAT(0) one if and only if the next inequality is satisfied for any points x_1, x_2, $x \in X$

$$d^2(x, \bar{x}) \leq \frac{1}{2}d^2(x, x_1) + \frac{1}{2}d^2(x, x_2) - \frac{1}{4}d^2(x_1, x_2),$$

where $\bar{x}$ is the midpoint of the segment joining the points x_1, and x_2.

This property can be used in order to obtain another inequality, for details please see [13], for the more general case of hyperbolic spaces.

Proposition 2.1. Let (X, d) be a CAT(0) space, x, y, $z \in X$, and $t \in [0, 1]$. The next inequality holds

$$d^2\big((1-t)x \oplus ty, z\big) \leq (1-t)d^2(x, y) + td^2(y, z) - t(1-t)d^2(x, y).$$

In our results we are going to use extensively a convexity-like property, recalled next.

Proposition 2.2. ([11]) Let (X, d) be a CAT(0) space, x, y, and $z \in X$. Then, for any $t \in [0, 1]$, the following inequality is satisfied

$$d\big((1-t)x \oplus ty, z\big) \leq (1-t)d(x, y) + td(y, z).$$

One important property which features the CAT(0) spaces is the fulfillment of the Opial condition. We recollect below the definition of the property introduced by Opial [16].

Definition 2.1. ([16]) A Banach space X satisfies the Opial property if for any sequence $\{x_n\}$ in X, which converges weakly to x, the next inequality holds

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for any $y \neq x$.

In the study of CAT(0) spaces, there are some sets which are very useful in proving convergence properties for sequences with suitable properties.

Definition 2.2. For a complete CAT(0) space (X, d), consider a bounded sequence $\{x_n\}$. The set

$$\mathcal{A}(x_n) = \{x \in X \big| \mathcal{R}(x, x_n) = \mathcal{R}(x_n)\}$$

is called the asymptotic center of the sequence $\{x_n\}$, where

$$\mathcal{R}(x_n) = \inf_{x\in X} \mathcal{R}(x, x_n)$$

is the asymptotic radius, and

$$\mathcal{R}(x, x_n) = \limsup_{n\to\infty} d(x, x_n).$$

Dhompongsa *et al.* [6] proved that, in case of complete CAT(0) spaces, the asymptotic center consists of one element only.

Kirk and Panyanak [12] introduced a type of convergence which they called the Δ-convergence, in the setting of Banach spaces. We recall the definition of such a convergence for the case of CAT(0) spaces.

Definition 2.3. Let $\{x_n\}$ be a sequence in the CAT(0) space (X, d). This sequence is Δ-convergent to $x \in X$ if its asymptotic center is a singleton formed by the element x.

With respect to this kind of convergence, the next lemma states two of its remarkable properties.

Lemma 2.1. i) ([12]) Any bounded sequence in a complete CAT(0) space possesses a Δ-convergent sequence.

ii) ([7]) The asymptotic center of a sequence bounded, included in a closed and convex subset K of a complete CAT(0) space belongs to K.

The asymptotic center plays a key role in the lemma below, which guarantees in an appropriate way the coincidence of some limits.

Lemma 2.2. ([7]) Suppose that $\{x_n\}$ is a bounded sequence in a complete CAT(0) space (X, d), $\mathcal{A}(x_n) = \{x\}$ and $\lim_{n\to\infty} d(x_n, x)$ is convergent. If $\{t_n\}$ is a subsequence of $\{x_n\}$ for which $\mathcal{A}(t_n) = \{t\}$, then $x = t$.

In 2011, Fuster and Gálvez [15] introduced a class of generalized nonexpansive mappings, the so-called operators endowed with the condition (L).

Definition 2.4. ([15]) Let K be a nonempty subset of a CAT(0) space X, and $T\colon K \to K$. T fulfills the property (L) if the next two conditions are satisfied

i) For any nonempty, closed, convex D of K, which is T-invariant (that is $TD \subseteq K$), there exists an almost fixed point sequence of T (i.e. a sequence $\{x_n\}$ so that $d(x_n, Tx_n)$ is convergent to zero);

ii) For any almost fixed point sequence $\{x_n\}$ of T in K, and $x \in K$, the following inequality holds true

$$\limsup_{n\to\infty} d(x_n, Tx) \leq \limsup_{n\to\infty} d(x_n, x).$$

Fuster and Gálvez [15] have also proved that the class of mappings which fulfill the property (L) properly contains that of the Suzuki operators [23], and also that of the García-Falset (E) mappings [8].

