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SOME COMPLEMENTARY q-BOUNDS VIA DIFFERENT CLASSES OF CONVEX FUNCTIONS

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The aim of this paper is to obtain some new bounds via different classes of convex functions which involve Riemann type quantum integrals. The results obtained in this paper become natural generalizations of classical results as we obtain classical results when $q \to 1$ where $0 < q < 1$.

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1. Introduction and Preliminaries

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad \forall x, y \in I, t \in [0, 1].$$

Breckner [1] defined s-convexity as follows:

Definition 1.1. *Let $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be s-convex in the second sense if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{1.1}$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

The class of s-convex functions is usually denoted by K_s^2. Note that when $s = 1$ s-convexity means just convexity.
Toader [19] generalized the class of convex functions by defining the class of m-convex functions.

Definition 1.2. *The function $f : [0, b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have:*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y). \tag{1.2}$$

Denote by $K_m(b)$ the set of the m-convex functions on $[0, b]$.

In [9], V. G. Miheşan introduced the notion of (s, m)-convex functions as:

Definition 1.3. *The function $f : [0, b] \to \mathbb{R}$ is said to be (s, m)-convex, where $(s, m) \in (0, 1]^2$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$, we have*

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t^s)f(y). \tag{1.3}$$

For more details on different generalizations of classical convexity, see [3, 4].
We now recall some preliminary concepts from quantum calculus on finite intervals. In [17, 18], J. Tariboon, S.K. Ntouyas, introduced the following concepts:

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Definition 1.4. *Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function and $x \in [a, b]$. Then q-derivative of f on $[a, b]$ at x is defined as*

$$ {}_a\mathrm{D}_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \ x \neq a. \tag{1.4} $$

A function f is q-differentiable on $[a, b]$ if ${}_a\mathrm{D}_q f(x)$ exists for all $x \in [a, b]$. If $a = 0$ in (1.4), we have ${}_0\mathrm{D}_q f(x) = \mathrm{D}_q f$, where D_q is the q-derivative of function f.

Definition 1.5. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. A second- order q-derivative on $[a, b]$, which is denoted as ${}_a\mathrm{D}_q^2 f$, provided ${}_a\mathrm{D}_q f$ is q-differentiable on $[a, b]$, is defined as ${}_a\mathrm{D}_q^2 f = {}_a\mathrm{D}_q({}_a\mathrm{D}_q f) : [a, b] \to \mathbb{R}$. Similarly higher order q-derivative on $[a, b]$ is defined by ${}_a\mathrm{D}_q^n f : [a, b] \to \mathbb{R}$.*

Tariboon et al. [17, 18] defined the q-integral as follows:

Definition 1.6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-integral on I is defined as*

$$ \int_a^x f(t)\mathrm{d}_q^{\mathscr{R}} t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), x \in I. \tag{1.5} $$

In this paper, we use the concepts of quantum calculus on finite intervals to derive some upper bounds for different classes of convex functions. In section 3, we obtain some q-bounds and these results can be viewed as quantum trapezium type inequalities. In section 4, we give some new q-bounds for the approximation of the integral average $\frac{1}{b-a} \int_a^b f(u)\mathrm{d}u$ by the value of $f(x)$ at point $x \in [a, b]$ which involves Riemann type quantum integrals. These results can be treated as quantum Ostrowski type inequalities. For some recent investigations on quantum integral inequalities, see [10, 11, 12, 15]. The classical version of trapezium and Ostrowski inequality respectively reads as:

Theorem 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex functions, then, for all $a, b \in I$ with $a < b$, we have*

$$ f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x \leq \frac{f(a) + f(b)}{2}. $$

Theorem 1.2. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval I, such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leq K$, then, the following inequality,*

$$ \left| f(x) - \frac{1}{b-a} \int_a^b f(u)\mathrm{d}u \right| \leq \frac{K}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right], \tag{1.6} $$

holds.

For more details on these integral inequalities, see [2, 4, 6, 7, 13, 14].

2. **Auxiliary Results**

In order to derive our main results, we need following auxiliary results.

Lemma 2.1 ([10])**.** *Let $f : I = [a, b] \subset\to \mathbb{R}$ be a q-differentiable on $]a, b[$, with ${}_a\mathrm{D}_q$ be continuous and integrable on I, where $0 < q < 1$, then*

$$ \frac{1}{b-a} \int_a^b f(x)\mathrm{d}_q^{\mathscr{R}} x - \frac{qf(a) + f(b)}{1+q} = \frac{q(b-a)}{1+q} \int_0^1 (1 - (1+q)t)\ {}_a\mathrm{D}_q f((1-t)a + tb)\mathrm{d}_q t. $$

Lemma 2.2 ([13]). *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathrm{D}_q$ be continuous and q-integrable on I where $0 < q < 1$, then*

$$\begin{aligned}
&f(x) - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}_q^{\mathscr{R}}u \\
&= \frac{q(x-a)^2}{b-a}\int_0^1 t\,{}_a\mathrm{D}_q f(tx+(1-t)a)\mathrm{d}_q t + \frac{q(b-x)^2}{b-a}\int_0^1 t\,{}_a\mathrm{D}_q f(tx+(1-t)b)\mathrm{d}_q t
\end{aligned}$$

Lemma 2.3 ([5]). *For $t \in [0,1]$, we have*

$$(1-t)^s \le 2^{1-s} - t^s,\ s \in [0,1],\ (1-t)^s \ge 2^{1-s} - t^s,\ s \in [1,\infty).$$

3. Quantum Hermite-Hadamard type inequalities

In this section, we derive some quantum Hermite-Hadamard type inequalities.

