CONSTITUTIVE MATERIAL LAWS IN THE MULTIFRACTAL THEORY OF MOTION (PART II)

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In the present paper, several properties of the stress/strain tensors are presented. More precisely, various operational procedures (algebraic and differentiable procedures, variational principles, harmonic mappings etc.) and their physical implications are highlighted (synchronization group between the structural units of any material structure, various types of stresses fields propagation through functionalities such as period doubling, modulation etc.).

Keywords: multifractal Theory of Motion, Schrödinger scenario, Madelung scenario, multifractal tensor, multifractal constitutive material laws

1. Introduction

In a recent paper [1], using the Fractal Theory of Motion in the Madelung Scenario [2,3], the presence of a permanent interaction between structural units of any complex system and a multifractal medium was shown. Therefore, the

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description of the multifractal medium through a multifractal tensor permits the obtainment of material constitutive laws. Additionally, particular types of material constitutive laws are discussed: strains exist even when no stresses are applied to the material, stresses which can be viewed as intrinsic or pure material properties (for example, Bell's constitutive laws for metallic materials). It is mentioned that the existence of the multifractal medium can be fundamented through the cosmic radiation background and the electromagnetic field in the Maxwell form.

In the present paper, several properties of the stress/strain tensors are presented. More precisely, various operational procedures (algebraic and differentiable procedures, variational principles, harmonic mappings etc.) and their physical implications are highlighted.

2. Algebraic properties induced by the stress/strain tensors

In continuum problems, it is of great importance the so-called quadric equation characteristics of a 3×3 matrix representing stresses or strains. That equation contains all the information related to the spatial distribution of the physical quantity represented by the respective matrix. To describe that distribution, a special reference system is needed any point in space, reference system generated by the eigenvectors of the matrix of considered physical quantity. Because this matrix is usually symmetric, its eigenvectors are mutually perpendicular. If the quantity is defined at any point in space, then any point from space is endowed with such an orthogonal reference system, which thus receives physical meaning through the quantity that the matrix represents. Indeed, in this system of reference there are always three numbers with physical meaning that characterize its origin, and these are the eigenvalues of the matrix. They uniquely characterize that point and, because they are the roots of a cubic equation (the so-called secular equation of the stress tensor), they can be taken in space as generalized elliptical coordinates ([4]).

In such a context, let it be considered the so-called secular equation of the stress tensor, in the so-called binomial form (i.e. the cubic equation):

$$a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3 = 0 \tag{1}$$

and it is assumed that the numbers a_k are real, representing by a_0 the possibility to adjust coefficients to take into account the arbitrariness allowed by the relationships between roots and coefficients. This fact makes the coefficients of equation (1) true homogeneous coordinates in a space three-dimensional, which helps not only to associate a 2×2 matrix to cubic, but also to characterize a specific group of movements. There also exists the possibility to consider the roots of the cubic equation as a special case of vector coordinates, which helps to interpret the group.

Equation (1), according to the mathematical methodology from [5,6] admits the real roots:

$$x_1 = \frac{h + h^* \cdot k}{1 + k},\ x_2 = \frac{h + \varepsilon \cdot h^* \cdot k}{1 + \varepsilon \cdot k},\ x_3 = \frac{h + \varepsilon^2 \cdot h^* \cdot k}{1 + \varepsilon^2 \cdot k} \tag{2}$$

with h, h^* the roots of the Hessian:

$$\Delta = (a_0 a_1 - a_1 a_2)^2 - 4(a_0 a_2 - a_1^2)(a_1 a_3 - a_2{}^2) \tag{3}$$

and k a complex factor of unity modulus and $\varepsilon \equiv (1 + i\sqrt{3})/2$ the cube root of the unit different from the unit itself. It is appropriate to give a physical and geometric interpretation of the pure external factor k, which appears when it is attempted to build the cubic having its Hessian ready to use. Let it be considered the vector of components x_1, x_2, x_3 as before. This vector is in relation to a special group [5] which will be mentioned later, but it represents a real spatial situation, which is revealed when the three roots are proper values of a symmetric matrix. Thus, the eigenvalues of the matrix can be represented by a column matrix:

$$|x\rangle \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{4}$$

Thus, any eigenvalue of a matrix is the component of a vector along the own appropriate direction. It is possible to decompose this vector in relation to the plane that cuts the axes of the reference system at points located at a distance of one unit from the origin. The standard vector is the normal to that plane. It is known as the octahedral plane, since it represents a face of an octahedron in space. The normal component on this plane of vector is given by:

$$\begin{gathered} |x_n\rangle \equiv |n\rangle\langle n \mid x\rangle \\ = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{3}}(1,1,1) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{x_1 + x_2 + x_3}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{gathered} \tag{5}$$

