U.P.B. Sci. Bull., Series A, Vol. 78, Iss. 3, 2016 ISSN 1223-7027

A LAGRANGE MULTIPLIER APPROACH USING INTERVAL FUNCTIONS FOR GENERALIZED NASH EQUILIBRIUM IN INFINITE DIMENSION

Bogdan BIOLAN[1]

In this paper we study a class of generalized Nash equilibrium problems in the interval analysis framework. Some characterizations of the solutions corresponding to players which share the same Lagrange Multipliers are also given. The types of functions used in this article are the so called interval applications or interval functions. According to [3], this kind of Nash equilibria concept was introduced by Rosen [12] in 1965 for finite dimensional spaces. In order to obtain the same property for the infinite dimensional approach, we use recent developments of a new duality theory. Regarding its usfulness new theorems are proven and similar kinds of equilibrium for pay-off interval type functions or their extended versions are approached.

Keywords: Variational inequalities; Nash equilibria; interval functions; Lagrange multipliers; infinite dimension;

1. Introduction

In [3] F.Faraci extended the Nash Equilibria concept defined by Rosen [12] in 1965 to infinite dimensional spaces. The aim of this paper is to extend this type of equilibria obtained in [3] to a class of functions, called interval functions. So the pay-off functions used in this article and the other functionals are described be interval functions. Generalized Nash equilibrium problems (GNEP's) are noncooperative games in which the strategy of each player can depend on the rival players' strategies. These problems have become popular recently because of their utility for modeling economic problems, as well as routing problems in communication networks. Recently, Facchinei *et al.* [2] have proved that for a large class of GNEP's, in finite dimension, certain solutions can be computed by solving a variational inequality rather than a quasi-variational inequality. Moreover, they have proved that the solutions of GNEP's which are preserved by switching to the variational inequality formulation are characterized by the fact that all players share the same vector of Lagrange multipliers. This kind of Nash equilibria was introduced by Rosen in his seminal paper [12] and its connection to variational inequalities has important consequences from computational point of view. In order to prove the main result from [2], assuming that some constraints qualification holds, the authors use the Knaster-Kuratowski-Mazurkievicz conditions associated to the GNEP. In this paper we are interested in extending

[1] PhD Candidate, Doctoral School of Mathematics, University of Bucharest, Romania, e-mail: bbiolanc@yahoo.com

the above mentioned result to an infinite-dimensional functional setting. This extension is motivated by the fact that modeling the all time dependent equilibrium problems requires the use of variational inequalities in L^p spaces. Moreover, the study of random equilibrium problems can be done using variational inequalities theory in probability spaces. Here can be mentioned the contributions of [6-11].

The application of the standard Lagrangian duality theory to the above mentioned infinite-dimensional problems is not possible, because the ordering cone which describes the inequality constraints (i.e. the cone of non-negative functions in some L^p space) has empty interior. However, very recently, a new duality theory developed in [3] has proved to be a powerful instrument to overcome this difficulty. The key tool in this theory is represented by the so-called Assumption S, which replaces, in the infinite-dimensional framework, the qualification constraints assumptions of the classical finite-dimensional setting. The main result has been improved and since then many works have been devoted to develop and apply the new duality theory. In this paper, by using the new theory, we are able to prove the existence of Lagrange Multipliers for GNEP's in general Banach spaces and to extend the results to the infinite dimension case. For more details concerning convex analysis in infinite dimension, see [13].

The paper is organized as follows. In Section 2 we introduce the setting of our problem and the variational inequality concept, which constitute the main object of our study. In Section 3 we prove our main result in an abstract Banach space, we denote attention to Assumption S and we show how abstract theory can be applied to concrete case of GNEP's in Lebesgue spaces. Finally, in Section 4, we conclude our work by revising the importance of the new results obtained in this paper.

2. The setting

Let X_1 and X_2 be two Banach spaces and let $u = (u^1, u^2)$ be an element of the product space $X = X_1 \times X_2$. The element u^1 corresponds to the first player and the element u^2 corresponds to the second one. Let $K \subset X$ be an non-empty and convex set and let $J_1, J_2: X \to R$ be two functionals, also known as the utility functions or the pay-off functions, so that $J_1(.\,, u^2)$ is convex and Gateaux differentiable for every $u^2 \in X_2$ and $J_2(u^1, .)$ is convex and Gateaux differentiable for every $u^1 \in X_1$.

