AN INERTIAL EXTRAGRADIENT METHOD FOR FINDING MINIMUM-NORM SOLUTIONS OF QUASIMONOTONE VARIATIONAL INEQUALITIES

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In this work, we propose a new self-adaptive extragradient projection method for solving variational inequalities with Lipschitz continuous and quasimonotone mapping in a real Hilbert space. Using the technique of inertial step into a single projection method, we obtained strong convergence theorem for the proposed algorithm. Our results extend and improve the existing results in the literature.

Keywords: Quasimonotone variational inequality; quasimonotone mapping; Lipschitz continuity; inertial technique; strong convergence.

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1. Introduction

The paper deals with a new numerical approach for finding a solution of the quasimonotone variational inequality problem (VI) [16, 17] in a real Hilbert space H.

Let C be a nonempty closed and convex subset in H and $F\colon H \to H$ be an operator. Recall that the problem (VI) for the operator F on C is stated as follows:

$$\text{Find } x^* \in C \text{ such that } \langle Fx^*, y - x^*\rangle \geq 0 \text{ for all } y \in C. \tag{1}$$

The solution set of the problem (VI) is denoted by S.

The dual variational inequality problem of (1) is to find a point $x^* \in C$ such that

$$\langle Fx, x - x^*\rangle \geq 0 \quad \forall x \in C. \tag{2}$$

We denote the solution set of dual variational inequality problem (2) by S_D. It is obvious that S_D is a closed convex set (possibly empty). In the case F is continuous and C is convex, we get

$$S_D \subset S.$$

If F is a pseudomonotone and continuous mapping, then $S = S_D$ (see, Lemma 2.1 in [12]). The inclusion $S \subset S_D$ is false, if F is a quasimonotone and continuous mapping (see, Example 4.2 in [38]).

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Variational inequality theory is an important tool in economics, engineering mechanics, mathematical programming, transportation and others (see, [1, 4, 5, 7, 14, 24, 25, 26]). Over the past decade, many numerical methods have been introduced for solving variational inequalities and related optimization problems, see [9, 10, 11, 13, 15, 18, 28, 30, 36].

The simplest one for solving VI (1) is the following gradient projection method:

$$\begin{cases} v_0 \in C, \\ v_{n+1} = P_C(v_n - \tau F v_n), \end{cases}$$

where P_C denotes the metric projection of H onto the set C and τ is a positive real number. The main restriction of gradient projection methods is that the operators require to be Lipschitz continuous and strongly monotone (or inverse strongly monotone). The extragradient method which introduced by Korpelevich [27] and Antipin [3] overcomes this disadvantage by performing an additional projection at each iteration in the following way:

$$\begin{cases} v_0 \in C, \\ u_n = P_C(v_n - \tau F v_n), \\ v_{n+1} = P_C(v_n - \tau F u_n), \end{cases} \tag{3}$$

where $F\colon C \to C$ is monotone and L-Lipschitz continuous, $\tau \in (0, \frac{1}{L})$. Recently, the extragradient method has given conclusive results assuming monotone and the Lipschitz continuous mappings (see, e.g., [9, 13, 32]). It is well known that to implement the extragradient method, one needs to calculate two projections onto C in each iteration. In [9] Censor et al. proposed the modified extragradient method which is called the subgradient extragradient method. In their method, they replaced the second projection onto C with a projection onto a half-space. However, in the subgradient extragradient method, it requires the cost mapping F to be defined on the whole H. This is a barrier if the mapping F is only Lipschitz continuous on C.

In recent years, the class of pseudomonotone mappings has been studied for solving the problem VI [8, 20, 35, 37]. In particular, when the mapping associated with variational inequality is pseudomonotone and sequentially weakly continuity, the extragradient method is introduced for solving variational inequalities in real Hilbert spaces [32].

Recently, some authors have investigated some weak convergence results of the extragradient methods when the assumption on F is quasimonotone (or non-monotone), which is weaker than the pseudomonotonicity assumption [2, 21, 22, 29, 33, 34, 38]. This is of interest because of the fact that the convergence analysis when F is pseudomonotone cannot be carried over to the case when F is quasimonotone. For instance, when F is quasimonotone, the dual variational inequality of problem (2) is not equivalent to problem (1). In [38], Ye and He proposed a double projection method and proved that it converges to a solution of problem (1) when F is only required to be continuous in a finite-dimensional space. Similar results are obtained by Izuchukwu et al. [21, 22], Alakoya et al. [2], Wang et al. [33, 34]. Recently, Liu et al. [29] proved that the forward-backward-forward method converges weakly to a solution of (1) when F is quasimonotone, Lipschitz continuous and sequentially weakly continuous in an infinite dimensional Hilbert space.

At the best knowledge of the authors, the study of the strong convergence of the extragradient method (3) for solving quasimonotone variational inequalities in the setting of Hilbert space is still unexplored. This leads us to the following question.

Question: *Can we give strong convergence results of the extragradient method* (3) *with the inertial technique for solving solving quasimonotone variational inequalities*?

Our aim in this paper is to answer the above question. We propose a new extragradient method with an inertial step and self-adaptive step sizes for solving the problem (1), with the following contributions: We introduce a novel extragradient method and obtain a strong convergence result when F is quasimonotone and Lipschitz continuous. Unlike the other projections, our proposed algorithm requires the cost mapping F to be calculated only on the closed convex set C in each iteration, rather than on the entire H, and utilizes self-adaptive step sizes to approximate a solution to the quasimonotone variational inequality problem.

This paper is organized as follows: In Sect. 2, we recall some definitions and preliminary results for further use. Sect. 3 deals with analyzing the convergence of the proposed algorithms. we prove strong convergence results under the conditions that F is Lipschitz continuous and quasi-monotone on H and the solution set S_D is nonempty.