In (5), $|n\rangle$ is the unit vector normal to the plane. The component from the octahedral plane (or tangent) of the vector (4) is then given by

$$|x_1\rangle \equiv |x\rangle - |x_n\rangle = \frac{1}{3}\begin{pmatrix} 2x_1 & -x_2 & -x_3 \\ 2x_2 & -x_3 & -x_1 \\ 2x_3 & -x_1 & -x_2 \end{pmatrix}. \tag{6}$$

This latter vector, in engineering applications, is also known as octahedral shear vector. It allows the interpretation of the complex number k introduced from external considerations earlier – see (2). For this, let it be noted that, by the mathematical methodology from [5,7,8], for the cubic (1), it is possible to give the binomial coefficients in relation to the quantities h, h^* and k, up to an arbitrary factor, through relationships

$$a_0 = 1 + k^3,\ a_1 = -(h + h^* \cdot k^3),$$
$$a_2 = h^2 + h^{*2} \cdot k^3,\ a_3 = -(h^3 + h^{*3} \cdot k^3). \tag{7}$$

From this identification, by direct calculation

$$\frac{1}{3}\sum x_1 = \frac{h + h^* \cdot k^3}{1 + k^3} \therefore |x_t\rangle = \frac{(h - h^*)k}{1 + k^3}\begin{pmatrix} k - 1 \\ \varepsilon(\varepsilon k - 1) \\ \varepsilon^2(\varepsilon^2 k - 1). \end{pmatrix} \tag{8}$$

Let it be taken as reference in the octahedral plane the vector corresponding to $k = 1$, which corresponds to the case when the cubic roots (2) are determined only by the Hessian roots, without any arbitrariness [9-12]. In this case, for the shear part of the vector

$$|x_1^0\rangle = \frac{(h - h^*)}{2}\begin{pmatrix} 0 \\ \varepsilon(\varepsilon - 1) \\ \varepsilon^2(\varepsilon^2 - 1) \end{pmatrix} \tag{9}$$

and it is possible to calculate the angle between this vector and a generic shear vector with $k \neq 1$, by the geometric formula:

$$\cos\ \theta = \frac{\langle x_t^0 \mid x_t\rangle}{\sqrt{\langle x_t^0 \mid x_t^0\rangle\langle x_t \mid x_t\rangle}} \tag{10}$$

Using (8) and (9), the factors of this equation are

$$\langle x_t^0 \mid x_t\rangle = -\frac{3(h - h^*)^2 k}{2(1 + k^3)}(k + 1),$$
$$\langle x_t \mid x_t\rangle = -6k\frac{(h - h^*)^2 k^2}{(1 + k^3)^2},\ \langle x_t^0 \mid x_t^0\rangle = -6\frac{(h - h^*)^2}{2^2} \tag{11}$$

so that (10) becomes

$$\cos\ \alpha \equiv \frac{1}{2}\left(\sqrt{k} + \frac{1}{\sqrt{k}}\right). \tag{12}$$

It is observed that, if the Hessian does not uniquely determine the respective cubic, at least when it has complex roots it underlines a family of cubics, with the roots depending on a single parameter [13-17]. This parameter results from the orientation angle of the octahedral shear vector in the octahedral plane from a given point in space.