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a q-differentiable function on $]a,b[$, such that ${}_a\mathrm{D}_q \in L[a,b]$. If $|{}_a\mathrm{D}_q f|$ is s-convex function, then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}_q^{\mathscr{R}}x - \frac{qf(a)+f(b)}{1+q}\right| \\
&\le \frac{q(b-a)}{1+q}\left(2^{2-s}|{}_a\mathrm{D}_q f(a)| + \left(\frac{1}{[s+1]_q} + \frac{1+q}{[s+2]_q}\right)(|{}_a\mathrm{D}_q f(b)| - |{}_a\mathrm{D}_q f(a)|)\right)
\end{aligned}$$

Proof. Since $|{}_a\mathrm{D}_q f|$ is s-convex function, so from Lemma 2.1, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}_q^{\mathscr{R}}x - \frac{qf(a)+f(b)}{1+q}\right| \\
&= \left|\frac{q(b-a)}{1+q}\int_0^1 (1-(1+q)t)\ {}_a\mathrm{D}_q f((1-t)a+tb)\mathrm{d}_q t\right| \\
&\le \frac{q(b-a)}{1+q}\int_0^1 (1+(1+q)t)|{}_a\mathrm{D}_q f((1-t)a+tb)|\mathrm{d}_q t \\
&\le \frac{q(b-a)}{1+q}\int_0^1 (1+(1+q)t)\left((1-t)^s|{}_a\mathrm{D}_q f(a)| + t^s|{}_a\mathrm{D}_q f(b)|\right)\mathrm{d}_q t.
\end{aligned}$$

Now, using Lemma 2.3, we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}_q^{\mathscr{R}}x - \frac{qf(a)+f(b)}{1+q}\right| \\
&\le \frac{q(b-a)}{1+q}\left(|{}_a\mathrm{D}_q f(a)|\int_0^1 (1+(1+q)t)(2^{1-s}-t^s)\mathrm{d}_q t \right. \\
&\qquad \left. + \int_0^1 (1+(1+q)t)t^s|{}_a\mathrm{D}_q f(b)|\mathrm{d}_q t\right) \\
&= \frac{q(b-a)}{1+q}\left(2^{2-s}|{}_a\mathrm{D}_q f(a)| + \left(\frac{1}{[s+1]_q} + \frac{1+q}{[s+2]_q}\right)(|{}_a\mathrm{D}_q f(b)| - |{}_a\mathrm{D}_q f(a)|)\right)
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on $]a,b[$ with ${}_a\mathrm{D}_q \in L[a,b]$. If $|{}_a\mathrm{D}_q f|^r$ is s-convex function where $r > 1$, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}_q^{\mathscr{R}}x - \frac{qf(a)+f(b)}{1+q}\right| \\ &\le \frac{q(b-a)}{1+q}\left(\frac{q}{1+q}\right)^{\frac{r}{r-1}}\left(2^{2-s}|{}_a\mathrm{D}_q f(a)|^r + \left(\frac{1}{[s+1]_q}\right.\right. \\ &\qquad\left.\left. + \frac{1+q}{[s+2]_q}\right)(|{}_a\mathrm{D}_q f(b)|^r - |{}_a\mathrm{D}_q f(a)|^r)\right)^{\frac{1}{r}}.\end{aligned}$$

Proof. Since $|{}_a\mathrm{D}_q f|^r$ is s-convex function and by Lemma 2.1, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}_q^{\mathscr{R}}x - \frac{qf(a)+f(b)}{1+q}\right| = \left|\frac{q(b-a)}{1+q}\int_0^1 (1-(1+q)t)\ {}_a\mathrm{D}_q f((1-t)a+tb)\mathrm{d}_q t\right| \\ &\le \left|\frac{q(b-a)}{1+q}\int_0^1 (1+(1+q)t)^{1-\frac{1}{r}}(1+(1+q)t)^{\frac{1}{r}}\ {}_a\mathrm{D}_q f((1-t)a+tb)\mathrm{d}_q t\right|\end{aligned}$$

Using q-Hölder's inequality and by Lemma 2.3, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}_q^{\mathscr{R}}x - \frac{qf(a)+f(b)}{1+q}\right| \le \frac{q(b-a)}{1+q}\left(\int_0^1 (1+(1+q)t)\mathrm{d}_q t\right)^{\frac{r}{r-1}} \\ &\times\left(\int_0^1 (1+(1+q)t)|{}_a\mathrm{D}_q f((1-t)a+tb)|^r\mathrm{d}_q t\right)^{\frac{1}{r}} \le \frac{q(b-a)}{1+q}\left(\frac{q}{1+q}\right)^{\frac{r}{r-1}} \\ &\times\left(\int_0^1 (1+(1+q)t)\left[(1-t)^s|{}_a\mathrm{D}_q f(a)|^r + t^s|{}_a\mathrm{D}_q f(b)|^r\right]\mathrm{d}_q t\right)^{\frac{1}{r}} \le \frac{q(b-a)}{1+q}\left(\frac{q}{1+q}\right)^{\frac{r}{r-1}} \\ &\times\left(\int_0^1 (1+(1+q)t)\left[(2^{1-s}-t^s)|{}_a\mathrm{D}_q f(a)|^r + t^s|{}_a\mathrm{D}_q f(b)|^r\right]\mathrm{d}_q t\right)^{\frac{1}{r}} = \frac{q(b-a)}{1+q}\left(\frac{q}{1+q}\right)^{\frac{r}{r-1}} \\ &\times\left(2^{2-s}|{}_a\mathrm{D}_q f(a)|^r + \left(\frac{1}{[s+1]_q} + \frac{1+q}{[s+2]_q}\right)(|{}_a\mathrm{D}_q f(b)|^r - |{}_a\mathrm{D}_q f(a)|^r)\right)^{\frac{1}{r}}.\end{aligned}$$