2. **Preliminaries**

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written as $x_n \to x$ as $n \to \infty$. For all $x, y \in H$, we have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \tag{4}$$

For all $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \ \ \forall y \in C.$$

P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive.

Lemma 2.1 ([19])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then we have*

$$z = P_C x \iff \langle x - z, z - y\rangle \geq 0 \ \ \forall y \in C.$$

Lemma 2.2 ([6, 19])**.** *Let C be a closed convex subset in a real Hilbert space H and $x \in H$. Then we have the following:*

(1) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \ \ \forall y \in H$;
(2) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2 \ \ \forall y \in C$.

Definition 2.1. Let $F\colon H \to H$ be a mapping. Then the mapping F is said to be:

(1) *L-Lipschitz continuous* with $L > 0$ if

$$\|Fx - Fy\| \leq L\|x - y\| \ \ \forall x, y \in H.$$

(2) *monotone* if

$$\langle Fx - Fy, x - y\rangle \geq 0 \ \ \forall x, y \in H.$$

(3) *pseudomonotone* in the sense of Karamardian [23] if

$$\langle Fx, y - x\rangle \geq 0 \Longrightarrow \langle Fy, y - x\rangle \geq 0 \ \ \forall x, y \in H.$$

(4) quasimonotone, if

$$\langle Fx, y - x\rangle > 0 \Longrightarrow \langle Fy, y - x\rangle \geq 0 \ \ \forall x, y \in H.$$

(5) *δ-strongly pseudomonotone* if there exists a constant $\delta > 0$ such that

$$\langle Fx, x - y\rangle \geq 0 \Longrightarrow \langle Fy, y - x\rangle \geq \delta\|x - y\|^2 \ \ \forall x, y \in H.$$

(6) *sequentially weakly continuous* if, for each sequence $\{x_n\}$ in H, $\{x_n\}$ converges weakly to a point $x \in H$ implies $\{Fx_n\}$ converges weakly to Fx.

It is easy to see that every implication (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) hold, but the converse is not true.

The following lemma gives a situation when S_D is nonempty.

Lemma 2.3 ([38])**.** *If either*

(1) *F is pseudomonotone on C and $S \neq \emptyset$,*
(2) *F is the gradient of G, where G is a differential quasiconvex function on an open set $K, C \subset K$ and attains its global minimum on C,*
(3) *F is quasi-monotone on C, $F \neq 0$ on C and C is bounded,*
(4) *F is quasi-monotone on C, $F \neq 0$ on C and there exists a positive number r such that, for every $v \in C$ with $\|v\| \geq r$, there exists $y \in C$ such that $\|y\| \leq r$ and $\langle Fv, y - v\rangle \leq 0$,*
(5) *F is quasimonotone on C and $S_N \neq \emptyset$, with $S_N := S \backslash S_T$, where $S_T := \{x^* \in C | \langle F(x^*), y - x^*\rangle = 0 \ \forall y \in C\}$.*
(6) *F is quasi-monotone on C, int$C$ is nonempty and there exists $v^* \in S$ such that $Fv^* \neq 0$.*

Then, S_D is nonempty.

Lemma 2.4 ([31])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, \quad \forall n \geq 1,$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ hold for every subsequence $\{a_{n_k}\}$ of sequence $\{a_n\}$ which satisfyies $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$, then $\lim_{n\to\infty} a_n = 0$.

3. The Main Results

We now introduce our algorithm.

Algorithm 3.1.

Initialization: *Given $\tau_1 > 0$, $0 < a \leq \lambda < \dfrac{1}{2}$, and $\{\alpha_n\}$, $\{\gamma_n\}$ are two nonnegative real numbers sequences such that $\sum_{n=1}^{\infty} \gamma_n < +\infty$. Let $x_0, x_1 \in H$ be arbitrary. We assume $\{\theta_n\} \subset (0, 1)$ is positive real numbers sequence that satisfies the following conditions:*

$$\lim_{n\to\infty}(1 - \theta_n) = 0, \quad \sum_{n=1}^{\infty}(1 - \theta_n) = \infty.$$

Iterative Steps: *Calculate x_{n+1} as follows:*

Step 1. *Given the current iterates x_{n-1} and x_n ($n \geq 1$), compute*

$$\begin{aligned} w_n &= x_n + \alpha_n(x_n - x_{n-1}), \\ y_n &= P_C(w_n - \tau_n F w_n). \end{aligned}$$

If $y_n = w_n$ then stop and w_n is a solution of problem (1)*. Otherwise, go to* **Step 2***.*

Step 2. *Compute*

$$x_{n+1} = (1 - \lambda)(\theta_n x_n) + \lambda P_C(w_n - \tau_n F y_n),$$

update

$$\tau_{n+1} = \begin{cases} \min\left\{\mu \dfrac{\|w_n - y_n\|}{\|Fw_n - Fy_n\|}, \tau_n + \gamma_n\right\} & \text{if } Fw_n \neq Fy_n, \\ \tau_n + \gamma_n & \text{otherwise.} \end{cases} \tag{5}$$

Set $n := n + 1$ and return to Setp 1.

3.1. Strong convergence

In order to analyze the convergence of the proposed algorithm, we assume the following conditions:

Condition 3.1. $S_D \neq \emptyset$.