In the following, by L_2 operators we refer to those mappings which satisfy condition ii) from the above definition.

We introduce now our numerical algorithm, as follows. Assume that K is a nonempty, and convex subset of a CAT(0) space X. For two mappings T, $S\colon K \to K$, and $x_0 \in K$, we introduce the next numerical scheme, inspired by [22]:

$$\begin{aligned} y_n &= (1-\beta_n)x_n \oplus \beta_n T x_n, \\ z_n &= (1-\gamma_n)Sx_n \oplus \gamma_n y_n, \\ x_{n+1} &= (1-\alpha_n)Tz_n \oplus \alpha_n T y_n, \quad n \geq 0. \end{aligned} \tag{2}$$

In order to prove the almost fixed point property with respect to this numerical algorithm, we need the next auxiliary result.

Lemma 2.3. ([14]) In the complete CAT(0) space (X, d), consider a point x, and two sequences $\{t_n\}$, $\{u_n\}$. $\{s_n\}$ is a sequence of real numbers bounded away by 0 and 1. Presume that there exists $\ell \in \mathbb{R}$ so that $\limsup_{n\to\infty} d(t_n, x) \leq \ell$, $\limsup_{n\to\infty} d(u_n, x) \leq \ell$, and $\lim_{n\to\infty} d(s_n t_n \oplus (1 - s_n)u_n) = \ell$. Then the sequence $\{d(t_n, u_n)\}$ is convergent to zero.

3. Main results

First, we give a result which states the convergence of a sequence linked with those in the proposed scheme (2), for quasi-nonexpansive mappings with common fixed points.

Lemma 3.1. Presume that the quasi-nonexpansive mappings T, $S\colon K \to K$ are defined on a nonempty, closed, convex subset K of a CAT(0) space (X, d), and possess at least one common fixed point. Then, for any common fixed point, the sequence $\{d(x_n, p)\}$ defined by algorithm (2) exists.

Proof. As both mappings are quasi-nonexpansive, it follows that

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)x_n \oplus \beta_n Tx_n) \leq (1-\beta_n)d(x_n, p) + \beta_n d(Tx_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) = d(x_n, p), \quad n \geq 0. \end{aligned} \tag{3}$$

Furthermore, by the use of this inequality, we get

$$\begin{aligned} d(z_n, p) &= d((1-\gamma_n)Sx_n \oplus \gamma_n y_n) \leq (1-\gamma_n)d(Sx_n, p) + \gamma_n d(y_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(y_n, p) \leq d(x_n, p), \quad n \geq 0. \end{aligned} \tag{4}$$

From the definition of algorithm (2), and the two previous inequalities, we obtain

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)Tz_n \oplus \alpha_n Ty_n) \leq (1-\alpha_n)d(Tz_n, p) + \alpha_n d(Ty_n, p) \\ &\leq (1-\alpha_n)d(z_n, p) + \alpha_n d(y_n, p) \leq d(x_n, p), \quad n \geq 0, \end{aligned}$$

that is $\{d(x_n, p)\}$ is a decreasing sequence of positive numbers, therefore it is convergent. □

We continue now with a theorem which refers to the existence of almost fixed point sequences associated to operators T and S, generated by algorithm (2).

Theorem 3.1. *Let K be a nonempty, closed, and convex subset of a complete* CAT(0) *space X, and $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ sequences obtained by the application of scheme* (2) *for the quasi-nonexpansive operators T, $S\colon K \to K$, whose common fixed point set is not void. Then, the next statements hold true:*

i) *The sequence $d(Ty_n, Tz_n)$ is convergent to zero.*
ii) *$\{x_n\}$ is an almost fixed point sequence for the mapping T.*
iii) *The sequence $\{d(Sx_n, y_n)\}$ converges to zero.*
iv) *The sequence $\{d(x_n, y_n)\}$ converges to zero.*
v) *$\{x_n\}$ is an almost fixed point sequence for the operator S.*