The result is thus proved. □

Theorem 3.3. *Let $f : [a,b] \to \mathbb{R}$ be a q-differentiable function on $]a,b[$ with ${}_a\mathrm{D}_q \in L[a,b]$. If $|{}_a\mathrm{D}_q f|$ is m-convex function, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}_q^{\mathscr{R}}x - \frac{qf(a)+f(b)}{1+q}\right| \\ &\le \frac{q(b-a)}{1+q}\left(\left(2 - \frac{1}{1+q} - \frac{1+q}{1+q+q^2}\right)|{}_a\mathrm{D}_q f(a)|\right. \\ &\qquad\left. + m\left(\frac{1}{1+q} + \frac{1+q}{1+q+q^2}\right)|{}_a\mathrm{D}_q f(\frac{b}{m})|\right).\end{aligned}$$

Proof. Since $|{}_a\mathrm{D}_q f|$ is m-convex function and using Lemma 2.1, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}_q^{\mathscr{R}}x - \frac{qf(a)+f(b)}{1+q}\right| \\
&= \left|\frac{q(b-a)}{1+q}\int_0^1 (1-(1+q)t)\ {}_a\mathrm{D}_q f((1-t)a+tb)\mathrm{d}_q t\right| \\
&\leq \frac{q(b-a)}{1+q}\int_0^1 |(1-(1+q)t)||{}_a\mathrm{D}_q f((1-t)a+tb)|\mathrm{d}_q t \\
&\leq \frac{q(b-a)}{1+q}\int_0^1 (1+(1+q)t)|{}_a\mathrm{D}_q f((1-t)a+m\frac{t}{m}b)|\mathrm{d}_q t \\
&\leq \frac{q(b-a)}{1+q}\int_0^1 (1+(1+q)t)\left((1-t)|{}_a\mathrm{D}_q f(a)|+mt|{}_a\mathrm{D}_q f(\frac{b}{m})|\right)\mathrm{d}_q t \\
&= \frac{q(b-a)}{1+q}\left(\left(2-\frac{1}{1+q}-\frac{1+q}{1+q+q^2}\right)|{}_a\mathrm{D}_q f(a)|\right. \\
&\qquad\qquad \left. +m\left(\frac{1}{1+q}+\frac{1+q}{1+q+q^2}\right)|{}_a\mathrm{D}_q f(\frac{b}{m})|\right).
\end{aligned}$$

The proof is completed. □

Theorem 3.4. *Let $f:[a,b]\to\mathbb{R}$ be a q-differentiable function on $]a,b[$ with ${}_a\mathrm{D}_q\in L[a,b]$. If $|{}_a\mathrm{D}_q f|$ is (s,m)-convex function, then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}_q^{\mathscr{R}}x - \frac{qf(a)+f(b)}{1+q}\right| \\
\leq\ & \frac{q(b-a)}{1+q}\left(2^{2-s}-\frac{1}{[s+1]_q}-\frac{1+q}{[s+2]_q}\right)|{}_a\mathrm{D}_q f(a)| \\
+\ & \frac{q(b-a)}{1+q}m\left(2+2^{2-s}-\frac{1}{[s+1]_q}-\frac{1+q}{[s+2]_q}\right)|{}_a\mathrm{D}_q f\left(\frac{b}{m}\right)|
\end{aligned}$$

Proof. Since $|{}_a\mathrm{D}_q f|$ is (s,m)-convex function and using Lemma 2.1, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}_q^{\mathscr{R}}x - \frac{qf(a)+f(b)}{1+q}\right| \\
&= \left|\frac{q(b-a)}{1+q}\int_0^1 (1-(1+q)t)\ {}_a\mathrm{D}_q f\left((1-t)a+m(1-(1-t))\frac{b}{m}\right)\mathrm{d}_q t\right| \\
&\leq \frac{q(b-a)}{1+q}\int_0^1 |1-(1+q)t|\ \left((1-t)^s|{}_a\mathrm{D}_q f(a)|\right. \\
&\qquad\qquad \left. +m(1-(1-t)^s)\left|{}_a\mathrm{D}_q f\left(\frac{b}{m}\right)\right|\right)\mathrm{d}_q t \\
&\leq \frac{q(b-a)}{1+q}\int_0^1 (1+(1+q)t)(1-t)^s|{}_a\mathrm{D}_q f(a)|\mathrm{d}_q t \\
&+ \frac{q(b-a)}{1+q}\int_0^1 m|(1-(1+q)t)(1-(1-t)^s)\left|{}_a\mathrm{D}_q f\left(\frac{b}{m}\right)\right|\mathrm{d}_q t.
\end{aligned}$$