The benefit of the above-mentioned method in the characterization of the cubic equation is especially physically. As noted above, in problems of physics and engineering, the issue is about quantities that can be taken as coefficients of the Hessian associated with a cubic (in particular, the cubic (1)), something much more encountered than dealing with the roots of the cubic itself. The general reasoning is that in the case of movements in a continuum it is rare to discern a located cause that can be associated with a force vector as in material point mechanics. Rather, it can be accepted that, at a given point in a continuum, what is done feels right as the cause of a movement is a certain average of influences from all possible directions. If those actions are characterized by a tensor of stress, or those movements through a strain tensor, which can also be observed and measured, they are actually the components of the tensor in a given direction or on a given plane [17-21]. Let it be referred to a certain plane of unity normal $\hat{n}$ through a point of the considered continuum. The component of the tensor $\boldsymbol{\sigma}$ - which can represent either stresses or strains - along the normal of this plane is calculated according to the formula

$$\sigma_n = \sigma_{ij} n_i n_j \tag{13}$$

In the same way, the component of the tensor in this plane will be given by

$$\sigma_t^2 \equiv \sigma^2 - \sigma_n^2 = (\sigma^2)_{ij} n_i n_j - \left(\sigma_{ij} n_i n_j\right)^2 \tag{14}$$

These equations are valid in the idea that the tensor has associated in each direction a vector defined through the relationship

$$\sigma_k = \sigma_{kl} n_l \tag{15}$$

Therefore, a tensor is defined here as a linear application between the unit sphere and the set of vectors in general. Referring to the previously used frame, these equations can be written in the form

$$\begin{aligned} \sigma_n &= x_1 n_1^2 + x_2 n_2^2 + x_3 n_3^2 \\ \sigma_t^2 &= x_1^2 n_1^2 + x_2^2 n_2^2 + x_3^2 n_3^2 - (x_1 n_1^2 + x_2 n_2^2 + x_3 n_3^2)^2 \end{aligned} \tag{16}$$

If now it is written

$$n_1 = \sin\,\theta \cos\,\phi,\ n_2 = \sin\,\theta \sin\,\phi,\ n_3 = \cos\,\theta, \tag{17}$$

where θ and ϕ are the usual spherical angles, it is possible to define the spatial average of some quantity $Q(\boldsymbol{\sigma})$, characteristic of the considered tensor, by the formula

$$\bar{Q}(\boldsymbol{\sigma}) = \frac{1}{\Omega} \iiint Q(\boldsymbol{\sigma}) d\Omega_n \tag{18}$$

where Ω is the solid angle around the given point and $d\Omega_n$ is the elementary solid angle relative to the direction $\hat{n}$, which are given by the relations

$$\Omega = 4\pi,\ d\Omega_n = \sin\,\theta d\theta d\phi \tag{19}$$

Applying this method to the quantities from equation (16) above, it is obtained [5]:

$$\begin{aligned} \sigma_n &= \frac{1}{3}(x_1 + x_2 + x_3) \\ \sigma_t^2 &= \frac{1}{15}[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2] \end{aligned} \tag{20}$$

The first of these expressions is obviously the magnitude of the normal component of the vector (4), and the second is, up to the factor of $1/5$, the square of the magnitude of the octahedral shear vector [21-23]. These quantities are directly related to the roots of the cubic Hessian representing the secular equation of the tensor in question.

3. Differentiable properties induced by the stress/strain tensors

A simple transitive group containing real parameters can be built by making use of the values of h, h^* and k from (2). The fundament of the theory is the notion that the transitive simple group with three real parameters [7,8]

$$x_j \leftrightarrow \frac{ax_j + b}{cx_j + d},\ a, b, c, d \in R \tag{21}$$

where x_j are the cube roots previously mentioned, produces a simple transitive group with real parameters for the complex variables of h, h^* and k:

$$h \leftrightarrow \frac{ah + b}{ch + d},\ h^* \leftrightarrow \frac{ah^* + b}{ch^* + d},\ k \leftrightarrow \frac{ch^* + d}{ch + d} \cdot k \tag{22}$$