Proof. i) Let p be a common fixed point of both T and S. According to Lemma 3.1, the sequence $\{d(x_n, p)\}$ is convergent. Denote by ℓ its limit. From the quasinonexpansiveness of the operators T, and S, we get that $d(Tx_n, p) \leq d(x_n, p)$, and $d(Sx_n, p) \leq d(x_n, p)$, $n \geq 1$, therefore it follows that

$$\limsup_{n\to\infty} d(Tx_n, p) \leq \ell,$$

and

$$\limsup_{n\to\infty} d(Sx_n, p) \leq \ell.$$

On the other hand, having in mind inequality (3), we obtain that

$$\limsup_{n\to\infty} d(y_n, p) \le \limsup_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d(x_n, p) = \ell. \tag{5}$$

Since T is quasi-nonexpansive, it follows that

$$\limsup_{n\to\infty} d(Ty_n, p) \le \limsup_{n\to\infty} d(y_n, p) \le \ell. \tag{6}$$

By the use of inequality (4), from Lemma 3.1 it can be observed that

$$\limsup_{n\to\infty} d(z_n, p) \le \limsup_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d(x_n, p) = \ell$$

which, jointly with the quasi-nonexpansiveness of the mapping T, implies that

$$\limsup_{n\to\infty} d(Tz_n, p) \le \limsup_{n\to\infty} d(z_n, p) \le \ell. \tag{7}$$

By the use of the last relation from algorithm (2), we have

$$\ell = \lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d\big((1-\alpha_n)Tz_n \oplus \alpha_n Ty_n, p\big),$$

which, together with inequalities (6), and (7) allow the application of Lemma 2.3, compelling that $\lim_{n\to\infty} d(Ty_n, Tz_n) = 0$.

ii) By the use of the properties of d, and the fact that there exists $\lim_{n\to\infty} d(Ty_n, Tz_n)$, we get

$$\begin{aligned}
\ell &= \lim_{n\to\infty} d(x_{n+1}, p) = \liminf_{n\to\infty} d\big((1-\alpha_n)Tz_n \oplus \alpha_n Ty_n, p\big)\\
&\le \liminf_{n\to\infty} \big((1-\alpha_n)d(Tz_n, p) + \alpha_n d(Ty_n, p)\big)\\
&\le \liminf_{n\to\infty} \big((1-\alpha_n)d(Tz_n, p) + \alpha_n d(Ty_n, Tz_n) + \alpha_n d(Tz_n, p)\big)\\
&\le \liminf_{n\to\infty} \big(d(Tz_n, p) + \alpha_n d(Ty_n, Tz_n)\big)\\
&\le \liminf_{n\to\infty} \big(d(Tz_n, p) + d(Ty_n, Tz_n)\big)\\
&= \liminf_{n\to\infty} d(Tz_n, p) + \lim_{n\to\infty} d(Ty_n, Tz_n)\\
&= \liminf_{n\to\infty} d(Tz_n, p) \le \ell,
\end{aligned}$$

hence $\liminf_{n\to\infty} d(Tz_n, p) = \ell$. Combining this relation with inequality (7), we obtain that the sequence $\lim_{n\to\infty} d(Tz_n, p) = \ell$.

Taking lim inf in the inequalities

$$d(Tz_n, p) \le d(Tz_n, Ty_n) + d(Ty_n, p) \le d(Tz_n, Ty_n) + d(y_n, p),$$

we get that $\liminf_{n\to\infty} d(y_n, p) \ge \ell$. By relation (5), it follows that there exists $\lim_{n\to\infty} d(y_n, p)$, and its value is ℓ. Since $y_n = (1-\gamma_n)x_n \oplus \alpha_n Tx_n$, applying Lemma 2.3, we obtain that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

iii) By the use of the triangle inequality, the fact that $\lim_{n\to\infty} d(Tz_n, p) = \ell$, and inequality (4), we obtain

$$\ell = \lim_{n\to\infty} d(Tz_n, p) = \limsup_{n\to\infty} d(Tz_n, p) \le \limsup_{n\to\infty} d(z_n, p) \le \limsup_{n\to\infty} d(x_n, p) = \ell,$$

therefore there exists $\lim_{n\to\infty} d(z_n, p) = \ell$. Since $z_n = (1-\gamma_n)Sx_n \oplus \gamma_n y_n$, $\limsup_{n\to\infty} d(Sx_n, p) \le \ell$, and $\lim_{n\to\infty} d(y_n, p) = \ell$, by Lemma 2.3 it follows that $\lim_{n\to\infty} d(Sx_n, y_n) = 0$.

