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GOOD GRADINGS OF CERTAIN ALGEBRAS OVER TORSION-FREE ABELIAN GROUPS

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The paper deals with good gradings over torsion-free Abelian groups. We derive first necessary conditions in order for two good gradings to be isomorphic with respect to the algebra of polynomial ring in n variables. Next we study the case of good gradings over the integer numbers group and give an equivalent condition in order to be isomorphic. Equivalent conditions are also obtained for the first Weyl algebra generated by two elements.

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1. Introduction

The origin of the concept of grading can be found in the theory of polynomials. In fact, the ring of polynomials in one variable with coefficients in some ring is a graded ring by the additive monoid of natural numbers. Also it can be viewed as a graded ring by the additive group $\mathbb{Z}$, where all components of negative degree are zero. Besides gradings by the additive group $\mathbb{Z}$, gradings over $\mathbb{Z}\times\mathbb{Z}\times...\times\mathbb{Z}$ were considered too. Such a grading of the ring of polynomials, where $\mathbb{Z}\times\mathbb{Z}\times...\times\mathbb{Z}$ is viewed as an ordered group, with the lexicographical order, played an important role in proving the fundamental theorem for symmetric polynomials.

There have been further investigated gradings by certain groups. Gradings of the algebra of skew polynomial ring in two variables, by the cyclic group C_n for any $n\in\mathbb{N}$ have been considered in [1]. In paper [2], the authors provide necessary and sufficient conditions for a general G-graded ring to be G-controlled, where G is a given group.

Gradings of the matrix algebra over the cyclic group C_n have been discussed by Dascalescu et al. in [3], such graded algebras also been known as

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superalgebras having many applications in geometry and theoretical physics (see [3], [4]).

We should also mention that the main source of the general concept of graded algebra by arbitrary groups is the representation theory of groups. Many results and methods used in this theory can be reformulated in terms of graded algebras which offer a better understanding of the phenomena. An open problem of remarkable importance was posed by E.Zelmanov ([5]) and it is the following: given a group G and a field K, find all G-gradings of the matrix algebra $M_n(K)$ for n a positive integer fixed. A systematic study of gradings on matrix algebra began in 1998 with the paper of S.Dascalescu et al. ([3]). It was investigated a special class of gradings, the so-called good gradings where the elementary matrices e_{ij} are homogeneous elements. In the same paper, the authors counted all the good gradings, described those which are strongly graded or crossed products and presented the conditions under which a grading is isomorphic to a good grading.

In paper [6], using results from descent theory, Caenepeel et al. have shown how to classify the gradings by a cyclic group of the matrix algebra over an arbitrary field. The classification problem of all gradings by a group of small order was recently approached in a series of works. For instance, in paper [6], Khazal et al. classify all the gradings of $M_2(K)$, over a field K.

Besides the G-gradings of matrix algebra, which are, in fact, structures of $K(G)$-comodule algebra, it could be interesting to study G-gradings of polynomial algebra, but there are a few approaches of this problem so far, one of the reasons being the fact that the problem of determining the automorphisms of $K[X_1,...,X_n]$ is still open. In this paper, we are concerned with good gradings of the polynomial algebra and of the Weyl algebra, over torsion-free Abelian groups.

The paper is organized as follows: basic definitions of graded algebras are presented first in Section 2. Necessary conditions in order for two good gradings to be isomorphic with respect to the algebra of polynomial ring in n variables are obtained and the particular case of the polynomial algebra over the integer numbers group is discussed in Section 3. Section 4 is devoted to the Weil algebra generated by two elements, while Section 5 concludes the paper.