The structure of this group is of type SL (2, R) [5,7,8]. In accordance with [5], the generators of SL (2, R) can be determined as components of the Cartan frame, in the form:

$$\begin{aligned} d(f) &\equiv \sum \frac{\partial f}{\partial x^k} dx^k \\ &= \left[\omega^1 \left(h^2 \frac{\partial}{\partial h} + h^{*2} \frac{\partial}{\partial h^*} + (h - h^*)k \frac{\partial}{\partial k}\right)\right. \\ &\left. + 2\omega^2 \left(h \frac{\partial}{\partial h} + h^* \frac{\partial}{\partial h^*}\right) + \omega^3 \left(\frac{\partial}{\partial h} + \frac{\partial}{\partial h^*}\right)\right](f) \end{aligned} \tag{23}$$

The differential 1-forms ω^k, as components of a Cartan coframe, can be determined by the following algebraic system:

$$\begin{aligned} dh &= \omega^1 h^2 + 2\omega^2 h + \omega^3 \\ dh^* &= \omega^1 h^{*2} + 2\omega^2 h^* + \omega^3 \\ dk &= \omega^1 k(h - h^*) \end{aligned} \tag{24}$$

Now, by identification of the right member of (23) through the standard scalar product of the group SL (2, R)

$$\omega^1 B_3 + \omega^3 B_1 - 2\omega^2 B_2 \tag{25}$$

both the infinitesimal generators

$$\begin{aligned} B_1 &= \frac{\partial}{\partial h} + \frac{\partial}{\partial h^*},\ B_2 = h\frac{\partial}{\partial h} + h^*\frac{\partial}{\partial h^*} \\ B_3 &= h^2\frac{\partial}{\partial h} + h^{*2}\frac{\partial}{\partial h^*} + (h-h^*)k\frac{\partial}{\partial k} \end{aligned} \tag{26}$$

and the Cartan coframe

$$\begin{aligned} \omega^1 &= \frac{dk}{(h-h^*)k},\ 2\omega^2 = \frac{dh-dh^*}{h-h^*} - \frac{h+h^*}{h-h^*}\frac{dk}{k}, \\ \omega^3 &= \frac{hdh^*-h^*dh}{h-h^*} + \frac{hh^*}{h-h^*}\frac{dk}{k}. \end{aligned} \tag{27}$$

are obtained.

In real values, given by $h \equiv u + iv, k = e^{\,i\phi}$, (26) and (27) can be expressed as

$$\begin{aligned} B_1 &= \frac{\partial}{\partial u},\ B_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} \\ B_3 &= (u^2 - v^2)\frac{\partial}{\partial u} + 2uv\frac{\partial}{\partial v} + 2v\frac{\partial}{\partial \phi} \\ \omega^1 &= \frac{d\phi}{2v},\ \omega^2 = \frac{dv}{v} - \frac{u}{v}d\phi \\ \omega^3 &= \frac{u^2+v^2}{2v}d\phi + \frac{vdu - udv}{v} \end{aligned} \tag{28}$$

Using the components (27) of the coframe, the invariant metric of SL (2, R) becomes

$$\mathrm{d}s^2 = -\left(d\phi + \frac{du}{v}\right)^2 + \frac{(du)^2 + (dv)^2}{v^2} \tag{29}$$

in which the Beltrami metric is identified. For such a metric, the condition

$$d\phi = -\frac{du}{v} \tag{30}$$

defines the parallelism angle in the Lobachevski plane, i.e. the connection form [23-25].

4. Harmonic mappings induced by the stress/strain tensors

In accordance with the previous results, the problem of the stress field becomes the equivalent to the existence of harmonic mappings. Indeed, let it be considered the variational principle [25-27]:

$$\delta \iiint \left[\frac{\gamma^{mn}\nabla_m h \nabla_n h^*}{(h-h^*)^2}\right](d^3x) = 0 \tag{31}$$

This variational principle corresponds to a harmonic map among the ordinary flat space of the metric γ^{mn} and the complex half-plane containing the Poincaré metric:

$$ds^2 = \frac{dhdh^*}{(h-h^*)^2} = \frac{du^2+dv^2}{v^2} \tag{32}$$

Now, the differential equation associated to the variational principle (31) is:

$$(h-h^*)\nabla^2 h = 2(\nabla h)(\nabla h) \tag{33}$$

and results that:

$$h = -i\frac{\cosh\chi - e^{-i\alpha}\sinh\chi}{\cosh\chi + e^{-i\alpha}\sinh\chi}$$
$$\nabla^2\left(\frac{\chi}{2}\right) = 0 \tag{34}$$

with α being real.