By Lemma 2.3, we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}_q^{\mathscr{R}}x - \frac{qf(a)+f(b)}{1+q}\right| \\
&\leq \frac{q(b-a)}{1+q}\int_0^1 (1+(1+q)t)(2^{1-s}-t^s)|_a\mathrm{D}_q f(a)|\mathrm{d}_q t \\
&+ \frac{q(b-a)}{1+q}m\left|_a\mathrm{D}_q f\left(\frac{b}{m}\right)\right|\int_0^1 \left(1+2^{1-s}-t^s+(1+q)t+(1+q)t(2^{1-s}-t^s)\right)\mathrm{d}_q t \\
&= \frac{q(b-a)}{1+q}\left(2^{2-s}-\frac{1}{[s+1]_q}-\frac{1+q}{[s+2]_q}\right)|_a\mathrm{D}_q f(a)| \\
&+ \frac{q(b-a)}{1+q}m\left(2+2^{2-s}-\frac{1}{[s+1]_q}-\frac{1+q}{[s+2]_q}\right)\left|_a\mathrm{D}_q f\left(\frac{b}{m}\right)\right|
\end{aligned}$$

This completes the proof. □

4. **Quantum Ostrowski type inequalities**

In this section, we derive some quantum Ostrowski type inequalities.

Theorem 4.1. *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathrm{D}_q$ be continuous and q-integrable on I where $0 < q < 1$. If $|_a\mathrm{D}_q|$ is s-convex function and $|\,_a\mathrm{D}_q f(.)| \leq K$, then, for $s \in [0, 1]$, we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}_q^{\mathscr{R}}u\right| \leq \frac{q2^{1-s}K[(x-a)^2+(b-x)^2]}{(b-a)1+q}.$$

Proof. Since it is given that $|_a\mathrm{D}_q f|$ is s-convex function, so from Lemma 2.2 and using the property of modulus, we have

$$\begin{aligned}
&\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}_q^{\mathscr{R}}u\right| = \left|\frac{q(x-a)^2}{b-a}\int_0^1 t_a\mathrm{D}_q f(tx+(1-t)a)\mathrm{d}_q t\right. \\
&\left.+ \frac{q(b-x)^2}{b-a}\int_0^1 t_a\mathrm{D}_q f(tx+(1-t)b)\mathrm{d}_q t\right| \\
&\leq \frac{q(x-a)^2}{b-a}\int_0^1 t|\,_a\mathrm{D}_q f(tx+(1-t)a)|\mathrm{d}_q t + \frac{q(b-x)^2}{b-a}\int_0^1 t|\,_a\mathrm{D}_q f(tx+(1-t)b)|\mathrm{d}_q t \\
&\leq \frac{q(x-a)^2}{b-a}\int_0^1 t[t^s|\,_a\mathrm{D}_q f(x)| + (1-t)^s|\,_a\mathrm{D}_q f(a)|]\mathrm{d}_q t \\
&\qquad + \frac{q(b-x)^2}{b-a}\int_0^1 t[t^s|\,_a\mathrm{D}_q f(x)| + (1-t)^s|\,_a\mathrm{D}_q f(b)|]\mathrm{d}_q t \\
&\leq \frac{q2^{1-s}K[(x-a)^2+(b-x)^2]}{(b-a)(1+q)}.
\end{aligned}$$

This completes the proof. □

Theorem 4.2. *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathrm{D}_q$ be continuous and q-integrable on I where $0 < q < 1$. If $|{}_a\mathrm{D}_q f|^r$ is s-convex function and $|{}_a\mathrm{D}_q f(x)| \le K$, then for $s \in [0, 1]$, $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, we have*

$$\left| f(x) - \frac{1}{b-a} \int\limits_a^b f(u) \mathrm{d}_q^{\mathscr{R}} u \right| \le \frac{q 2^{\frac{1-s}{r}} K[(x-a)^2 + (b-x)^2]}{b-a} \Big(\frac{1-q}{1-q^{p+1}} \Big)^{\frac{1}{p}}.$$