Condition 3.2. *The mapping* $F\colon C \to C$ *is* L*-Lipschitz continuous on* H*. However, the information of* L *is not necessary to be known.*

Condition 3.3. *The mapping* F *is sequentially weakly continuous on* C*, i.e., for each sequence* $\{x_n\} \subset C$*,* $\{x_n\}$ *converges weakly to* x^* *implies* $\{Fx_n\}$ *converges weakly to* Fx^**.*

Condition 3.4. *The mapping* F *is quasimonotone on* C*.*

We need the following lemmas.

Lemma 3.1 ([29])**.** *Let* $\{\tau_n\}$ *be a sequence generated by* (5)*. Then*

$$\lim_{n\to\infty} \tau_n = \tau \text{ with } \tau \in \left[\min\left\{\tau_1, \frac{\mu}{L}\right\}, \tau_1 + \alpha^*\right],$$

where $\alpha^* = \sum_{n=1}^{\infty} \gamma_n$*. Moreover, we also obtain*

$$\|Fw_n - Fy_n\| \leq \frac{\mu}{\tau_{n+1}}\|w_n - y_n\|.$$

Lemma 3.2. *Assume that Conditions* 3.1 - 3.4 *hold. Let* $\{w_n\}$ *be a sequence generated by Algorithm* 3.1*. If there exists a subsequence* $\{w_{n_k}\}$ *convergent weakly to* $z \in H$ *and* $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$*, then* $z \in S_D$ *or* $Fz = 0$*.*

Proof. First, we see that $\{w_{n_k}\} \rightharpoonup z$ and $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$ this implies that $y_{n_k} \rightharpoonup z$ and since $y_n \in C$ we get $z \in C$.

Now, we divide the proof into two cases.

Case 1: If $\limsup_{k\to\infty} \|Fy_{n_k}\| = 0$, then we have

$$\lim_{k\to\infty} \|Fy_{n_k}\| = \liminf_{k\to\infty} \|Fy_{n_k}\| = 0.$$

Since y_{n_k} converges weakly to $z \in C$ and F satisfies Condition 3.3 we get

$$0 \leq \|Fz\| \leq \liminf_{k\to\infty} \|Fy_{n_k}\| = 0.$$

This implies that $Fz = 0$.

Case 2: If $\limsup_{k\to\infty} \|Fy_{n_k}\| > 0$. Without loss of generality, we take

$$\lim_{k\to\infty} \|Ay_{n_k}\| = M > 0.$$

It then follows that there exists a $K \in \mathbb{N}$ such that $\|Fy_{n_k}\| > \dfrac{M}{2}$ for all $k \geq K$. Since $y_{n_k} = P_C(w_{n_k} - \tau_{n_k} Fw_{n_k})$, we have

$$\langle w_{n_k} - \tau_{n_k} Fw_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0 \ \ \forall x \in C,$$

or, equivalently,

$$\frac{1}{\tau_{n_k}}\langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq \langle Fw_{n_k}, x - y_{n_k}\rangle \ \ \forall x \in C.$$

Consequently, we have

$$\frac{1}{\tau_{n_k}}\langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle Fw_{n_k}, y_{n_k} - w_{n_k}\rangle \leq \langle Fw_{n_k}, x - w_{n_k}\rangle \ \ \forall x \in C. \tag{6}$$

Since $\{w_{n_k}\}$ is weakly convergent, $\{w_{n_k}\}$ is bounded. Then, by the Lipschitz continuity of F, $\{Fw_{n_k}\}$ is bounded. As $\|w_{n_k} - y_{n_k}\| \to 0$, $\{y_{n_k}\}$ is also bounded and $\tau_{n_k} \geq \min\{\tau_1, \dfrac{\mu}{L}\}$. Passing (6) to the limit as $k \to \infty$, we get

$$\liminf_{k\to\infty} \langle Fw_{n_k}, x - w_{n_k}\rangle \geq 0 \ \ \forall x \in C. \tag{7}$$

Moreover, we have

$$\begin{aligned}\langle Fy_{n_k}, x - y_{n_k}\rangle \ &= \ \langle Fy_{n_k} - Fw_{n_k}, x - w_{n_k}\rangle + \langle Fw_{n_k}, x - w_{n_k}\rangle \\ &\quad + \langle Fy_{n_k}, w_{n_k} - y_{n_k}\rangle. \end{aligned} \tag{8}$$

Since $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$ and F is L-Lipschitz continuous on H, we get

$$\lim_{k\to\infty} \|Fw_{n_k} - Fy_{n_k}\| = 0$$

which, together with (7) and (8), implies that

$$\liminf_{k\to\infty} \langle Fy_{n_k}, x - y_{n_k}\rangle \geq 0. \tag{9}$$

• If $\limsup_{k\to\infty} \langle Fy_{n_k}, x - y_{n_k}\rangle > 0$, then there exists a subsequence $\{y_{n_{k_j}}\}$ such that $\lim_{j\to\infty} \langle Fy_{n_{k_j}}, x - y_{n_{k_j}}\rangle > 0$. Consequently, there exists $j_0 \in \mathbb{N}$ such that

$$\langle Fy_{n_{k_j}}, x - y_{n_{k_j}}\rangle > 0 \ \forall j \geq j_0.$$

Letting $j \to \infty$, we have $z \in S_D$.