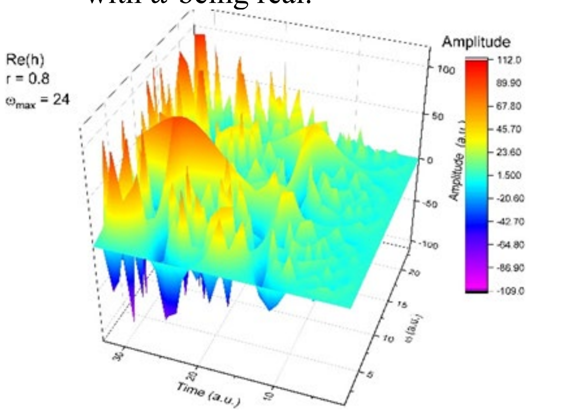

Fig. 1: 3D representation of real (Re(h)) for r=0.8 and $\omega_{max} = 24$

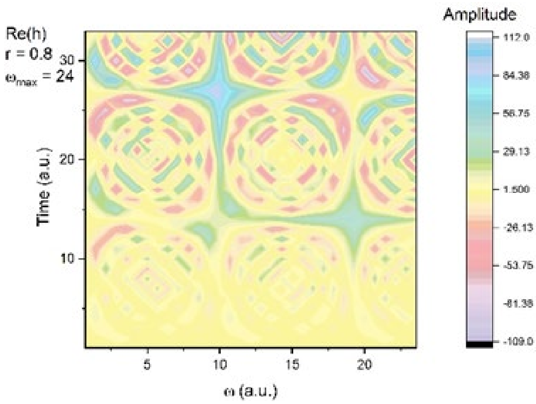

Fig. 2: Contour plot of real (Re(h)) for r=0.8 and $\omega_{max} = 24$

In what follows, the harmonic mapping modes are explained, based both on scale resolution [28] and temporal ordering. In the following figures, the 3D representation and contour plot for the real (Re(h)) and imaginary (Im(h)) part of h are shown, together with the full representation of the signal (Se(h)), where $r = tanh\left(\frac{\chi}{2}\right)$, $\alpha = \omega t$ and ω is the pulsation of motion and t is the time.