Proof. Using Lemma 2.2, Hölder's inequality and the given hypothesis that $|{}_a\mathrm{D}_q f|^r$ is s-convex function, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a} \int\limits_a^b f(u) \mathrm{d}_q^{\mathscr{R}} u \right| \\
&= \left| \frac{q(x-a)^2}{b-a} \int\limits_0^1 t \, {}_a\mathrm{D}_q f(tx + (1-t)a) \mathrm{d}_q t + \frac{q(b-x)^2}{b-a} \int\limits_0^1 t \, {}_a\mathrm{D}_q f(tx + (1-t)b) \mathrm{d}_q t \right| \\
&\le \frac{q(x-a)^2}{b-a} \Big(\int\limits_0^1 t^p \mathrm{d}_q t \Big)^{\frac{1}{p}} \Big(\int\limits_0^1 |{}_a\mathrm{D}_q f(tx + (1-t)a)|^r \mathrm{d}_q t \Big)^{\frac{1}{r}} \\
&\qquad + \frac{q(b-x)^2}{b-a} \Big(\int\limits_0^1 t^p \mathrm{d}_q t \Big)^{\frac{1}{p}} \Big(\int\limits_0^1 |{}_a\mathrm{D}_q f(tx + (1-t)b)|^r \mathrm{d}_q t \Big)^{\frac{1}{r}} \\
&\le \frac{q(x-a)^2}{b-a} \Big(\frac{1-q}{1-q^{p+1}} \Big)^{\frac{1}{p}} \Big(\int\limits_0^1 [t^s |{}_a\mathrm{D}_q f(x)|^r + (1-t)^s |{}_a\mathrm{D}_q f(a)|^r] \mathrm{d}_q t \Big)^{\frac{1}{r}} \\
&\qquad + \frac{q(b-x)^2}{b-a} \Big(\frac{1-q}{1-q^{p+1}} \Big)^{\frac{1}{p}} \Big(\int\limits_0^1 [t^s |{}_a\mathrm{D}_q f(x)|^r + (1-t)^s |{}_a\mathrm{D}_q f(b)|^r] \mathrm{d}_q t \Big)^{\frac{1}{r}} \\
&\le \frac{q 2^{\frac{1-s}{r}} K[(x-a)^2 + (b-x)^2]}{b-a} \Big(\frac{1-q}{1-q^{p+1}} \Big)^{\frac{1}{p}}.
\end{aligned}$$

This completes the proof. □

Theorem 4.3. *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathrm{D}_q f$ be continuous and q-integrable on I where $0 < q < 1$. If $|{}_a\mathrm{D}_q f|^r$ is s-convex function and $|{}_a\mathrm{D}_q f(x)| \le K$, then, for $r \ge 1$, we have*

$$\left| f(x) - \frac{1}{b-a} \int\limits_a^b f(u) \mathrm{d}_q^{\mathscr{R}} u \right| \le \frac{qK[(x-a)^2 + (b-x)^2]}{b-a} \Xi(s; r; q),$$

where

$$\Xi(s; r; q) = \Big(\frac{1}{1+q} \Big)^{1-\frac{1}{r}} 2^{\frac{1-s}{r}}.$$

Proof. Using Lemma 2.2, power means inequality and hypothesis that $|{}_a\mathrm{D}_q f|^r$ is s-convex function, we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}_q^{\mathscr{R}}u\right|\\
&=\left|\frac{q(x-a)^2}{b-a}\int_0^1 t\,{}_a\mathrm{D}_q f(tx+(1-t)a)_0 d_q t+\frac{q(b-x)^2}{b-a}\int_0^1 t\,{}_a\mathrm{D}_q f(tx+(1-t)b)\mathrm{d}_q t\right|\\
&\leq \frac{q(x-a)^2}{b-a}\Big(\int_0^1 t\mathrm{d}_q t\Big)^{1-\frac{1}{r}}\Big(\int_0^1 |{}_a\mathrm{D}_q f(tx+(1-t)a)|^r \mathrm{d}_q t\Big)^{\frac{1}{r}}\\
&\quad+\frac{q(b-x)^2}{b-a}\Big(\int_0^1 t\mathrm{d}_q t\Big)^{1-\frac{1}{r}}\Big(\int_0^1 |{}_a\mathrm{D}_q f(tx+(1-t)b)|^r \mathrm{d}_q t\Big)^{\frac{1}{r}}\\
&\leq \frac{q(x-a)^2}{b-a}\Big(\frac{1}{1+q}\Big)^{1-\frac{1}{r}}\Big(\int_0^1 [t^s|{}_a\mathrm{D}_q f(x)|^r+(1-t)^s|{}_a\mathrm{D}_q f(a)|^r]\mathrm{d}_q t\Big)^{\frac{1}{r}}\\
&\quad+\frac{q(b-x)^2}{b-a}\Big(\frac{1}{1+q}\Big)^{1-\frac{1}{r}}\Big(\int_0^1 [t^s|{}_a\mathrm{D}_q f(x)|^r+(1-t)^s|{}_a\mathrm{D}_q f(b)|^r]\mathrm{d}_q t\Big)^{\frac{1}{r}}\\
&\leq \frac{qK2^{\frac{1-s}{r}}[(x-a)^2+(b-x)^2]}{b-a}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}.
\end{aligned}$$

This completes the proof. □

Theorem 4.4. *Let $f: I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathrm{D}_q f$ be continuous and q-integrable on I where $0<q<1$. If $|{}_a\mathrm{D}_q|$ is (s,m)-convex function, then, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}_q^{\mathscr{R}}u\right|\leq \min\{\Delta_1(a,b;x;m;s;q),\Delta_2(a,b;x;m;s;q)\},$$