For every $u = (u^1, u^2)$ the sets of optimal strategies of the two players can be expressed as follows:

$K_1(u) = \{v^1 \in X_1: (v^1, u^2) \in K\} \subset X_1,$

$K_2(u) = \{v^2 \in X_2: (u^1, v^2) \in K\} \subset X_2\ .$

Note that if $u \in K$, then the above sets are convex and non-empty, as $u^i \in K_i(u)$. The purpose of each player i, given the strategy of the rival, is to choose a strategy which minimizes the function J_i on its optimal set.

The following definition describes the goal of the game, which consists in finding an equilibrium point for both players, represented by a vector $\bar{u} = (\bar{u}^1, \bar{u}^2)$, such that no player can decrease his utility function by changing unilaterally $\bar{u}^i$ to any other optimal point. We will recall this definition from [3] and afterwards we will show how an interval function can change this setting.

Definition 2.1. We say that $\bar{u} = (\bar{u}^1, \bar{u}^2)$ is a generalized Nash equilibrium point or a solution of the GNEP (Generalised Nash Equilibrium Problems, see [3]) if $\bar{u} \in K$ and the following conditions hold:

$$J_1(\bar{u}^1, \bar{u}^2) = \min\{J_1(u^1, \bar{u}^2); u^1 \in K_1(\bar{u})\},$$
$$J_2(\bar{u}^1, \bar{u}^2) = \min\{J_2(\bar{u}^1, u^2); u^2 \in K_2(\bar{u})\}.$$

Let $a, b, a', b' \in \mathbb{R}$.

Definition 2.2. We say that $[a,b] \leq [a', b']$ if: $\begin{Bmatrix} a \leq a' \\ b \leq b' \end{Bmatrix}$.

We say that $[a,b] < [a', b']$ if: $\begin{Bmatrix} a \leq a' \\ b < b' \end{Bmatrix}$ or $\begin{Bmatrix} a < a' \\ b \leq b' \end{Bmatrix}$ or $\begin{Bmatrix} a < a' \\ b < b' \end{Bmatrix}$.

Let I be a non-empty set. Then we define $f: I \to MI(\mathbb{R}), f = [f^L, f^U]$, with $f^L(x) \leq f^U(x), \forall x \in I$, the so-called interval function, where

$$MI(\mathbb{R}) = \{J : J \subset \mathbb{R} \text{ is a closed interval}\}.$$

We say that $\overline{x}$ is a minimum for f if the following constraints hold:

$$[f^L(\bar{x}), f^U(\bar{x})] < [f^L(x), f^U(x)], \forall x \in I.$$

Let $g: I \to \mathbb{R}^m$ be a vectorial application, where $m \geq 1$.

Consider the optimization problem:

$$\begin{Bmatrix} \inf\,[f^L, f^U] \\ g(x) \leq 0 \\ x \in I \end{Bmatrix}.$$

We say that $\bar{x}$ is an optimum interval point iff $\bar{x}$ is an optimal solution for: $p^L(\bar{x})$ and $p^U(\bar{x})$, where:

$$p^L(\bar{x}): \begin{Bmatrix} \min f^L(x) \\ g(x) \leq 0 \\ f^U(x) \leq f^U(\bar{x}) \\ x \in I \end{Bmatrix} \quad \text{and:} \quad p^U(\bar{x}): \begin{Bmatrix} \min f^U(x) \\ g(x) \leq 0 \\ f^L(x) \leq f^L(\bar{x}) \\ x \in I \end{Bmatrix}.$$

We will give the definition of Nash equilibrium point for this class of functions.

Let J_1 and J_2 be two interval functions, $J_1, J_2: X \to MI(\mathbb{R})$ the utility functions or pay-off functions so that $J_1(., u^2)$ is convex and Gateaux differentiable for every $u^2 \in X_2$ and $J_2(u^1, .)$ is convex and Gateaux differentiable, for every $u^1 \in X_1$.