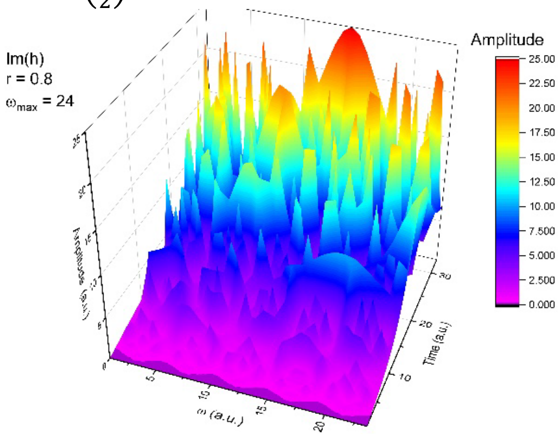

Fig. 3: 3D representation of imaginary (Im(h)) for r=0.8 and $\omega_{max} = 24$

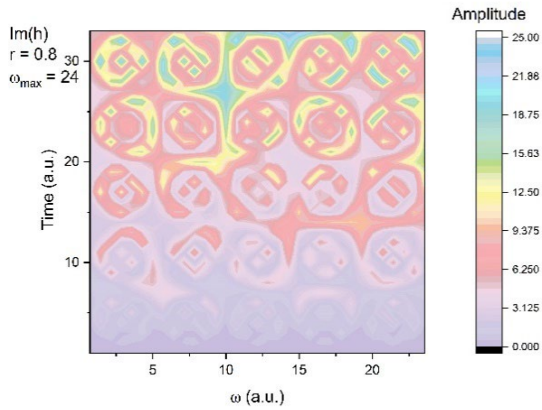

Fig. 4: Contour plot of imaginary (Im(h)) for r=0.8 and $\omega_{max} = 24$

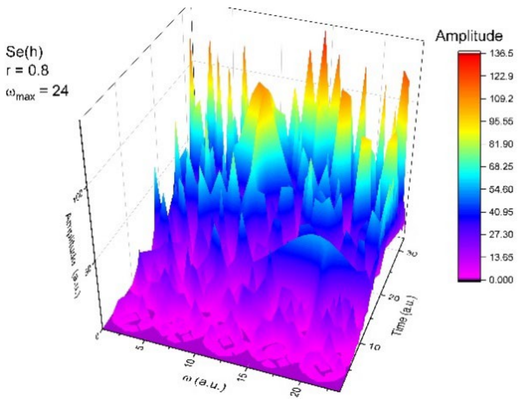

Fig. 5: 3D representation of signal (Se(h)) for r=0.8 and $\omega_{max} = 24$

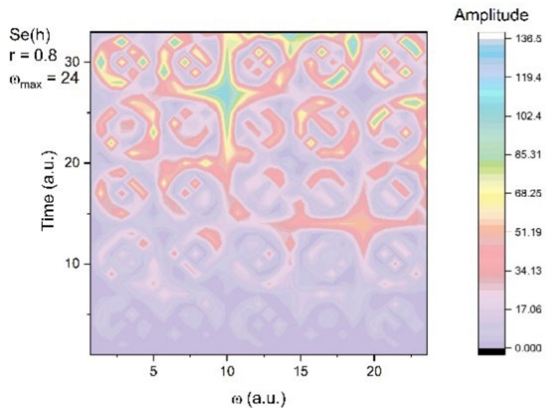

Fig. 6: Contour plot of signal (Se(h)) for r=0.8 and $\omega_{max} = 24$

In Figures 1-6, three separate illustrations of the function which describes the explicit (Re(h)), implicit (Im(h)) and the measurable (Se(h)) factors are shown.

When studying the influence of the external factors, by means of the control parameter r, it can be observed that diverse dynamics can be induced at various scale resolutions (ω_{max}) [29].