• If $\limsup_{k\to\infty} \langle Fy_{n_k}, x - y_{n_k}\rangle = 0$. From (9) implies that

$$\lim_{k\to\infty} \langle Fy_{n_k}, x - y_{n_k}\rangle = 0.$$

Let $\epsilon_k := |\langle Fy_{n_k}, x - y_{n_k}\rangle| + \dfrac{1}{k+1}$. Then we obtain

$$\langle Fy_{n_k}, x - y_{n_k}\rangle + \epsilon_k > 0 \ \ \forall k \geq K. \tag{10}$$

Furthermore, for each $k \geq 1$, since $\{y_{n_k}\} \subset C$, we can suppose $Fy_{n_k} \neq 0$ (otherwise, y_{n_k} is a solution) and, setting

$$q_{n_k} = \frac{Fy_{n_k}}{\|Fy_{n_k}\|^2},$$

we have $\langle Fy_{n_k}, q_{n_k}\rangle = 1$ for each $k \geq K$. Now, we can deduce from (10) that, for each $k \geq K$,

$$\langle Fy_{n_k}, x + \epsilon_k q_{n_k} - y_{n_k}\rangle > 0.$$

Since F is quasimonotone on H, we get

$$\langle F(x + \epsilon_k q_{n_k}), x + \epsilon_k q_{n_k} - y_{n_k}\rangle \geq 0. \tag{11}$$

Now, for all $k \geq K$, using (11) we get

$$\begin{aligned}\langle Fx, x + \epsilon_k q_{n_k} - y_{n_k}\rangle &= \langle Fx - F(x + \epsilon_k q_{n_k}), x + \epsilon_k q_{n_k} - y_{n_k}\rangle \\ &\quad + \langle F(x + \epsilon_k q_{n_k}), x + \epsilon_k q_{n_k} - y_{n_k}\rangle \\ &\geq \langle Fx - F(x + \epsilon_k q_{n_k}), x + \epsilon_k q_{n_k} - y_{n_k}\rangle \\ &\geq -\|Fx - F(x + \epsilon_k q_{n_k})\| \|x + \epsilon_k q_{n_k} - y_{n_k}\| \\ &\geq -\epsilon_k L \|q_{n_k}\| \|x + \epsilon_k q_{n_k} - y_{n_k}\| \\ &= -\epsilon_k L \frac{1}{\|Fy_{n_k}\|} \|x + \epsilon_k q_{n_k} - y_{n_k}\| \\ &\geq -\epsilon_k L \frac{2}{M} \|x + \epsilon_k q_{n_k} - y_{n_k}\|. \end{aligned} \tag{12}$$

In (12), letting $k \to \infty$ and using the fact that $\lim_{k\to\infty} \epsilon_k = 0$ and the boundedness of $\{\|x + \epsilon_k q_{n_k} - y_{n_k}\|\}$ we get

$$\langle Fx, x - z\rangle \geq 0 \ \ \forall x \in C.$$

This implies that $z \in S_D$. □

Theorem 3.1. *Assume that Conditions 3.1 - 3.4 hold and $Fx \neq 0$, for all $x \in C$. Then, the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to an element $x^* \in S_D \subset S$, where $\|x^*\| = \min\{\|u\| : u \in S_D\}$ provided that*

$$\lim_{n\to+\infty} \frac{\alpha_n}{1-\theta_n}\|x_n - x_{n-1}\| = 0. \tag{13}$$

Remark 3.1. Note that S_D is a closed and convex set, thus $\|x^*\| = \min\{\|u\| : u \in S_D\}$ implies that $x^* = P_{S_D}(0)$.

Proof. Let $z_n := P_C(w_n - \tau_n F y_n)$.

Claim 1.

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - \Big(1 - \mu\frac{\tau_n}{\tau_{n+1}}\Big)(\|y_n - w_n\|^2 + \|z_n - y_n\|^2). \tag{14}$$

Indeed, since $x^* \in S_D \subset S \subset C$, we have

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|P_C(w_n - \tau_n F y_n) - P_C x^*\|^2\\
&\leq \langle z_n - x^*, w_n - \tau_n F y_n - x^*\rangle\\
&= \frac{1}{2}\|z_n - x^*\|^2 + \frac{1}{2}\|w_n - \tau_n F y_n - x^*\|^2 - \frac{1}{2}\|z_n - w_n + \tau_n F y_n\|^2\\
&= \frac{1}{2}\|z_n - x^*\|^2 + \frac{1}{2}\|w_n - x^*\|^2 + \frac{1}{2}\tau_n^2\|Fy_n\|^2 - \langle w_n - x^*, \tau_n F y_n\rangle\\
&- \frac{1}{2}\|z_n - w_n\|^2 - \frac{1}{2}\tau_n^2\|Fy_n\|^2 - \langle z_n - w_n, \tau_n F y_n\rangle\\
&= \frac{1}{2}\|z_n - x^*\|^2 + \frac{1}{2}\|w_n - x^*\|^2 - \frac{1}{2}\|z_n - w_n\|^2 - \langle z_n - x^*, \tau_n F y_n\rangle.
\end{aligned}$$

This implies that

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - x^*, \tau_n F y_n\rangle. \tag{15}$$

Since $x^* \in S_D$ and $y_n \in C$ we get

$$\langle Fy_n, x^* - y_n\rangle \leq 0.$$

Thus we have

$$\langle Fy_n, x^* - z_n\rangle = \langle Fy_n, x^* - y_n\rangle + \langle Fy_n, y_n - z_n\rangle \leq \langle Fy_n, y_n - z_n\rangle. \tag{16}$$

From (15) and (16), we obtain

$$\begin{aligned}
\|z_n - x^*\|^2 &\leq \|w_n - x^*\|^2 - \|z_n - w_n\|^2 + 2\tau_n\langle Fy_n, y_n - z_n\rangle\\
&= \|w_n - x^*\|^2 - \|z_n - y_n\|^2 - \|y_n - w_n\|^2 - 2\langle z_n - y_n, y_n - w_n\rangle\\
&\quad + 2\tau_n\langle Fy_n, y_n - z_n\rangle\\
&= \|w_n - x^*\|^2 - \|z_n - y_n\|^2 - \|y_n - w_n\|^2\\
&\quad + 2\langle w_n - \tau_n F y_n - y_n, z_n - y_n\rangle.
\end{aligned} \tag{17}$$