where

$$\begin{aligned}
\Delta_1(a,b;x;m;s;q)&=\frac{q(x-a)^2}{b-a}\left\{\varphi_1(s;q)\,|{}_a\mathrm{D}_q f(x)|+m\varphi_2(s;q)\left|{}_a\mathrm{D}_q f\Big(\frac{a}{m}\Big)\right|\right\}\\
&\quad+\frac{q(b-x)^2}{b-a}\left\{\varphi_1(s;q)\,|{}_a\mathrm{D}_q f(x)|+m\varphi_2(s;q)\left|{}_a\mathrm{D}_q f\Big(\frac{b}{m}\Big)\right|\right\}\\
\Delta_2(a,b;x;m;s;q)&=\frac{q(x-a)^2}{b-a}\left\{m\varphi_1(s;q)\left|{}_a\mathrm{D}_q f\Big(\frac{x}{m}\Big)\right|+\varphi_2(s;q)|{}_a\mathrm{D}_q f(a)|\right\}\\
&\quad+\frac{q(b-x)^2}{b-a}\left\{m\varphi_1(s;q)\left|{}_a\mathrm{D}_q f\Big(\frac{x}{m}\Big)\right|+\varphi_2(s;q)|{}_a\mathrm{D}_q f(b)|\right\}
\end{aligned}$$

$$\varphi_1(s;q)=\frac{1}{[s+2]_q},\tag{4.1}$$

and

$$\varphi_2(s;q)=\frac{1}{1+q}-\frac{1}{[s+2]_q},\tag{4.2}$$

respectively.

Proof. Since it is given that $|{}_a\mathrm{D}_q f|$ is (s,m)-convex function, so from Lemma 2.2 and using the property of modulus, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}_q^{\mathscr{R}}u\right|$$

$$\begin{aligned}
&=\left|\frac{q(x-a)^2}{b-a}\int_0^1 t\,{}_a\mathrm{D}_q f(tx+(1-t)a)\mathrm{d}_q t+\frac{q(b-x)^2}{b-a}\int_0^1 t\,{}_a\mathrm{D}_q f(tx+(1-t)b)\mathrm{d}_q t\right|\\
&\le \frac{q(x-a)^2}{b-a}\int_0^1 t|\,{}_a\mathrm{D}_q f\,(tx+(1-t)a)\,|\mathrm{d}_q t+\frac{q(b-x)^2}{b-a}\int_0^1 t|\,{}_a\mathrm{D}_q f(tx+(1-t)b)|\mathrm{d}_q t\\
&\le \frac{q(x-a)^2}{b-a}\int_0^1 t\left[t^s|\,{}_a\mathrm{D}_q f(x)|+m(1-t^s)\left|{}_a\mathrm{D}_q f\Big(\frac{a}{m}\Big)\right|\right]\mathrm{d}_q t\\
&\qquad+\frac{q(b-x)^2}{b-a}\int_0^1 t\left[t^s|\,{}_a\mathrm{D}_q f(x)|+m(1-t^s)\left|{}_a\mathrm{D}_q f\Big(\frac{b}{m}\Big)\right|\right]\mathrm{d}_q t\\
&=\frac{q(x-a)^2}{b-a}\left\{\frac{1}{[s+2]_q}|{}_a\mathrm{D}_q f(x)|+m\left(\frac{1}{1+q}-\frac{1}{[s+2]_q}\right)\left|{}_a\mathrm{D}_q f\Big(\frac{a}{m}\Big)\right|\right\}\\
&\quad+\frac{q(b-x)^2}{b-a}\left\{\frac{1}{[s+2]_q}|{}_a\mathrm{D}_q f(x)|+m\left(\frac{1}{1+q}-\frac{1}{[s+2]_q}\right)\left|{}_a\mathrm{D}_q f\Big(\frac{b}{m}\Big)\right|\right\}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}_q^{\mathscr{R}}u\right|&\le\frac{q(x-a)^2}{b-a}\left\{\frac{m}{[s+2]_q}\left|{}_a\mathrm{D}_q f\Big(\frac{x}{m}\Big)\right|+\Big(\frac{1}{1+q}\right.\\
&\left.-\frac{1}{[s+2]_q}\Big)|\,{}_a\mathrm{D}_q f(a)|\right\}+\frac{q(b-x)^2}{b-a}\left\{\frac{m}{[s+2]_q}\left|{}_a\mathrm{D}_q f\Big(\frac{x}{m}\Big)\right|+\left(\frac{1}{1+q}-\frac{1}{[s+2]_q}\right)|\,{}_a\mathrm{D}_q f(b)|\right\}.
\end{aligned}$$

This completes the proof. $\square$

Theorem 4.5. *Let $f: I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathrm{D}_q f$ be continuous and q-integrable on I where $0<q<1$. If $|{}_a\mathrm{D}_q f|^r$ is (s,m)-convex function, then for, $p,r>1$, $\frac{1}{p}+\frac{1}{r}=1$, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}_q^{\mathscr{R}}u\right|\le\min\{\Theta_1(a,b;x;m;s;q),\Theta_2(a,b;x;m;s;q)\},$$

where

$$\begin{aligned}
\Theta_1(a,b;x;m;s;q)&=\frac{q}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\\
&\quad\times\left[(x-a)^2\left(\vartheta_1(q)|{}_a\mathrm{D}_q f(x)|^r+m\vartheta_2(q)\left|{}_a\mathrm{D}_q f\Big(\frac{a}{m}\Big)\right|^r\right)^{\frac{1}{r}}\right.\\
&\quad\left.+(b-x)^2\left(\left[\vartheta_1(q)|{}_a\mathrm{D}_q f(x)|^r+m\vartheta_2(q)\left|{}_a\mathrm{D}_q f\Big(\frac{b}{m}\Big)\right|^r\right]\right)^{\frac{1}{r}}\right],
\end{aligned}$$