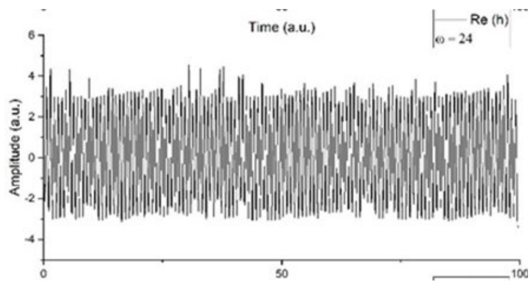

Fig. 7: Time series of Re(h) for ω = 24

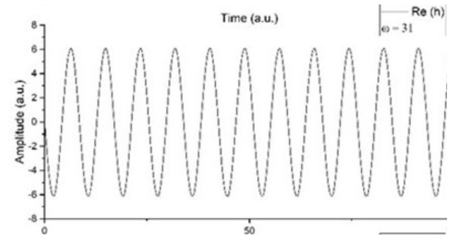

Fig. 8: Time series of Re(h) for ω = 31

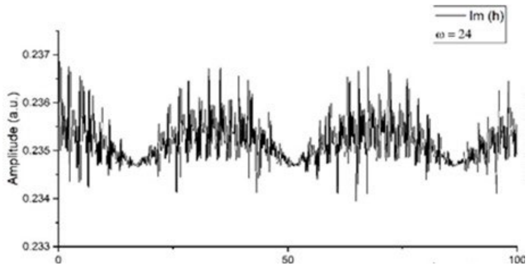

Fig. 9: Time series of Im(h) for ω = 24

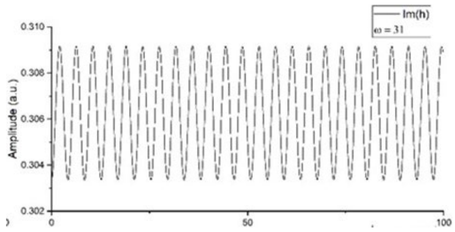

Fig. 10: Time series of Im(h) for ω = 31

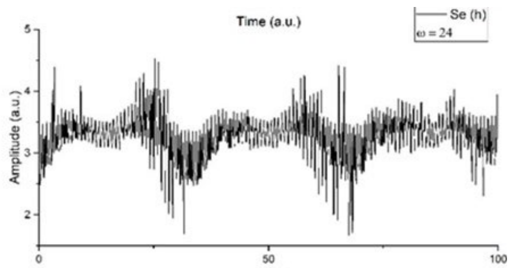

Fig. 11: Time series of Se(h) for ω = 24

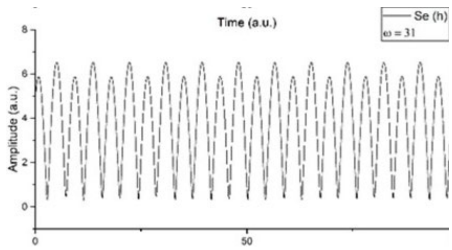

Fig. 12: Time series of Se(h) for ω = 31

In Figures 7-12 the time-series of the three previously mentioned functions are illustrated, for a fixed control parameter, r = 0.8. This showcases a dynamic unperturbed by external factors, for ω = 24 and ω = 31. For example, the period doubling is more of a superposition of Re(h) and Im(h). This can be viewed as a classical oscillatory behavior, where different frequencies are reflected in the structure of the time series. It is also possible to identify quasi-chaotic, non-linear behaviors, for some systems – especially in the imaginary part.

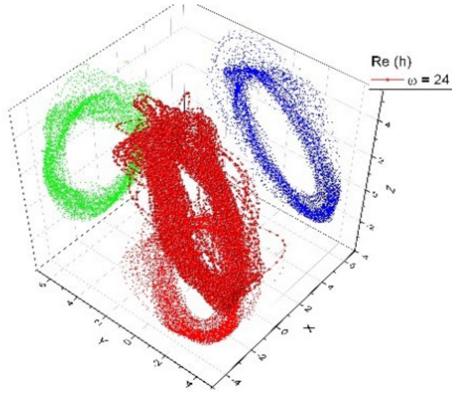

Fig. 13: Reconstruction of the system attractors in the phase space for the Re(h), $\omega = 24$

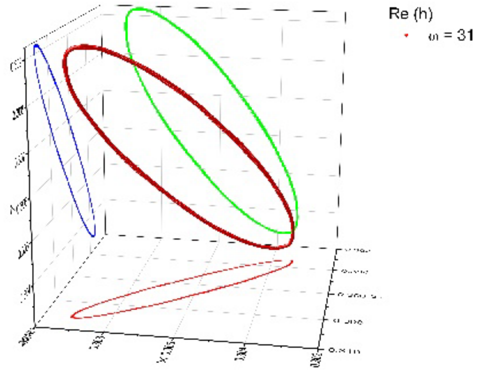

Fig. 14: Reconstruction of the system attractors in the phase space for the Re(h), $\omega = 31$

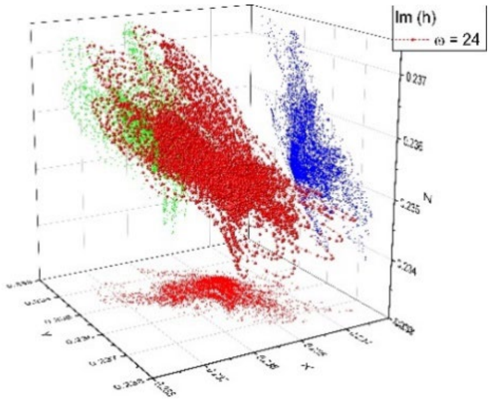

Fig. 15: Reconstruction of the system attractors in the phase space for the Im(h), $\omega = 24$

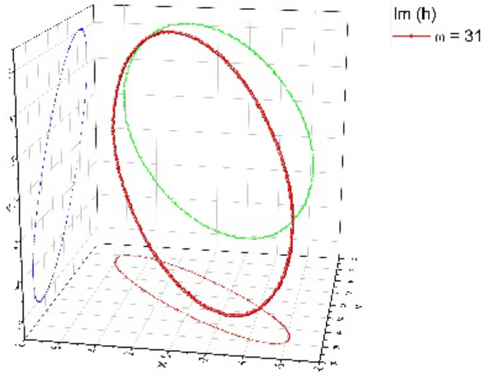

Fig. 16: Reconstruction of the system attractors in the phase space for the Im(h), $\omega = 31$