2. Preliminaries

Let K be a field, A be a K-algebra and $(G,*)$ be an Abelian group. G is a torsion-free group if the subgroup $tG=\{g\in G:\exists n\in\mathbb{N}^*, \underbrace{g*g*...*g}_{n}=0\}$

reduces to $\{0\}$. A G- grading of A is a vector space decomposition $A=\bigoplus_{g\in G} A_g$ such that $A_g A_h \subseteq A_{g*h}$, for any $g,h\in G$. In this case, the set $h(A)=\bigcup_{g\in G} A_g$ is the set of homogeneous elements of A and any nonzero element $a\in A_g$ is said to be homogeneous of degree g and we write $\deg(a)=g$. An isomorphism $\varphi: S\to T$ between two G- gradings of A, $S=\bigoplus_{g\in G} S_g$ and $T=\bigoplus_{h\in G} T_h$ is an isomorphism between the corresponding algebras, such that for any $s\in S_g, g\in G$ we have $\varphi(s)\in T_g$.

If A is the algebra of polynomial ring in n variables, $R=K[X_1,X_2,...,X_n]$, a G- grading of R is said to be a good grading if $X_1,X_2,...,X_n$ are homogeneous elements, that is $X_1,X_2,...,X_n \subset h(R)$. We recall that if G is a finite group with m elements, there are m^n good gradings of R.

If A is the first Weil algebra $A_1(K)$ generated by two variables, x,y satisfying $yx-xy=1$, a G- grading of $A_1(K)$ is said to be a good grading if the generators x,y are homogeneous elements, that is $x,y\subset h(A_1(K))$. If G is a finite group with m elements, there are m^2 good gradings of $A_1(K)$.

In the sequel, we study the good gradings over torsion-free Abelian groups, which are not finite groups.

3. Good gradings on the algebra of the polynomial ring

In this section we are concerned with gradings of the algebra of polynomial ring in n variables, $R=K[X_1,X_2,...,X_n]$. In what follows we derive necessary conditions in order for two good gradings over a torsion- free Abelian group to be isomorphic.

Hypothesis 1. Let $S=\bigoplus_{g\in G} S_g$ and $T=\bigoplus_{h\in G} T_h$ be two good gradings of R such that $\deg(X_i)=g_i$ for $i=\overline{1,n}$ with respect to S and $\deg(X_i)=h_i$ for $i=\overline{1,n}$ with respect to T.

Theorem 2. Suppose that Hypothesis 1 is verified and S and T are isomorphic. Denote by F_1 and F_2 the subgroup generated by $\{g_1,g_2,...,g_n\}$ and $\{h_1,h_2,...,h_n\}$, respectively. The following assertions hold:

i) $F_1 = F_2$;

ii) If $rang(F_1) = rang(F_2) = n$ then $\{g_1, g_2, ..., g_n\} = \{h_1, h_2, ..., h_n\}$. Moreover, S and T are isomorphic via the isomorphism $X_i \mapsto X_{\sigma(i)}$, $i = \overline{1,n}$, for any permutation $\sigma \in S_n$.

Proof. i) Any isomorphism of graded algebras S and T, $\varphi : S \to T$ is, in particular, a K-automorphism of $R = K[X_1, X_2, ..., X_n]$. Taking $X_l \in S_{g_l}$ we have $\varphi(X_l) = \sum a_{i_{l_1}} ... a_{i_{l_n}} X_1^{i_{l_1}} X_2^{i_{l_2}} \cdot ... \cdot X_n^{i_{l_n}} \in T_{g_l}$, for any $l = \overline{1,n}$, so we obtain

$$\begin{cases} i_{11}h_1 + ... + i_{1n}h_n = g_1 \\ i_{21}h_1 + ... + i_{2n}h_n = g_2 \\ .... \\ i_{n1}h_1 + ... + i_{nn}h_n = g_n \end{cases}$$

with $i_{rs} \in \mathbb{N}$, for any $r, s = \overline{1,n}$. From the above relation we deduce $F_1 \subset F_2$. Conversely, consider the inverse isomorphism from T to S, $\psi : T \to S$ and we get $F_2 \subset F_1$. Therefore, $F_1 = F_2$.