$$\Theta_2(a,b;x;m;s;q) = \frac{q}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}$$
$$\times \Bigg[(x-a)^2\left(m\vartheta_1(q)\left|{}_a\mathrm{D}_q f\Big(\frac{x}{m}\Big)\right|^r + \vartheta_2(q)|{}_a\mathrm{D}_q f(a)|^r\right)^{\frac{1}{r}}$$
$$+(b-x)^2\left(\left[m\vartheta_1(q)\left|{}_a\mathrm{D}_q f\Big(\frac{x}{m}\Big)\right|^r + \vartheta_2(q)|{}_a\mathrm{D}_q f(b)|^r\right]\right)^{\frac{1}{r}}\Bigg],$$

$$\vartheta_1(s;q) = \frac{1}{[s+1]_q}, \text{ and } \vartheta_2(s;q) = 1 - \frac{1}{[s+1]_q}.$$

Proof. Using Lemma 2.2, Hölder's inequality and the given hypothesis that $|{}_a\mathrm{D}_q f|^r$ is (s,m)-convex function, we have

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}_q^{\mathscr{R}}u\right| = \left|\frac{q(x-a)^2}{b-a}\int_0^1 t\,{}_a\mathrm{D}_q f(tx+(1-t)a)\mathrm{d}_q t\right.$$
$$\left. + \frac{q(b-x)^2}{b-a}\int_0^1 t\,{}_a\mathrm{D}_q f(tx+(1-t)b)\mathrm{d}_q t\right| \le \frac{q(x-a)^2}{b-a}\Big(\int_0^1 t^p\mathrm{d}_q t\Big)^{\frac{1}{p}}\Big(\int_0^1 |{}_a\mathrm{D}_q f(tx+(1-t)a)|^r\mathrm{d}_q t\Big)^{\frac{1}{r}}$$
$$+ \frac{q(b-x)^2}{b-a}\Big(\int_0^1 t^p\mathrm{d}_q t\Big)^{\frac{1}{p}}\Big(\int_0^1 |{}_a\mathrm{D}_q f(tx+(1-t)b)|^r\mathrm{d}_q t\Big)^{\frac{1}{r}}$$

$$\le \frac{q(x-a)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\left(\int_0^1\left[t^s|{}_a\mathrm{D}_q f(x)|^r + m(1-t^s)\left|{}_a\mathrm{D}_q f\Big(\frac{a}{m}\Big)\right|^r\right]\mathrm{d}_q t\right)^{\frac{1}{r}}$$
$$+ \frac{q(b-x)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\left(\int_0^1\left[t^s|{}_a\mathrm{D}_q f(x)|^r + m(1-t^s)\left|{}_a\mathrm{D}_q f\Big(\frac{b}{m}\Big)\right|^r\right]\mathrm{d}_q t\right)^{\frac{1}{r}}$$
$$\le \frac{q(x-a)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}} \times \left(\left(\frac{1}{[s+1]_q}\right)|{}_a\mathrm{D}_q f(x)|^r + m\left(1-\frac{1}{[s+1]_q}\right)\left|{}_a\mathrm{D}_q f\Big(\frac{a}{m}\Big)\right|^r\right)^{\frac{1}{r}}$$
$$+ \frac{q(b-x)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}} \times \left(\left[\left(\frac{1}{[s+1]_q}\right)|{}_a\mathrm{D}_q f(x)|^r + m\left(1-\frac{1}{[s+1]_q}\right)\left|{}_a\mathrm{D}_q f\Big(\frac{b}{m}\Big)\right|^r\right]\right)^{\frac{1}{r}}.$$

Similarly

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}_q^{\mathscr{R}}u\right| \le \frac{q(x-a)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}$$
$$\times\left(\left(\frac{m}{[s+1]_q}\right)\left|{}_a\mathrm{D}_q f\Big(\frac{x}{m}\Big)\right|^r + \left(1-\frac{1}{[s+1]_q}\right)|{}_a\mathrm{D}_q f(a)|^r\right)^{\frac{1}{r}}$$
$$+ \frac{q(b-x)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}} \times \left(\left[\left(\frac{m}{[s+1]_q}\right)\left|{}_a\mathrm{D}_q f\Big(\frac{x}{m}\Big)\right|^r + \left(1-\frac{1}{[s+1]_q}\right)|{}_a\mathrm{D}_q f(b)|^r\right]\right)^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 4.6. *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathrm{D}_q f$ be continuous and q-integrable on I where $0 < q < 1$. If $|{}_a\mathrm{D}_q f|^r$ is (s, m)-convex function, then, for $r \geq 1$, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}_q^{\mathscr{R}} u \right| \leq \min\{\Omega_1(a, b; x; m; s; q), \Omega_2(a, b; x; m; s; q)\},$$

where

$$\Omega_1(a, b; x; m; s; q) = \frac{q}{b-a}\Big(\frac{1}{1+q}\Big)^{1-\frac{1}{r}} \times \Bigg[(x-a)^2\Bigg(\varphi_1(s; q)|{}_a\mathrm{D}_q f(x)|^r + m\varphi_2(s; q)\Big|{}_a\mathrm{D}_q f\Big(\frac{a}{m}\Big)\Big|^r\Bigg)^{\frac{1}{r}}$$
$$+ (b-x)^2\Bigg(\varphi_1(s; q)|{}_a\mathrm{D}_q f(x)|^r + m\varphi_2(s; q)\Big|{}_a\mathrm{D}_q f\Big(\frac{b}{m}\Big)\Big|^r\Bigg)^{\frac{1}{r}}\Bigg],$$