iv) The following inequalities hold true

$$\begin{aligned}
d(y_n, x_n) &= d\big((1-\beta_n)x_n \oplus \gamma_n Tx_n, x_n\big)\\
&\le (1-\beta_n)d(x_n, x_n) + \beta_n d(Tx_n, x_n) \le d(Tx_n, x_n),\ n \ge 0.
\end{aligned}$$

By considering $n \to \infty$, we get that the sequence $\{d(x_n, y_n)\}$ converges to zero.

v) Eventually, it can be noticed that

$$d(Sx_n, x_n) \leq d(Sx_n, y_n) + d(y_n, x_n),$$

and, by the use of points iii) and iv) already proved, it follows that $\{x_n\}$ is an almost fixed point sequence. $\square$

We are now in a position to provide a Δ-convergence result for algorithm (2), with respect to a common fixed point for two operators which fulfill condition L_2.

Theorem 3.2. *Let K be a nonempty, closed, convex subset of the complete* CAT(0) *space (X, d), and T, $S\colon K \to K$ two operators endowed with the property* L_2*, which have at least one common fixed point. Then the iterative sequence $\{x_n\}$ generated by algorithm* (2) *is Δ-convergent to a common fixed point of S and T.*

Proof. Consider that $\mathcal{W}(x_n)$ is the reunion of all the asymptotic centers over all the subsequences of the sequence $\{x_n\}$.

The first stage in our proof is to show that each element from $\mathcal{W}(x_n)$ is also a fixed point of the mappings T, and S, respectively.

Consider $t \in \mathcal{W}(x_n)$, and $\{t_n\}$ a subsequence of $\{x_n\}$ whose asymptotic center is the set formed by the element t, that is $\mathcal{R}(t, t_n) = \mathcal{R}(t_n)$. As $\{d(x_n, p)\}$ is convergent, where p is a common fixed point of T and S, it means that $\{x_n\}$ is bounded, so $\{t_n\}$ is bounded. Then, there exists a subsequence $\{s_n\}$ of $\{t_n\}$, so that $\Delta - \lim_{n\to\infty} s_n = s \in K$. Since $\lim_{n\to\infty} d(Tx_n, x_n) = 0$, and $\{s_n\}$ is a subsequence of $\{x_n\}$, it follows that $\lim_{n\to\infty} d(s_n, Ts_n) = 0$, so $\{s_n\}$ is an almost fixed point sequence. By the use of the (L_2) property, and of that of Opial, we obtain

$$\limsup_{n\to\infty} d(s_n, Ts) \leq \limsup_{n\to\infty} d(s_n, s) \leq \limsup_{n\to\infty} d(s_n, Ts),$$

hence

$$\limsup_{n\to\infty} d(s_n, Ts) = \limsup_{n\to\infty} d(s_n, s).$$

Having in view the property of the asymptotic center, we get that s is a fixed point of T. Furthermore, as the sequence $\{d(x_n, s)\}$ is convergent, by Lemma 2.2, we get that $s = t$, so $\mathcal{W}(x_n)$ is included into the set of the fixed points of the mapping T.

Similarly, a corresponding property may be proved for the operator S, hence each element of the set $\mathcal{W}(x_n)$ is a common fixed point of the two mappings T and S.

Let us prove now that $\mathcal{W}(x_n)$ contains one element only. Presuming the contrary, there can be found a subsequence $\{s_n\}$ of $\{x_n\}$ so that its Δ-limit is $s \neq t$. Then Lemma 2.2 necessarily implies that $s = t$, therefore the supposition was false, and $\mathcal{W}(x_n)$ is a singleton, and the conclusion follows. $\square$

In case T is the identity mapping, scheme (2) becomes the Ishikawa algorithm for the CAT(0) spaces, and Theorem 3.2 reduces to the Δ-convergence of this scheme in the setting of CAT(0) spaces, for operators endowed with the L_2 property (so subsequently for Suzuki nonexpansive operators, or García-Falset ones).

If we consider S as the identity operator in (2), we obtain the numerical algorithm (1), and Theorem 3.2 reduces to a result on its Δ-convergence, in case of mappings which fulfill the condition (L_2) (that is it refers also to Suzuki type mappings or García-Falset operators).

We continue with a theorem which gives necessary and sufficient conditions for the strong convergence of the sequence generated by scheme (2).