ii) Suppose that $F_1 = F_2 = F$. Since $rang(F_1) = rang(F_2) = n$, the sets $\{g_1, g_2, ..., g_n\}$ and $\{h_1, h_2, ..., h_n\}$ are basis in F. Let $A = (i_{rl})_{r,l=\overline{1,n}} \in M_n(\mathbb{N})$ and $B = (j_{rl})_{r,l=\overline{1,n}} \in M_n(\mathbb{N})$ be two invertible matrices such that $B\begin{pmatrix} g_1 \\ g_2 \\ ... \\ g_n \end{pmatrix} = A\begin{pmatrix} h_1 \\ h_2 \\ ... \\ h_n \end{pmatrix}$ and $AB = BA = I_n$. We argue by induction over $n \in \mathbb{N}$ that such invertible matrices should have just one non-zero element namely one, on every row and every column, all the others being zero.

Consider first $n = 2$. From the equality $\begin{pmatrix} i_{11} & i_{12} \\ i_{21} & i_{22} \end{pmatrix}\begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, one has: $i_{11}j_{11} + i_{12}j_{21} = 1 = i_{21}j_{12} + i_{22}j_{22}$ and $i_{11}j_{12} + i_{12}j_{22} = 0 = i_{21}j_{11} + i_{22}j_{21}$. Simple computations lead to $A = I_2 = B$ or $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Suppose now that the assertion is true for some $n \in \mathbb{N}$ and we prove that it is also true for $n+1$. From $AB = I_n$ we have $i_{11}j_{11} + i_{12}j_{21} + ... + i_{1n}j_{n1} = 1$. If $i_{11} = j_{11} = 1$ then we get $i_{21} = ... = i_{n1} = 0$ and $j_{12} = ... = j_{1n} = 0$. From $BA = I_n$ we have also that $j_{21}=...=j_{n1}=0$ and $i_{12}=...=i_{1n}=0$. Using again the equality $AB=I_n$ we obtain

$$\begin{pmatrix} i_{22} & \cdots & i_{2n} \\ \cdots & \cdots & \cdots \\ i_{n2} & \cdots & i_{nn} \end{pmatrix}\begin{pmatrix} j_{22} & \cdots & j_{2n} \\ \cdots & \cdots & \cdots \\ j_{n2} & \cdots & j_{nn} \end{pmatrix}=I_{n-1}.$$

According to the inductive hypothesis it results that the matrices from the above equation have the property that on every row and every column, there is just one element equal to 1 while all the others equal zero. Hence we may conclude that $\{g_1,g_2,...,g_n\}=\{h_1,h_2,...,h_n\}$. Similarly, if $i_{11}j_{11}=0$ there exists $l=\overline{1,n}$ such that $i_{1l}=j_{l1}=1$ and, by the same reason as above we get the conclusion.

Let us now investigate the case when $G=\mathbb{Z}^m$. The following result describes the homogeneous elements of a good grading on the algebra of polynomial ring in two variables, $K[X,Y]$.

Proposition 3. Let $G=(\mathbb{Z}^m,+)$ and let $S=\underset{(p_1,...,p_m)\in\mathbb{Z}^m}{\oplus} S_{(p_1,...,p_m)}$ be a $\mathbb{Z}^m$-good grading of polynomial ring in two variables, $R=K[X,Y]$ such that $\deg(X)=(g_1,g_2,...,g_m)$ and $\deg(Y)=(h_1,h_2,...,h_m)$. If $d=(g_1,h_1)\in\mathbb{N}$, there exist $i_0,j_0\in\mathbb{Z}$ such that $S_{(p_1,...,p_m)}\subset\left\langle\left\{X^{i_0+\frac{|h_1|}{d}t}Y^{j_0-\frac{g_1}{d}\mathrm{sgn}(h_1)t},\ t\in\mathbb{N}\right\}\right\rangle$.