Using Lemma 2.1, it's easy to see that from $y_n = P_C(w_n - \tau_n F w_n)$, and $z_n \in T_n$, we obtain

$$2\langle y_n + \tau_n F w_n - w_n, y_n - z_n\rangle \leq 0.$$

This is equivalent to

$$2\langle w_n - \tau_n F w_n - y_n, z_n - y_n\rangle \leq 0. \tag{18}$$

Using (18) we deduce

$$\begin{aligned}2\langle w_n - \tau_n F y_n - y_n, z_n - y_n\rangle &= 2\langle w_n - \tau_n F w_n - y_n, z_n - y_n\rangle + 2\tau_n\langle F w_n - F y_n, z_n - y_n\rangle \\ &\le 2\tau_n\langle F w_n - F y_n, z_n - y_n\rangle \\ &\le \mu\frac{\tau_n}{\tau_{n+1}}\|w_n - y_n\|^2 + \mu\frac{\tau_n}{\tau_{n+1}}\|y_n - z_n\|^2. \end{aligned} \tag{19}$$

Substituting (19) into (17), we obtain

$$\begin{aligned}\|z_n - x^*\|^2 &\le \|w_n - x^*\|^2 - \Big(1 - \mu\frac{\tau_n}{\tau_{n+1}}\Big)\|y_n - w_n\|^2 - \Big(1 - \mu\frac{\tau_n}{\tau_{n+1}}\Big)\|z_n - y_n\|^2 \\ &= \|w_n - x^*\|^2 - \Big(1 - \mu\frac{\tau_n}{\tau_{n+1}}\Big)(\|y_n - w_n\|^2 + \|z_n - y_n\|^2).\end{aligned}$$

From $\lim_{n\to\infty}\Big(1 - \mu\frac{\tau_n}{\tau_{n+1}}\Big) = 1 - \mu > 0$, it follows that there exists $n_0 \in \mathbb{N}$ such that

$$1 - \mu\frac{\tau_n}{\tau_{n+1}} > 0 \;\; \forall n \ge n_0.$$

Thus, we have

$$\|z_n - x^*\| \le \|w_n - x^*\| \quad \forall n \ge n_0.$$

Claim 2. The sequence $\{x_n\}$ is bounded. Indeed, by Claim 1 then there exists $n_0 \in \mathbb{N}$ such that

$$\|z_n - x^*\| \le \|w_n - x^*\| \quad \forall n \ge n_0. \tag{20}$$

On the other hand, from the definition of w_n, we get

$$\begin{aligned}\|w_n - x^*\| &= \|x_n + \alpha_n(x_n - x_{n-1}) - x^*\| \\ &\le \|x_n - x^*\| + \alpha_n\|x_n - x_{n-1}\| \\ &= \|x_n - x^*\| + (1 - \theta_n)\frac{\alpha_n}{1 - \theta_n}\|x_n - x_{n-1}\|.\end{aligned} \tag{21}$$

By the condition $\lim_{n\to+\infty}\frac{\alpha_n}{1-\theta_n}\|x_n - x_{n-1}\| = 0$, it follows that there exists a constant $M_1 > 0$ such that

$$\frac{\alpha_n}{1-\theta_n}\|x_n - x_{n-1}\| \le M_1, \quad \forall n \ge 1. \tag{22}$$

Combining (20), (21) and (22), we obtain

$$\|z_n - x^*\| \le \|w_n - x^*\| \le \|x_n - x^*\| + (1 - \theta_n)M_1. \tag{23}$$

Now, from the definition of $\{x_n\}$, we get

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|(1-\lambda)(\theta_n x_n) + \lambda - x^*\| \\ &= \|(1-\lambda)\theta_n(x_n - x^*) + \lambda(z_n - x^*) - (1-\theta_n)(1-\lambda)x^*\| \\ &\le \|(1-\lambda)\theta_n(x_n - x^*) + \lambda(z_n - x^*)\| + (1-\theta_n)(1-\lambda)\|x^*\|.\end{aligned} \tag{24}$$

Now, we estimate $\|(1-\lambda)\theta_n(x_n - x^*) + \lambda(z_n - x^*)\|$

$$\begin{aligned}&\|(1-\lambda)\theta_n(x_n - x^*) + \lambda(z_n - x^*)\|^2 \\ &= (1-\lambda)^2\theta_n^2\|x_n - x^*\|^2 + 2(1-\lambda)\theta_n\lambda\langle x_n - x^*, z_n - x^*\rangle + \lambda^2\|z_n - x^*\|^2 \\ &\le (1-\lambda)^2\theta_n^2\|x_n - x^*\|^2 + 2(1-\lambda)\theta_n\lambda\|x_n - x^*\|\|z_n - x^*\| + \lambda^2\|z_n - x^*\|^2 \\ &= [(1-\lambda)\theta_n\|x_n - x^*\| + \lambda\|z_n - x^*\|]^2.\end{aligned}$$

Thus

$$\|(1-\lambda)\theta_n(x_n - x^*) + \lambda(z_n - x^*)\| \le (1-\lambda)\theta_n\|x_n - x^*\| + \lambda\|z_n - x^*\|. \tag{25}$$