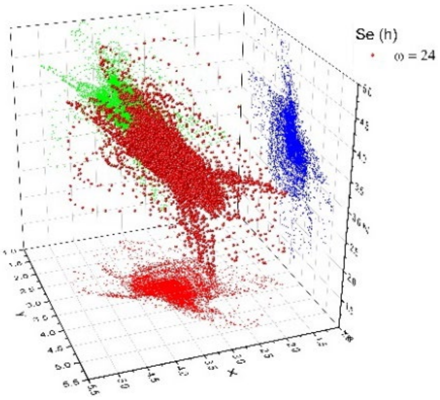

Fig. 17: Reconstruction of the system attractors in the phase space for the Se(h), $\omega = 24$

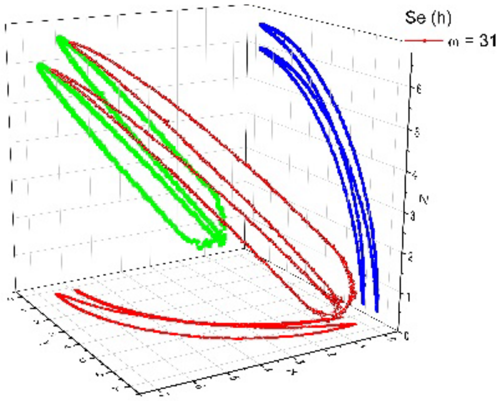

Fig. 18: Reconstruction of the system attractors in the phase space for the Se(h), $\omega = 31$

In order to illustrate the quasi-chaoticity/implicit chaos which can be identified, the attractors for all the time series from Figures 7-12 can be consulted in Figures 13-18. Some attractors appear to have converging trajectories (i.e. towards a certain area), while others appear to have diverging trajectories (i.e. outwards of their representation plane). These non-linear behaviors underline the implicit existence of quasi-chaoticity.

5. Conclusions

The main conclusions of the present paper are the following:

i) Algebraic properties of the stress/strain tensors and their physical implications are presented. In such a context, if actions are characterized by a tensor of stress, or those movements through a strain tensor, which can also be observed and measured, they are actually the components of the tensor in a given direction or on a given plane. Such a situation is explained.

ii) Differentiable properties of the stress/strain tensors and their physical implications are presented. It is shown that the existence of stress/strain cubics induces a particular group of SL (2, R)-type and moreover, a differentiable geometry associated to such a group (in the form of the differential 1-forms, of the differential 2-forms, of parallelism of direction in the Poincaré metrics etc.) becomes functional. In such a conjecture, SL (2, R) can behave as a synchronization group between the structural units of any material structure and various types of stresses fields propagation can be described.

iii) Harmonic mappings properties of the stress/strain tensors and their physical implications are presented. The existence of a variational principle induces a harmonic mapping between the ordinary flat space and the complex half-plane possessing the Poincaré metric, situation in which various functionalities of the stresses fields are represented (period doubling, modulation etc.).

iv) Usually, material constitutive laws are considered to have a semi-empirical character, due to the inherent complexity of the interactions which take place within the materials (i.e. when a material is subjected to mechanical loading). As an example, plastic modelling of unidirectional composite materials can be described using the Drucker-Prager model. Moreover, when discussing metal machining modelling, Johnson-Cook (JC), Modified Johnson-Cook, power law etc. are all models based on material constitutive laws. The present formalism allows the obtainment of various classes of material constitutive laws, which can potentially characterize several types of materials (metallic materials, composite materials, shape memory alloys etc.), as well as machining/processing modelling. Such an approach - i.e. obtaining classes of material constitutive laws - could lead to the increase of mechanical performance of materials. It is noted that the analysis of "non-linearities" (based on harmonic mappings), which occur when a material is subjected to mechanical loading, make possible the explaining of various material constitutive laws. In a future paper, such a constitutive law will be explained for a particular class of materials.

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