Proof. We study first the homogeneous element S_e, $e=(0,0,...,0)\in\mathbb{Z}^m$. From the very definition of a grading we have that X^iY^j is of degree $i(g_1,g_2,...,g_m)+j(h_1,h_2,...,h_m)$ and hence, $X^iY^j\in S_e$ if and only if $i(g_1,g_2,...,g_m)+j(h_1,h_2,...,h_m)=e$. In order to have $R_e\neq K$, the system should admit a non-zero solution. This implies $\begin{vmatrix} g_k & h_k \\ g_l & h_l \end{vmatrix}=0,\ \forall k,l=\overline{1,m}$, that is $\frac{g_k}{h_k}=\frac{g_l}{h_l},\ \forall k,l=\overline{1,m}$. It is sufficient to consider the equation $ig_1+jh_1=0$ and, without loss of generality, we can suppose that $(i,j)=1$. Notice that since $i,j\in\mathbb{N}$, we have $g_1h_1<0$. Putting $d=(g_1,h_1)$ we obtain: $i\frac{g_1}{d}=-j\frac{h_1}{d}$, with

$\frac{g_1}{d}, \frac{h_1}{d} \in \mathbb{Z}$, from where we deduce $i = \frac{|h_1|}{d}$ and $j = \frac{|g_1|}{d}$. We obtain that $S_e = \left\langle \left\{ X^{\frac{|h_1|}{d}t} Y^{\frac{|h_1|}{d}t}, t \in \mathbb{N} \right\} \right\rangle = K\left[X^{\frac{|h_1|}{d}} Y^{\frac{|h_1|}{d}} \right]$.

We are left with the homogeneous element $S_{(p_1,\ldots,p_m)}$, for some $(p_1,\ldots,p_m) \in \mathbb{Z}^m \setminus \{e\}$. As before, $X^i Y^j \in S_{(p_1,\ldots,p_m)}$ if and only if $i(g_1, g_2, \ldots, g_m) + j(h_1, h_2, \ldots, h_m) = (p_1, \ldots, p_m)$. This is a system of m equations in two unknowns which should satisfy the following conditions: $\frac{g_k}{g_l} = \frac{h_k}{h_l} = \frac{p_k}{p_l}, \forall k, l = \overline{1, m}$. The system above reduces to the diophantical equation of degree 1:

$$ig_1 + jh_1 = p_1$$

This equation has integer solutions if and only if $d \mid p_1$, where $d = (g_1, h_1)$. Assume that (i_0, j_0) is a particular solution with $i_0, j_0 \in \mathbb{N}$, i_0 the smallest natural value. Then, equation () is equivalent to $(i - i_0)g_1 + (j - j_0)h_1 = 0$, $i - i_0, j - j_0 \in \mathbb{Z}$. So

$$(i - i_0)\frac{g_1}{d} = -(j - j_0)\frac{h_1}{d} \Rightarrow i - i_0 = \frac{|h_1|}{d}t \text{ and } j - j_0 = -\frac{g_1}{d}\operatorname{sgn}(h_1)t,\ t \in \mathbb{N},$$

provided that $j \in \mathbb{N}$. Finally, we get $i = i_0 + \frac{|h_1|}{d}t$, $j = j_0 - \frac{g_1}{d}\operatorname{sgn}(h_1)t$ and we are done.

Hypothesis 4. Hypothesis 1 is verified for $R = K[X, Y]$, $n = 2$ and $G = (\mathbb{Z}, +)$.

Theorem 5. Suppose that Hypothesis 4 is verified and $S_e, T_e \neq K$. Then S and T are isomorphic if and only if $\{g_1, g_2\} = \{h_1, h_2\}$.