Combining (23) and (25), we deduce

$$\begin{aligned}
&\|(1-\lambda)\theta_n(x_n-x^*)+\lambda(z_n-x^*)\| \\
&\qquad \le (1-\lambda)\theta_n\|x_n-x^*\|+\lambda\|x_n-x^*\|+(1-\theta_n)\lambda M_1 \\
&\qquad = (1-(1-\theta_n)(1-\lambda))\|x_n-x^*\|+(1-\theta_n)\lambda M_1 \\
&\qquad \le (1-(1-\theta_n)(1-\lambda))\|x_n-x^*\|+(1-\theta_n)(1-\lambda)M_1 \qquad (26)\\
&\qquad\quad (\text{ by } 0<a\le\lambda\le\frac{1}{2}).
\end{aligned}$$

Substituting (26) into (24), we get

$$\begin{aligned}
\|x_{n+1}-x^*\| &\le (1-(1-\theta_n)(1-\lambda))\|x_n-x^*\|+(1-\theta_n)(1-\lambda)(\|x^*\|+M_1) \\
&\le \max\{\|x_n-x^*\|,\|x^*\|+M_1\} \\
&\le ... \le \max\{\|x_0-x^*\|,\|x^*\|+M_1\}.
\end{aligned}$$

Therefore, the sequence $\{x_n\}$ is bounded. So, $\{z_n\}$ is also bounded.
Claim 3.

$$\begin{aligned}
\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)(\|y_n-w_n\|^2+\|z_n-y_n\|^2) &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\
&\quad +(1-\theta_n)[\lambda(1-(1-\lambda)(1-\theta_n))M_1+(1-\lambda)A_{x^*}],
\end{aligned}$$

for some $A_{x^*}>0$. Indeed, we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 = \|(1-\lambda)\theta_n(x_n-x^*)+\lambda(z_n-x^*)-(1-\theta_n)(1-\lambda)x^*\|^2 \\
&= \|(1-\lambda)\theta_n(x_n-x^*)+\lambda(z_n-x^*)\|^2+(1-\theta_n)^2(1-\lambda)^2\|x^*\|^2 \\
&-2(1-\theta_n)(1-\lambda)\langle(1-\lambda)\theta_n(x_n-x^*)+\lambda(z_n-x^*),x^*\rangle \\
&\le \|(1-\lambda)\theta_n(x_n-x^*)+\lambda(z_n-x^*)\|^2+(1-\theta_n)(1-\lambda)\Bigg[(1-\theta_n)(1-\lambda)\|x^*\| \\
&+2\langle(1-\lambda)\theta_n\|(x_n-x^*)+\lambda(z_n-x^*)\|\|x^*\|\Bigg] \\
&\le \|(1-\lambda)\theta_n(x_n-x^*)+\lambda(z_n-x^*)\|^2+(1-\theta_n)(1-\lambda)A_{x^*}. \qquad (27)
\end{aligned}$$

The last inequality obtains by the boundness of $\{x_n\}$, $\{z_n\}$, and $\{\theta_n\}$, implies there exists $A_{x^*}>0$ such that

$$(1-\theta_n)(1-\lambda)\|x^*\|+2\langle(1-\lambda)\theta_n\|(x_n-x^*)+\lambda(z_n-x^*)\|\|x^*\|\le A_{x^*}\quad\forall n.$$

Now, we estimate $\|(1-\lambda)\theta_n(x_n-x^*)+\lambda(z_n-x^*)\|^2$. We have

$$\begin{aligned}
&\|(1-\lambda)\theta_n(x_n-x^*)+\lambda(z_n-x^*)\|^2 \\
&\quad = (1-\lambda)^2\theta_n^2\|x_n-x^*\|^2+2(1-\lambda)\theta_n\lambda\langle x_n-x^*,z_n-x^*\rangle+\lambda^2\|z_n-x^*\|^2 \\
&\quad \le (1-\lambda)^2\theta_n^2\|x_n-x^*\|^2+2(1-\lambda)\theta_n\lambda\|x_n-x^*\|\|z_n-x^*\|+\lambda^2\|z_n-x^*\|^2 \\
&\quad \le (1-\lambda)^2\theta_n^2\|x_n-x^*\|^2+(1-\lambda)\theta_n\lambda\|x_n-x^*\|^2+(1-\lambda)\theta_n\lambda\|z_n-x^*\|^2 \\
&\quad +\lambda^2\|z_n-x^*\|^2 \\
&\quad \le (1-\lambda)\theta_n(1-(1-\lambda)(1-\theta_n))\|x_n-x^*\|^2+\lambda(1-(1-\lambda)(1-\theta_n))\|z_n-x^*\|^2. \qquad (28)
\end{aligned}$$

Combining (14) and (28), we get

$$\begin{aligned}
&\|(1-\lambda)\theta_n(x_n-x^*)+\lambda(z_n-x^*)\|^2\\
&\le (1-\lambda)\theta_n(1-(1-\lambda)(1-\theta_n))\|x_n-x^*\|^2+\lambda(1-(1-\lambda)(1-\theta_n))\|w_n-x^*\|^2\\
&\quad -\lambda(1-(1-\lambda)(1-\theta_n))\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)(\|y_n-w_n\|^2+\|z_n-y_n\|^2).\\
&\le (1-\lambda)\theta_n(1-(1-\lambda)(1-\theta_n))\|x_n-x^*\|^2+\lambda(1-(1-\lambda)(1-\theta_n))\|x_n-x^*\|^2\\
&\quad +\lambda(1-(1-\lambda)(1-\theta_n))(1-\theta_n)M_1\\
&\quad -\lambda(1-(1-\lambda)(1-\theta_n))\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)(\|y_n-w_n\|^2+\|z_n-y_n\|^2)\\
&=(1-(1-\lambda)(1-\theta_n))^2\|x_n-x^*\|^2+\lambda(1-(1-\lambda)(1-\theta_n))(1-\theta_n)M_1\\
&\quad -\lambda(1-(1-\lambda)(1-\theta_n))\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)(\|y_n-w_n\|^2+\|z_n-y_n\|^2).
\end{aligned}$$