Now, we will define the equilibrium point for the problems with the interval functions: $[{J_1}^L, {J_1}^U]$, $[{J_2}^L, {J_2}^U]$.

Definition 2.3. We say that $\bar{u} = (\bar{u}^1, \bar{u}^2)$ is an interval equilibrium point for GNEP if the following conditions hold:

(1) $J_1(\bar{u}^1, \bar{u}^2) = \min\{J_1(u^1, \bar{u}^2); u^1 \in K_1(\bar{u})\}$, where $\bar{u}^2$ is fixed;

(2) $J_2(\bar{u}^1, \bar{u}^2) = J_2(\bar{u}^1, \bar{u}^2) = \min\{J_2(\bar{u}^1, u^2); u^2 \in K_2(\bar{u})\}$, where $\bar{u}^1$ is fixed,

i.e., $\bar{u}^1$ is optimal for the problems:

$$p^L(\bar{u}^2): \begin{Bmatrix} \min {J_1}^L(u^1, \bar{u}^2) \\ u^1 \in K_1(\bar{u}) \\ {J_1}^U(u^1, \bar{u}^2) \leq {J_1}^U(\bar{u}^1, \bar{u}^2) \end{Bmatrix}, p^U(\bar{u}^2): \begin{Bmatrix} \min {J_1}^U(u^1, \bar{u}^2) \\ u^1 \in K_1(\bar{u}) \\ {J_1}^L(u^1, \bar{u}^2) \leq {J_1}^L(\bar{u}^1, \bar{u}^2) \end{Bmatrix}$$

and $\bar{u}^2$ is optimal for the problems:

$$p^L(\bar{u}^1): \begin{Bmatrix} \min {J_2}^L(\bar{u}^1, u^2) \\ u^2 \in K_2(\bar{u}) \\ {J_2}^U(\bar{u}^1, u^2) \leq {J_2}^U(\bar{u}^1, \bar{u}^2) \end{Bmatrix}, p^U(\bar{u}^1): \begin{Bmatrix} \min {J_2}^U(\bar{u}^1, u^2) \\ u^2 \in K_2(\bar{u}) \\ {J_2}^L(\bar{u}^1, u^2) \leq {J_2}^L(\bar{u}^1, \bar{u}^2) \end{Bmatrix},$$

respectively.

Remark 2.1. (1) and (2) are the equilibrium conditions for the so-called interval functions.

Remark 2.2. $\bar{u}$ is an interval equilibrium point for GNEP iff $\bar{u}$ is an optimum for: $p^L(\bar{u}^1), p^U(\bar{u}^1)$ and $p^L(\bar{u}^2), p^U(\bar{u}^2)$.

Now we recall the concept of Gateaux differentiability. Let Y be a Banach space and Y^* the dual of the topological space Y.

Definition 2.4. The function $h: Y \to \mathbb{R}$ is said to be Gateaux differentiable in $\bar{u} \in Y$ if there exists $\varphi \in Y^*$ such that we have:

$$\lim_{\alpha \to 0^+} \frac{h(\bar{u} + \alpha u) - h(\bar{u})}{\alpha} = \varphi(u), (\forall) u \in Y.$$

The functional φ is called the Gateaux derivative of h and it will be denoted by $\varphi \equiv Dh(\bar{u})$.

From well-known results of convex analysis (see e.g. Theorem 3.8 of [4]), $\bar{u} = (\bar{u}^1, \bar{u}^2)$ is considered to be optimum interval for a GNEP interval game if and only if:

$$D_1{J_1}^L(\bar{u}^1, \bar{u}^2)(u^1 - \bar{u}^1) \geq 0,\ (\forall)\ u^1 \in K_1(\bar{u}) \cap \{u^1: {J_1}^U(u^1, \bar{u}^2) \leq {J_1}^U(\bar{u}^1, \bar{u}^2)\},$$
$$D_1{J_1}^U(\bar{u}^1, \bar{u}^2)(u^1 - \bar{u}^1) \geq 0,\ (\forall)\ u^1 \in K_1(\bar{u}) \cap \{u^1: {J_1}^L(u^1, \bar{u}^2) \leq {J_1}^L(\bar{u}^1, \bar{u}^2)\},$$
(2.2)
$$D_2{J_2}^L(\bar{u}^1, \bar{u}^2)(u^2 - \bar{u}^2) \geq 0,\ (\forall)\ u^2 \in K_2(\bar{u}) \cap \{u^2: {J_2}^U(\bar{u}^1, u^2) \leq {J_2}^U(\bar{u}^1, \bar{u}^2)\},$$
$$D_2{J_2}^U(\bar{u}^1, \bar{u}^2)(u^2 - \bar{u}^2) \geq 0,\ (\forall)\ u^2 \in K_2(\bar{u}) \cap \{u^2: {J_2}^L(\bar{u}^1, u^2) \leq {J_2}^L(\bar{u}^1, \bar{u}^2)\},$$

where D_1 and D_2 stand for the Gateaux derivative of ${J_1}^L(.,\bar{u}^2), {J_1}^U(.,\bar{u}^2)$ and ${J_2}^U(\bar{u}^1,.), {J_2}^L(\bar{u}^1,.)$, respectively.
Denote by $\Gamma: X \to {X_1}^* \times {X_2}^*$,

$$\Gamma(u^1,u^2) = \begin{pmatrix} D_1{J_1}^L(u^1,u^2) \\ D_1{J_1}^U(u^1,u^2) \\ D_2{J_2}^L(u^1,u^2) \\ D_2{J_2}^U(u^1,u^2) \end{pmatrix}.$$

Now we will recall the concept of underlevel subset.
Definition 2.5. We say that $\boldsymbol{L}_\psi(\alpha) = \{x: \psi(x) \le \alpha\}$, where $\alpha \in \mathbb{R}$ is the underlevel subset of the function $\psi: X \to \mathbb{R}$.
Considering this, it is clear that (2.2) is equivalent with:

$$\Gamma(\bar{u})^T(\mathrm{u} - \bar{u}) \ge 0, (\forall)u \in \left(K_1(\bar{u}) \cap \boldsymbol{L}_{{J_1}^U}\left({J_1}^U(\bar{u}^1,\bar{u}^2)\right) \cap \boldsymbol{L}_{{J_1}^L}\left({J_1}^L(\bar{u}^1,\bar{u}^2)\right)\right) \times \left(K_2(\bar{u}) \cap \boldsymbol{L}_{{J_2}^U}\left({J_2}^U(\bar{u}^1,\bar{u}^2)\right) \cap \boldsymbol{L}_{{J_2}^L}\left({J_2}^L(\bar{u}^1,\bar{u}^2)\right)\right).$$

Obvious: $\boldsymbol{L}_{{J_1}^L}\left({J_1}^L(\bar{u}^1,\bar{u}^2)\right) = \{u^1: {J_1}^L(u^1,\bar{u}^2) \le {J_1}^L(\bar{u}^1,\bar{u}^2)\}$

$$\boldsymbol{L}_{{J_1}^U}\left({J_1}^U(\bar{u}^1,\bar{u}^2)\right) = \{u^1: {J_1}^U(u^1,\bar{u}^2) \le {J_1}^U(\bar{u}^1,\bar{u}^2)\},$$

and the same for the others two involving J_2.
Since the convex sets $K_i(\bar{u})$ depend on the solution, one obtains that GNEP for interval games can be formulated equivalently as a quasi-variational inequality. The nature of the optimal sets allows us to reduce the problem to variational inequalities. Solving this associated to Γ and the set K (in short : $VI(\Gamma, \mathrm{K})$), means finding a point $\bar{u} = (\bar{u}^1, \bar{u}^2) \in K$ such that we have the following inequality:

$$\Gamma(\bar{u})^T(\mathrm{u} - \bar{u}) \ge 0, (\forall)u \in K.$$

Theorem 2.1. Every solution of the variational inequality $VI(\Gamma, \mathrm{K})$) is a solution of GNEP interval games.
Proof:
Let $\bar{u} = (\bar{u}^1, \bar{u}^2) \in K$ be solution of (2.4.) where Γ is as in (2.3). If $u^1 \in K_1(\bar{u})$, then $u = (u^1, \bar{u}^2) \in K$ and from the definition of Γ we have that $0 \le \Gamma(\bar{u})^T(\mathrm{u} - \bar{u}) = D_1{J_1}^L(\bar{u}^1,\bar{u}^2)(u^1 - \bar{u}^1)$. In a similar way we get the other three inequalities of (2.2).
A solution of the GNEP interval games that is also a solution of $VI(\Gamma, \mathrm{K})$) is usually referred in Nash equilibrium theory as a variational equilibrium.
Theorem 2.2. ([7], Corollary 3.7). If K is a convex, closed and bounded subset of a reflexive space X and $\Gamma: \mathrm{K} \to \mathrm{X}^*$ is a monotone map which is continuous on finite-dimensional subspaces of K, then $VI(\Gamma, \mathrm{K})$) has a solution.

3. The Lagrange multipliers rule

A solution of the GNEP interval games can be obtained as a solution of the $VI(\Gamma, \mathrm{K})$). By adopting the reduction method, we can lose solutions of the GNEP interval game.

We want to see now which kind of solutions are preserved for a special set of constraints. We follow the finite dimensional case [5] and prove that a solution of the GNEP interval game is a variational equilibrium iff the shared constraints have the same multipliers. The result is true under any constraints qualification condition.

The setting is the same like in Section 2. We assume that Y is a Banach space ordered by a convex cone, let's say C, $g: X \to Y$ is convex, continuously Gateaux differentiable mapping and:

$$K = \{w \in Y^* :< w, z >_{Y^*,Y} \geq 0, (\forall) z \in C\},$$

$<.,.>_{Y^*,Y}$ denotes the duality between Y^* and Y.

If $f: X \to \mathbb{R}$ and $\bar{u} \in K$, we say that $\bar{u}$ is a solution of the minimal problem $(P_{f,K})$([3]) if:

$$f(\bar{u}) = \min\{f(x) |\ x \in K\}.$$

The following theorem is the main result of our research.

Theorem 3.1. (i) Let $\bar{u}$ be a solution of the $VI(\Gamma, \mathrm{K})$ so that a suitable constraints qualification condition for the $VI(\Gamma, \mathrm{K})$ takes place at $\bar{u}$. Then $\bar{u}$ is a solution of the GNEP–interval game such that both players have the same Lagrange multipliers.

(ii) $\bar{u}$ is a solution of the GNEP-interval game such that a constraints qualification condition takes place at $\bar{u}$ and both players have the same Lagrange multipliers. Then $\bar{u}$ is a solution of the $VI(\Gamma, \mathrm{K})$.

Proof:

(i) Suppose that $\bar{u}$ is a solution of the $VI(\Gamma, \mathrm{K})$. Then, if $f: X \to \mathbb{R}$ is the function defined by: $f(u) = \Gamma(\bar{u})^T(\mathrm{u} - \bar{u})$ (3.1),

then f is convex, Gateaux differentiable with the derivative given by:

$$Df(u)(z) = \Gamma(\bar{u})^T(z) \text{ for all } z \in X \text{ and for all } u \in X \text{ and:}$$

$f(\bar{u}) = \min\{f(x) |\ x \in K\} = 0$. Under a suitable constraints qualification condition, there exists $\bar{w} \in C^*$ such that:

(3.2): $0 = Df(\bar{u}) + \bar{w}Dg(\bar{u}) = \Gamma(\bar{u})^T + \bar{w}Dg(\bar{u})$, and (3.3): $< \bar{w}, g(\bar{u}) >_{Y^*,Y}$.

Since $g \in C^1(X, Y)$, $Dg(\bar{u})u = D_1g(\bar{u})u^1 + D_2g(\bar{u})u^2$, $(\forall)(u^1, u^2) \in X$

and for the arbitrariness of $(u^1, u^2) \in X$, (3.2) and (3.3) can be rewritten as:

$\left(D_1J_1{}^L(\bar{u}) + D_1J_1{}^U(\bar{u})\right)u^1 + \left(D_2J_2{}^L(\bar{u}) + D_2J_2{}^U(\bar{u})\right)u^2 + \bar{w}(D_1g(\bar{u})u^1)$