Proof. If $\{g_1, g_2\} = \{h_1, h_2\}$ the result is obvious. Suppose now that S and T are isomorphic via the isomorphism $\varphi : S \to T$. Since $T_e \neq K$, the equation $ih_1 + jh_2 = 0$ has non-zero solution in $\mathbb{N}$. If $h_1 = h_2 = 0$ we get the trivial grading and so, $g_1 = g_2 = 0$. If $(h_1, h_2) \neq (0,0)$, then $\operatorname{sgn}(h_1 h_2) = -1$. Without loss of generality we may consider $h_1 < 0$ and $h_2 > 0$. Writing $\varphi(X) = \sum_{i,j\in\mathbb{N}} a_{ij} X^i Y^j \in T_{g_1}$ and $\varphi(Y) = \sum_{i',j'\in\mathbb{N}} b_{i'j'} X^{i'} Y^{j'} \in T_{g_2}$, we obtain the following system:

$$\begin{cases} ih_1 + jh_2 = g_1 \\ i'h_1 + j'h_2 = g_2 \end{cases}$$

If (i_0, j_0) is the smallest solution of the first equation, with respect to the lexicographical order and $(i_0^{\prime}, j_0^{\prime})$ is the smallest solution of the second equation, from Proposition 3, it follows that:

$$i = i_0 + \frac{h_2}{d}t,\ j = j_0 - \frac{h_1}{d}t,\ t \in \mathbb{N},$$
$$i' = i_0^{\prime} + \frac{h_2}{d}t,\ j' = j_0^{\prime} - \frac{h_1}{d}t,\ t \in \mathbb{N},$$

where $d = (h_1, h_2)$ satisfying $d \mid (g_1, g_2)$. Analogously, considering the inverse isomorphism $\varphi^{-1} : T \to S$, we obtain $(h_1, h_2) = (g_1, g_2)$ and

$$\varphi(X) = X^{i_0} Y^{j_0} \sum_{t\in\mathbb{N}} a_t X^{\frac{h_2}{d}t} Y^{-\frac{h_1}{d}t}$$
$$\varphi(Y) = X^{i_0^{\prime}} Y^{j_0^{\prime}} \sum_{t\in\mathbb{N}} b_t X^{\frac{h_2}{d}t} Y^{-\frac{h_1}{d}t}$$

As φ is a K-automorphism of $K[X,Y] \Rightarrow K[X,Y] = K[\varphi(X), \varphi(Y)]$ and we may write $X = \sum_{p,q\in\mathbb{N}} \varphi^p(X)\varphi^q(Y) = X^{i_0 p + i_0^{\prime} q} Y^{j_0 p + j_0^{\prime} q} \sum_{t\in\mathbb{N}} c_t X^{\frac{h_2}{d}t} Y^{-\frac{h_1}{d}t}$, leading to the following system:

$$\begin{cases} i_0 p + i_0^{\prime} q + \dfrac{h_2}{d} t = 1 \\ j_0 p + j_0^{\prime} q - \dfrac{h_1}{d} t = 0 \end{cases} \quad (1)$$

with $p^2 + q^2 \neq 0$. There are now only three possibilities.

First case. $i_0 p = 1 \Rightarrow i_0 = p = 1;$ *then* $p = 1 \Rightarrow j_0 = 0$. Obviously, we have that $h_1 = g_1$. If $t \neq 0 \Rightarrow h_1 = h_2 = 0$ which is a contradiction with our supposition. We obtain $\varphi(X)=a_0X+X\sum_{t\in\mathbb{N}^*}a_tX^{\frac{h_2}{d}t}Y^{-\frac{h_1}{d}t}$ and writing Y in the form $Y=\sum_{r,s\in\mathbb{N}}\varphi^r(X)\varphi^s(Y)$, yields the following system:

$$\begin{cases} r+i_0^{,}s+\dfrac{h_2}{d}t=0 \\ j_0^{,}s-\dfrac{h_1}{d}t=1 \end{cases}$$

which immediately gives $r=0$. If $j_0^{,}s=1$, then $j_0^{,}=s=1, i_0^{,}=0$ and $h_2=g_2$. If $j_0^{,}s=0$, then $j_0^{,}=i_0^{,}=h_2=g_2=0$.