Substituting (28) into (27), we get

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le (1-(1-\lambda)(1-\theta_n))^2\|x_n-x^*\|^2\\
&\quad +(1-\theta_n)(\lambda(1-(1-\lambda)(1-\theta_n))M_1+(1-\lambda)A_{x^*})\\
&\quad -\lambda(1-(1-\lambda)(1-\theta_n))\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)(\|y_n-w_n\|^2+\|z_n-y_n\|^2)\\
&\le \|x_n-x^*\|^2 \text{ (by } (1-(1-\lambda)(1-\theta_n))^2\le 1 \text{ and } \lambda\ge a)\\
&\quad +(1-\theta_n)[\lambda(1-(1-\lambda)(1-\theta_n))M_1+(1-\lambda)A_{x^*}]\\
&\quad -a(1-(1-\lambda)(1-\theta_n))\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)(\|y_n-w_n\|^2+\|z_n-y_n\|^2).
\end{aligned}$$

This implies that

$$\begin{aligned}
a(1-(1-\lambda)(1-\theta_n))\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)(\|y_n-w_n\|^2+\|z_n-y_n\|^2)&\\
\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2&\\
+(1-\theta_n)[\lambda(1-(1-\lambda)(1-\theta_n))M_1+(1-\lambda)A_{x^*}].&
\end{aligned}$$

Claim 4.

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\le(1-(1-\lambda)(1-\theta_n))\|x_n-x^*\|^2\\
&+(1-\lambda)(1-\theta_n)\Big[\frac{\alpha_n}{1-\theta_n}\|(x_n-x_{n-1}\|(1-(1-\lambda)(1-\theta_n))\frac{\lambda M_3}{1-\lambda}+2\langle x^*,x_{n+1}-x^*\rangle\Big],
\end{aligned}$$

for some $M_3>0$. Indeed, using the inequality (4) and (28) we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2&=\|(1-\lambda)\theta_n(x_n-x^*)+\lambda(z_n-x^*)-(1-\theta_n)(1-\lambda)x^*\|^2\\
&\le\|(1-\lambda)\theta_n(x_n-x^*)+\lambda(z_n-x^*)\|^2+2(1-\theta_n)(1-\lambda)\langle x^*,x_{n+1}-x^*\rangle\\
&\le(1-\lambda)\theta_n(1-(1-\lambda)(1-\theta_n))\|x_n-x^*\|^2\\
&\quad+\lambda(1-(1-\lambda)(1-\theta_n))\|w_n-x^*\|^2+2(1-\theta_n)(1-\lambda)\langle x^*,x_{n+1}-x^*\rangle.
\end{aligned}\tag{29}$$

On the other hand, by the definition of $\{w_n\}$, we have

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|x_n - x^* + \alpha_n(x_n - x_{n-1})\|^2 \\
&= \|x_n - x^*\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle x_n - x^*, x_n - x_{n-1}\rangle \\
&\le \|x_n - x^*\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\|x_n - x^*\|\|x_n - x_{n-1}\| \\
&\le \|x_n - x^*\|^2 + \alpha_n\|x_n - x_{n-1}(\|\alpha_n\|x_n - x_{n-1}\| + 2\|x_n - x^*\|) \\
&\le \|x_n - x^*\|^2 + \alpha_n\|x_n - x_{n-1}\|M_3, \qquad (30)
\end{aligned}$$

for some $M_3 > 0$.
Substituting (30) into (29) we deduce

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \le (1-\lambda)\theta_n(1-(1-\lambda)(1-\theta_n))\|x_n - x^*\|^2 \\
&\quad + \lambda(1-(1-\lambda)(1-\theta_n))\|x_n - x^*\|^2 \\
&\quad + \lambda(1-(1-\lambda)(1-\theta_n))\alpha_n\|x_n - x_{n-1}\|M_3, +2(1-\theta_n)(1-\lambda)\langle x^*, x_{n+1} - x^*\rangle \\
&= (1-(1-\lambda)(1-\theta_n))^2\|x_n - x^*\|^2 \\
&+ (1-\lambda)(1-\theta_n)\left[\frac{\alpha_n}{1-\theta_n}\|x_n - x_{n-1}\|(1-(1-\lambda)(1-\theta_n))\frac{\lambda M_3}{1-\lambda} + 2\langle x^*, x_{n+1} - x^*\rangle\right] \\
&\le (1-(1-\lambda)(1-\theta_n))\|x_n - x^*\|^2 \\
&+ (1-\lambda)(1-\theta_n)\left[\frac{\alpha_n}{1-\theta_n}\|x_n - x_{n-1}\|(1-(1-\lambda)(1-\theta_n))\frac{\lambda M_3}{1-\lambda} + 2\langle x^*, x_{n+1} - x^*\rangle\right].
\end{aligned}$$

Claim 5. $\{\|x_n - x^*\|^2\}$ converges to zero. Indeed, by Lemma 2.4 it suffices to show that $\limsup_{k\to\infty}\langle x^*, x_{n_k+1} - x^*\rangle \le 0$ for every subsequence $\{\|x_{n_k} - x^*\|\}$ of $\{\|x_n - x^*\|\}$ satisfying

$$\liminf_{k\to\infty}(\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|) \ge 0.$$