$+ \bar{w}(D_2g(\bar{u})u^2) = 0$, $(\forall)(u^1, u^2) \in X$, and for the arbitrariness of $(u^1, u^2) \in X$, (3.2) and (3.3) read as:

(α) $D_1{J_1}^L(\bar{u}) + \bar{w}D_1 g(\bar{u}) = 0,$ $D_1{J_1}^U(\bar{u}) + \bar{w}D_1 g(\bar{u}) = 0,$ $D_2{J_2}^L(\bar{u}) + \bar{w}D_2 g(\bar{u}) = 0,$ $D_2{J_2}^U(\bar{u}) + \bar{w}D_2 g(\bar{u}) = 0.$

$(\beta) < \bar{w}, g(\bar{u}) >_{Y^*,Y} = 0.$

If $g_1: X_1 \to Y$ is the mapping $g_1(u^1) = g_1(u^1, \bar{u}^2)$, then the set $K_1(\bar{u})$ has the following expression: $K_1(\bar{u})^L = \{u^1: g(u^1) \in -C\} \cap \boldsymbol{L}_{{J_1}^L}({J_1}^L(\bar{u}^1, \bar{u}^2))$ and:
$K_1(\bar{u})^U = \{u^1: g(u^1) \in -C\} \cap \boldsymbol{L}_{{J_1}^U}({J_1}^U(\bar{u}^1, \bar{u}^2))$.

Similarly, if $g_2: X_2 \to Y$ is defined by $g_2(u^2) = g_1(\bar{u}^1, u^2)$, then:
$K_2(\bar{u})^L = \{u^2: g(u^2) \in -C\} \cap \boldsymbol{L}_{{J_2}^L}({J_2}^L(\bar{u}^1, \bar{u}^2))$,
$K_2(\bar{u})^U = \{u^2: g(u^2) \in -C\} \cap \boldsymbol{L}_{{J_2}^U}({J_2}^U(\bar{u}^1, \bar{u}^2))$. One also has that:
$Dg_i(\bar{u}^i) = D_i g(\bar{u})$, and $g_i(\bar{u}) = g(\bar{u}), i = 1,2.$

Then (α) and (β) can be rewritten as:
$D_1{J_1}^L(\bar{u}) + \bar{w}Dg_1(\bar{u}^1) = 0, D_1{J_1}^U(\bar{u}) + \bar{w}Dg_1(\bar{u}^1) = 0, D_2{J_2}^L(\bar{u}) + \bar{w}Dg_2(\bar{u}^2) = 0, D_2{J_2}^U(\bar{u}) + \bar{w}Dg_2(\bar{u}^2) = 0,$

$$< \bar{w}, g_1(\bar{u}) >_{Y^*,Y} = < \bar{w}, g_2(\bar{u}) >_{Y^*,Y} = 0.$$

The above condition means that $\bar{u}$ satisfies the Lagrange multipliers rule for the GNEP interval games with $\bar{w}$ the common multiplier for both players.

This condition guarantees (e.g. Corollary 5.15 of [4]) that $\bar{u}$ is the minimal solution of the following problems $(P_{f,K})$, with $(f, K) = ({J^L}_1, K_1(\bar{u})^L)$ and $(f, K) = ({J^U}_1, K_1(\bar{u})^U)$ and $(f, K) = ({J^L}_2, K_2(\bar{u})^L)$ and $(f, K) = ({J^U}_2, K_2(\bar{u})^U)$, respectively, that is $\bar{u}$ is a GNEP interval solution and both players have the same Lagrange multipliers.

(ii) Suppose that $\bar{u}$ is a GNEP interval solution and some constraints qualification takes place at $\bar{u}$. If the two players have the same Lagrange multipliers, then:
(α_1) $D_1{J_1}^L(\bar{u}) + \bar{w}Dg_1(\bar{u}^1) = 0, D_1{J_1}^U(\bar{u}) + \bar{w}Dg_1(\bar{u}^1) = 0,$
$(\beta_1) < \bar{w}, g_1(\bar{u}^1) >_{Y^*,Y} = 0$
and
(α_2) $D_2{J_2}^L(\bar{u}) + \bar{w}Dg_2(\bar{u}^2) = 0, D_2{J_2}^U(\bar{u}) + \bar{w}Dg_2(\bar{u}^2) = 0,$
$(\beta_2) < \bar{w}, g_1(\bar{u}^2) >_{Y^*,Y} = 0.$

In conclusion it is clear that (α) and (β) are satisfied. From Corollary 5.15 of [4], we get that $\bar{u}$ is a minimal solution of the problem $(P_{f,K})$ with f as in (3.1). This means that $\bar{u}$ is a solution of the $VI(\Gamma, \mathrm{K})$.