Second case. $i_0^{,}q=1\Rightarrow i_0^{,}=q=1; j_0^{,}=0,\ h_1=g_2$. Like in the first case we can prove that $h_2=g_1$.

Third case. $\frac{h_2}{d}t=1\Rightarrow h_2=d$ and $t=1$. From (1) we get $h_1=0$ and $i_0p=i_0^{,}q=j_0p=j_0^{,}q=0$. Since $p^2+q^2\neq 0$, we may suppose that $p\neq 0$. It comes that $i_0=j_0=0\Rightarrow g_1=0=h_1$. Hence $\varphi(X)=\sum_{t\in\mathbb{N}}a_tX^t$ and $Y=\sum_{r,s\in\mathbb{N}}\varphi^r(X)\varphi^s(Y)=\sum_{r,s\in\mathbb{N}}\left(\sum_{t\in\mathbb{N}}a_tX^t\right)^r X^{i_0^{,}s}Y^{j_0^{,}s}\left(\sum_{l\in\mathbb{N}}b_lX^l\right)^s$ which implies $j_0^{,}=1$ and $i_0^{,}=0$, so $h_2=g_2$.

The three cases above show that $\{g_1,g_2\}=\{h_1,h_2\}$.

4. Good gradings on the Weyl algebra generated by two elements

In this section we are concerned with good gradings of the first Weyl algebra $A_1(K)$ generated by two variables, x,y satisfying $yx-xy=1$. This property implies that if $\deg(x)=g$ and $\deg(y)=h$ then $h=g^{-1}$. The next result gives an equivalent condition in order for two good gradings over a torsion- free Abelian group G to be isomorphic.

Hypothesis 6. Let $S=\bigoplus_{g\in G} S_g$ and $T=\bigoplus_{h\in G} T_h$ be two good gradings of $A_1(K)$ such that $\deg(x)=g$ and $\deg(y)=g^{-1}$ with respect to S and $\deg(x)=h$ and $\deg(y)=h^{-1}$ with respect to T.

Theorem 7. Suppose that Hypothesis 6 is verified. Then S and T are isomorphic if and only if $g=h$ or $g=h^{-1}$.

Proof. If $g=h$ we define the isomorphism φ as the identity map; if $g=h^{-1}$ the isomorphism is $\varphi(x)=y$, $\varphi(y)=x$ and the result is obvious.

Conversely, assume that S and T are isomorphic via the isomorphism $\varphi: S\to T$. Let $\varphi(x)=\sum_{i,j\in\mathbb{N}} a_{ij}x^i y^j$ and $\varphi(y)=\sum_{p,r\in\mathbb{N}} b_{pr}x^p y^r$. Since $\varphi(x)\in T_g$ and $\varphi(y)\in T_{g^{-1}}$ we get $h^{i-j}=g$ and $h^{p-r}=g^{-1}$. By using the inverse isomorphism ψ from T to S, $\psi(x)=\sum_{u,v\in\mathbb{N}} a_{uv}x^u y^v$ and $\psi(y)=\sum_{z,t\in\mathbb{N}} b_{zt}x^z y^t$, we also have that $g^{u-v}=h$ and $g^{z-t}=h^{-1}$. Then $g^{(u-v)(i-j)}=g$ and $g^{(z-t)(p-r)}=g$. Since G is torsion free, for $g\neq e$ we obtain: $(u-v)(i-j)=1$ and $(z-t)(p-r)=1$ which implies either $g=h$ or $g=h^{-1}$. In the case in which $g=e$ it is straightforward that $h=e=g$. The proof is over.

5. Conclusions

In this paper we studied the good gradings of the polynomial ring and of the Weyl algebra generated by two elements, over torsion- free Abelian groups. In general, proving that two good gradings are isomorphic is rather difficult. However, there are some necessary conditions that must be met in order for them to be isomorphic to each other. Considering good gradings over the additive group of integer numbers, equivalent conditions for two good gradings to be isomorphic are obtained.

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