For this, suppose that $\{\|x_{n_k} - x^*\|\}$ is a subsequence of $\{\|x_n - x^*\|\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|) \ge 0$. Then

$$\begin{aligned}
&\liminf_{k\to\infty}(\|x_{n_k+1} - x^*\|^2 - \|x_{n_k} - x^*\|^2) \\
&\qquad = \liminf_{k\to\infty}[(\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|)(\|x_{n_k+1} - x^*\| + \|x_{n_k} - x^*\|)] \ge 0.
\end{aligned}$$

By Claim 3 we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}(1-\theta_{n_k})\Big(1-\mu\frac{\tau_{n_k}}{\tau_{n_k+1}}\Big)(\|y_{n_k} - w_{n_k}\|^2 + \|z_{n_k} - y_{n_k}\|^2) \\
&\le \limsup_{k\to\infty}[\|x_{n_k} - x^*\|^2 - \|x_{n_k+1} - x^*\|^2 \\
&\quad + (1-\theta_{n_k})[\lambda(1-(1-\lambda)(1-\theta_{n_k}))A_1 + (1-\lambda)A_{x^*}] \\
&\le \limsup_{k\to\infty}[\|x_{n_k} - x^*\|^2 - \|x_{n_k+1} - x^*\|^2] \\
&\quad + \limsup_{k\to\infty}(1-\theta_{n_k})[\lambda(1-(1-\lambda)(1-\theta_{n_k}))M_1 + (1-\lambda)A_{x^*}] \\
&= -\liminf_{k\to\infty}[\|x_{n_k+1} - x^*\|^2 - \|x_{n_k} - x^*\|^2] \le 0.
\end{aligned}$$

This implies that

$$\lim_{k\to\infty}(\|y_{n_k} - w_{n_k}\|^2 + \|z_{n_k} - y_{n_k}\|^2) = 0. \qquad (31)$$

It follows from (31) that

$$\lim_{k\to\infty}\|y_{n_k} - w_{n_k}\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\|z_{n_k} - y_{n_k}\| = 0. \qquad (32)$$

Now, we show that

$$\|x_{n_k+1} - x_{n_k}\| \to 0 \quad \text{as} \quad n \to \infty. \tag{33}$$

Indeed, from (31), it follows that

$$\|z_{n_k} - w_{n_k}\| \le \|z_{n_k} - y_{n_k}\| + \|y_{n_k} - w_{n_k}\| \to 0. \tag{34}$$

and

$$\|x_{n_k} - w_{n_k}\| = \alpha_{n_k}\|x_{n_k} - x_{n_k-1}\| = (1-\theta_{n_k})\frac{\alpha_{n_k}}{1-\theta_{n_k}}\|x_{n_k} - x_{n_k-1}\| \to 0. \tag{35}$$

Combining (34) and (35), we deduce

$$\lim_{k\to+\infty}\|z_{n_k} - x_{n_k}\| = 0.$$

Therefore, we have

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| &= \|(1-\lambda)\theta_{n_k}x_{n_k} + \lambda z_{n_k} - x_{n_k}\| \\ &= \|\lambda(z_{n_k} - x_{n_k}) - (1-\theta_{n_k})(1-\lambda)x_{n_k}\| \\ &\le \lambda\|z_{n_k} - x_{n_k}\| + (1-\theta_{n_k})(1-\lambda)\|x_{n_k}\| \to 0.\end{aligned}$$

Since the sequence $\{x_{n_k}\}$ is bounded, without any loss of generality we may assume that $\{s_{n_k}\}$ converges weakly to some $z \in H$, such that

$$\limsup_{k\to\infty}\langle x^*, x_{n_k} - x^*\rangle = \langle x^*, z - x^*\rangle. \tag{36}$$

From (35) we get

$$w_{n_k} \rightharpoonup z,$$

this together with (32), using Lemma 3.2 and assumption $Fx \neq 0\ \forall x \in C$ we obtain $z \in S_D$ and, from (36) and the definition of $x^* = P_{S_D}(0)$, we have

$$\limsup_{k\to\infty}\langle x^*, x_{n_k} - x^*\rangle = \langle x^*, z - x^*\rangle \le 0. \tag{37}$$

Combining (33) and (37), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle x^*, x_{n_k+1} - x^*\rangle &\le \limsup_{k\to\infty}\langle x^*, x_{n_k} - x^*\rangle \\ &= \langle x^*, z - x^*\rangle \\ &\le 0. \end{aligned}\tag{38}$$

Hence, by (38), $\lim_{n\to\infty}\frac{\alpha_n}{1-\theta_n}\|x_n - x_{n-1}\| = 0$. Apply Lemma 2.4 to Claim 4, we obtain $\lim_{n\to\infty}\|x_n - x^*\| = 0$. That is the desired result. □

Remark 3.2. 1. We can choose the sequence $\{\alpha_n\}$ that satisfies condition (13) as follows:

$$\alpha_n = \begin{cases}\min\left\{\alpha, \dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise,}\end{cases}$$

where $\alpha > 0$ and $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = 0(1-\theta_n)$. This means that $\lim_{n\to\infty}\frac{\epsilon_n}{1-\theta_n} = 0$.

2. The parameter $\{\theta_n\}$ satisfies Algorithm 3.1 as follows: $\theta_n = 1 - \frac{1}{(n+1)^p}, 0 < p \le 1$. Then, it is easy to see that $\lim_{n\to+\infty}(1-\theta_n) = 0$ and $\sum_{n=1}^{\infty}(1-\theta_n) = +\infty$.

4. **Conclusions**

In this paper, we investigate a new version of the inertial extragradient algorithm for finding a solution of the variational inequality problem in Hilbert spaces where the operator is assumed to be Lipschitz continuous and quasimonotone. The strong convergence theorem of the proposed algorithm is presented under assumptions of the quasimonotonicity and the Lipschitz continuity of the variational inequality mapping.

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