Remark 3.1. $\bar{w}$ is the common Lagrange multiplier for both players.

4. Conclusions

In this paper we have studied a special type of Nash equilibria, corresponding to the case when the pay-off functions associated to the two players who want to maximize their winning chances are described by interval functions.

Results from Convex Analysis and Duality Theory were used for obtaining new original results concerning these type of equilibria. The results obtained can be applied in several fields such as Economics. For investors who seek to improve their available wealth (where the available wealth is considered to be an interval function) at the end of a period, the equilibrium results obtained in this paper show that this available wealth has an optimum interval point under some given conditions. Furthermore, these results can be applied in other fields of mathematics such as Optimization, Optimal Control Theory and Differential Geometry.

Acknowledgements

This paper has been financially supported within the project entitled Programe doctorale si postdoctorale **-** suport pentru cresterea competitivitatii cercetarii în domeniul Stiintelor exacte, contract number POSDRU/159/1.5/S/ 137750. This project is co-financed by European Social Fund through Sectoral Operational Programme for Human Resources Development 2007-2013. Investing in people!

R E F E R E N C E S

[1]. *P. Daniele, S. Giuffré, G. Idone, A. Maugeri*, "Infinite dimensional duality and applications", Matematische Annalen, **vol. 339**, May 2007, pp. 221-239.
[2]. *F. Facchinei, A. Fisher, V. Picialli*, "On generalized Nash games and variational inequalities", Operations Research Letters, **vol. 35**, no. 2, March 2007, pp. 159-164.
[3]. *F. Faraci, F. Raciti*, "On generalized Nash equilibrium in infinite dimension: the Lagrange multipliers approach", Optimization: A Journal of Mathematical Programming and Operations Research, Dec. 2012, pp. 1-18, DOI: 10.1080/02331934.2012.747090.
[4]. *J. Jahn*, Introduction to the Theory of Nonlinear Optimization, Springer, Berlin, 2006.
[5]. *A. Maugeri, F. Raciti*, "On existence theorems for monotone and no-monotone variational inequalities", Journal of Convex Analysis, **vol. 16**, 2009, pp. 899-911.
[6]. *V. Preda*, "On nonlinear-programming and matrix game equivalence", Australian Mathematical Society Series B-Applied Mathematics, **vol. 35**, 1994, pp. 429-438.
[7]. *V. Preda, S. Dedu, M. Sheraz*, "New measure selection for Hunt-Devolder regime switching interest rate models", Physica A, **vol. 413**, Aug. 2014, pp. 350-359.
[8]. *V. Preda, C. Balcau*, "On maxentropic reconstruction of countable Markov chains and matrix scaling problems", Studies in Applied Mathematics, **vol. 111**, 1, Jul. 2003, pp. 85-100.
[9]. *V. Preda, C. Balcau*, "On maxentropic reconstruction of multiple Markov chains", Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie, **vol. 50**, 4, 2007, pp. 295-304.
[10]. *V. Preda*, "On duality with generalized convexity", Bolletino Della Unione Matematica Italiana, **vol. 5A**, 3, Oct. 1991, pp. 291-305.
[11]. *V. Preda*, "Nondifferentiable mathematical programs. Optimality and higher-order duality results", Proceedings of the Romanian Academy Series A – Mathematics Physics Technical Sciences Information Science, **vol. 9**, 3, 2008, pp. 179-183.
[12]. *J.B. Rosen*, "Existence and uniqueness of equilibrium points for concave *n*-person games", Econometrica, **vol. 33**, 3, Jul. 1965, pp. 520-534.
[13]. *C. Zălinescu*, Convex Analysis in General Vector Spaces, World Scientific, Singapore, 2002.