$$\Omega_2(a, b; x; m; s; q) = \frac{q}{b-a}\Big(\frac{1}{1+q}\Big)^{1-\frac{1}{r}} \times \Big[(x-a)^2\Big(m\varphi_1(s; q)\Big|{}_a\mathrm{D}_q f\Big(\frac{x}{m}\Big)\Big|^r$$
$$+ \varphi_2(s; q)|{}_a\mathrm{D}_q f(a)|^r\Big)^{\frac{1}{r}} + (b-x)^2\Big(m\varphi_1(s; q)\Big|{}_a\mathrm{D}_q f\Big(\frac{x}{m}\Big)\Big|^r + \varphi_2(s; q)\Big|{}_a\mathrm{D}_q f(b)\Big|^r\Big)^{\frac{1}{r}}\Big],$$

and $\varphi_1(s; q)$, $\varphi_2(s; q)$ are given by (4.1) and (4.2) respectively.

Proof. Using Lemma 2.2, power means inequality and hypothesis that $|{}_a\mathrm{D}_q f|^r$ is s-convex function, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}_q^{\mathscr{R}} u \right| \\
&= \left| \frac{q(x-a)^2}{b-a}\int_0^1 t\,{}_a\mathrm{D}_q f(tx + (1-t)a)_0 d_q t + \frac{q(b-x)^2}{b-a}\int_0^1 t\,{}_a\mathrm{D}_q f(tx + (1-t)b)\mathrm{d}_q t \right| \\
&\leq \frac{q(x-a)^2}{b-a}\Big(\int_0^1 t\mathrm{d}_q t\Big)^{1-\frac{1}{r}}\Big(\int_0^1 |{}_a\mathrm{D}_q f(tx + (1-t)a)|^r \mathrm{d}_q t\Big)^{\frac{1}{r}} \\
&\quad + \frac{q(b-x)^2}{b-a}\Big(\int_0^1 t\mathrm{d}_q t\Big)^{1-\frac{1}{r}}\Big(\int_0^1 |{}_a\mathrm{D}_q f(tx + (1-t)b)|^r \mathrm{d}_q t\Big)^{\frac{1}{r}} \\
&\leq \frac{q(x-a)^2}{b-a}\Big(\frac{1}{1+q}\Big)^{1-\frac{1}{r}}\Bigg(\int_0^1 [t^s|{}_a\mathrm{D}_q f(x)|^r + m(1-t^s)\Big|{}_a\mathrm{D}_q f\Big(\frac{a}{m}\Big)\Big|^r]\mathrm{d}_q t\Bigg)^{\frac{1}{r}} \\
&\quad + \frac{q(b-x)^2}{b-a}\Big(\frac{1}{1+q}\Big)^{1-\frac{1}{r}}\Bigg(\int_0^1 [t^s|{}_a\mathrm{D}_q f(x)|^r + (1-t^s)\Big|{}_a\mathrm{D}_q f\Big(\frac{b}{m}\Big)\Big|^r]\mathrm{d}_q t\Bigg)^{\frac{1}{r}} \\
&\leq \frac{q}{b-a}\Big(\frac{1}{1+q}\Big)^{1-\frac{1}{r}} \\
&\times \Bigg[(x-a)^2\Bigg(\Bigg(\frac{1}{[s+1]_q}\Bigg)|{}_a\mathrm{D}_q f(x)|^r + m\Bigg(\frac{1}{1+q} - \frac{1}{[s+1]_q}\Bigg)\Big|{}_a\mathrm{D}_q f\Big(\frac{a}{m}\Big)\Big|^r\Bigg)^{\frac{1}{r}}
\end{aligned}$$

$$+(b-x)^2\left(\left(\left(\frac{1}{[s+1]_q}\right)|{}_a\mathrm{D}_q f(x)|^r + m\left(\frac{1}{1+q}-\frac{1}{[s+1]_q}\right)\left|{}_a\mathrm{D}_q f\Big(\frac{b}{m}\Big)\right|^r\right)^{\frac{1}{r}}\right].$$

Similarly

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}_q^{\mathscr{R}}u\right| \le \frac{q}{b-a}\Big(\frac{1}{1+q}\Big)^{1-\frac{1}{r}}$$

$$\times\left[(x-a)^2\left(m\left(\frac{1}{[s+1]_q}\right)\left|{}_a\mathrm{D}_q f\Big(\frac{x}{m}\Big)\right|^r+\left(\frac{1}{1+q}-\frac{1}{[s+1]_q}\right)|{}_a\mathrm{D}_q f(a)|^r\right)^{\frac{1}{r}}\right.$$

$$\left.+(b-x)^2\left(m\left(\frac{1}{[s+1]_q}\right)\left|{}_a\mathrm{D}_q f\Big(\frac{x}{m}\Big)\right|^r+\left(\frac{1}{1+q}-\frac{1}{[s+1]_q}\right)\left|{}_a\mathrm{D}_q f(b)\right|^r\right)^{\frac{1}{r}}\right].$$

This completes the proof. □

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