Theorem 3.3. *Let K be a nonempty, closed, and convex subset of a* CAT(0) *space* (X, d), *and* S, $T\colon K \to K$, *which satisfies condition* L_2. *Then the sequence* $\{x_n\}$, *generated by* (2) *converges to a point from* $\mathcal{F}$, *the set of common fixed points of the two operators, if and only if* $\liminf_{n\to\infty} d(x_n, \mathcal{F})$.

Proof. Suppose first that the sequence $\{x_n\}$, generated by (2), is strongly converges to a common fixed point $p \in \mathcal{F}$. Then, it follows that

$$\begin{aligned} \liminf_{n\to\infty} d(x_n, \mathcal{F}) &= \liminf_{n\to\infty} \inf_{q\in\mathcal{F}} d(x_n, q) \\ &\leq \liminf_{n\to\infty} d(x_n, p) = 0, \end{aligned}$$

and the conclusion has been proved.

Presume now that $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. By Lemma 3.1, we get that $\{d(x_n, p)\}$ is a decreasing sequence, for any $p \in \mathcal{F}$. We obtain

$$d(x_{n+1}, \mathcal{F}) = \inf_{q\in\mathcal{F}} d(x_{n+1}, q) \leq \inf_{q\in\mathcal{F}} d(x_n, q) = d(x_n, \mathcal{F}),$$

hence the sequence $\{d(x_n, \mathcal{F})\}$ is decreasing, so convergent, let us say to ℓ. As $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$, we get that $\ell = 0$. Consider $\epsilon > 0$. Then, there exists a rank N so that for any $n > N$, $d(x_n, \mathcal{F}) < \frac{\epsilon}{4}$, that is

$$\inf_{q\in\mathcal{F}} d(x_n, \mathcal{F}) < \frac{\epsilon}{4}.$$

It follows that we can find $p \in \mathcal{F}$ so that $d(x_N, p) < \frac{\epsilon}{2}$. By the triangle inequality, for m, $n > N$, we have

$$d(x_n, x_m) \leq d(x_m, p) + d(x_n, p),$$

which, as $d(x_n, p)$ is decreasing, implies

$$d(x_m, x_n) \leq d(x_N, p) + d(x_N, p) \leq \epsilon,$$

so $\{x_n\}$ is a Cauchy sequence in a complete CAT(0) space, so it converges to $x \in K$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, we get that $d(x, \mathcal{F}) = 0$. It can be easily check that $\mathcal{F}$ is a closed set, therefore $x \in \mathcal{F}$, and the conclusion follows. □

As particular cases of this theorem, we emphasize that of T being the identity mapping, which leads to a result on the strong convergence of the Ishikawa numerical scheme, and that of S being the identity operator, which gives the strong convergence of process (1).

We will give now another strong convergence result, based on a condition involving a nondecreasing sequence.

Theorem 3.4. *Consider that K is a nonempty, closed, and convex subset of a* CAT(0) *space* (X, d) *and* S, T *selfmappings on* K *which satisfy condition* L_2, *and* $\mathcal{F}$ *is nonempty. Suppose that there exists an increasing function* $f\colon [0,\infty) \to [0,\infty)$, *with* $f(0) = 0$, $f(a) > 0$ *for* $a > 0$, *and at least one of the following assumptions hold:*

i) $d(x, Tx) \geq f(d(x, \mathcal{F}))$, *for any* $x \in K$;

ii) $d(x, Sx) \geq f(d(x, \mathcal{F}))$, *for any* $x \in K$.

Then, the sequence generated by algorithm (2) *is convergent to a common fixed point of mappings* S *and* T.

Proof. As in the proof of Theorem 3.3, we get that the sequence $\{d(x_n, \mathcal{F})\}$ is decreasing, so it has a limit.

Presume that condition i) holds. Then, since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we obtain

$$\lim_{n\to\infty} f(d(x_n, \mathcal{F})) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0,$$

which, combined with the properties of the function f, leads to $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Using Theorem 3.3, we get the conclusion.

The case when property ii) holds can be solved similarly. □

4. Conclusions

We have introduced a numerical scheme for the determination of a common fixed associated with two mappings which satisfy a property defined by means of lim sup, which we called L_2 property, in the framework of CAT(0) spaces. Properties regarding almost fixed point sequences related to it are studied. A necessary and sufficient condition of Δ-convergence is stated. Two theorems on its strong convergence are stated and proved, provided additional conditions are fulfilled. These results generalize existing results in specialized literature.

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