COMMON FIXED POINT THEOREMS VIA INTEGRAL TYPE CONTRACTION IN MODULAR METRIC SPACE

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The Banach fixed point theory is one of the important results in pure mathematics that Banach proved in 1922. This theory was expanded by several authors in different areas by introducing different contraction conditions. In this work, we extend the Banach fixed point theorem in modular metric spaces by investigating contractive conditions involving integral types. More precisely, we prove some existence and uniqueness theorems of a common fixed point of self mappings satisfying contraction conditions of the integral type. Then, we state some corollaries, and examples to illustrate the validity of our results.

Keywords: Contraction mappings of integral types, Complete modular metric space, Fixed point, Common fixed point.

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1. Introduction

The Banach fixed point Theorem [6] is the first result in the fixed point theory formulated and proven in mathematics by the pioneering mathematician Banach. In functional analysis, a lot of research has contained fixed point theory in different spaces. For some works in fixed point theory, see the following references [2, 3, 4, 5, 9, 16, 18, 24, 26, 27, 28, 29, 30, 31, 32, 35]. In 2002, Branciari [7] presented a new idea for the contraction condition of a fixed point theorem. He proved the existence of a fixed point for mapping satisfying a general contractive condition of integral type on a complete metric space. Then after, Liu, Li, Kang and Cho [22] expanded the idea of Branciari by giving a new result and stated illustrative examples. In 2012, Gupta et al. [16] introduced the idea of a common fixed point theorem for contraction of integral type as below:

Theorem 1.1. [16] *Let $S, T : X \to X$ be self compatible maps of a complete metric space (X, d) satisfying the following conditions:*

(1) $S(X) \subseteq T(X)$,

(2)
$$\int_0^{d(Sx,Sy)} \varphi(t)dt \leq \int_0^{d(Tx,Ty)} \varphi(t)dt - \phi\Big(\int_0^{d(Tx,Ty)} \varphi(t)dt\Big)$$

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$\forall x, y \in X$, where, $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue integrable function which is summable, non negative, and $\int_0^{\epsilon} \varphi(t)dt > 0$ for all $\epsilon > 0$, $\phi : [0, +\infty) \to [0, +\infty)$ is a lower semi continuous and non decreasing function, such that, $\phi(t) = 0$ if and only if $t = 0$.
Then S and T have a unique common fixed point.

For the first time, Chistyakov [10] defined the notion of modular space and presented some ideal applications. He used the theory of modular metric space. Later, several mathematicians extended their study of a fixed point theory in modular metric space, see for examples [8, 11, 12, 13, 14, 15, 21, 25, 33, 34].
In this paper, we investigate some existence and uniqueness theorems of the common fixed point for mappings satisfying contractive condition of the integral type on complete modular metric space.

2. **Preliminaries**

Definition 2.1. [10] *A metric modular on a non empty set X is a function $\omega : (0, \infty) \times X \times X \to [0, \infty)$ that will be written as $\omega_\nu(x, y) = \omega(\nu, x, y)$; for all $x, y, z \in X$ and for all $\nu > 0$, satisfies the following three conditions:*

(1) *$\omega_\nu(x, y) = 0$ if and only if $x = y$, $\forall \nu > 0$ and $x, y \in X$.*
(2) *$\omega_\nu(x, y) = \omega_\nu(y, x)$, $\forall \nu > 0$ and $x, y \in X$.*
(3) *$\omega_{\nu+\sigma}(x, y) \leq \omega_\nu(x, z) + \omega_\sigma(z, y)$; for all $\nu, \sigma > 0$ and $x, y, z \in X$.*

Remark 2.1. *Let ω be a modular on a set X. Then for given $x, y \in X$, the function $0 < \nu \to \omega_\nu(x, y) \in (0, \infty)$ is non increasing on $(0, \infty)$.*
In fact if $0 < \nu < \sigma$, then by above definition

$$\omega_\sigma(x, y) \leq \omega_{\sigma-\nu}(x, x) + \omega_\nu(x, y) = \omega_\nu(x, y)$$

for all $x, y, z \in X$.

Definition 2.2. [12] *Given a modular ω on X, a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω is said to be modular convergent to an element $x \in X_\omega$ if there exists a number $\nu > 0$, possibly depending on $\{x_n\}$ and x, such that $\lim_{n\to\infty} \omega_\nu(x_n, x) = 0$. i.e $x_n \to x$ as $n \to \infty$.*

Definition 2.3. [12] *Given a modular ω on X, a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω is said to be modular Cauchy if there exists a number $\nu = \nu(\{x_n\}) > 0$, such that*

$$\lim_{n,m\to\infty} \omega_\nu(x_n, x_m) = 0.$$

Definition 2.4. [12] *A modular space X_ω is said to be modular complete if each Cauchy sequence in X_ω is modular convergent. In fact, if $\{x_n\} \subset X_\omega$ and there exists $\nu = \nu(\{x_n\}) > 0$ such that*

$$\lim_{n,m\to\infty} \omega_\nu(x_n, x_m) = 0,$$

then there exists $x \in X_\omega$, such that $\lim_{n\to\infty} \omega_\nu(x_n, x) = 0$.

The definition of the coincidence point is given as follows:

Definition 2.5. *Let S and T be two self maps on a set X. If $Sx = Tx$ for some $x \in X$, then x is called a coincidence point of S and T.*

Definition 2.6. [19, 20] *Let S and T be two self maps on a set X. Then S and T are said to be weakly compatiable if they commute at their coincidence point. i.e $TSx = STx$ whenever $Tx = Sx$.*

Lemma 2.1. [19, 20] *Let S and T be weakly compatiable self mappings on a set X. If S and T have a unique point of coincidence u, then u is the unique common fixed point of S and T.*

Notation:
In the rest of this paper, we will consider the following notations:

- Φ_1 is denoted to the family of all functions $\phi : [0, +\infty) \to [0, +\infty)$ such that:
  (1) ϕ is continuous and nondecreasing.
  (2) $\phi(t) = 0$ if and only if $t = 0$.
- Φ_2 is denoted to the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ such that: φ is a Lebesgue integrable function which is summable, non negative, and $\int_0^{\epsilon} \varphi(t)dt > 0$ for all $\epsilon > 0$.
- Φ_3 is denoted to the family of all functions $\psi : [0, +\infty) \to [0, +\infty)$ such that: $\psi(0) = 0$.

Lemma 2.2. [17] *Let $\varphi \in \Phi_2$ and $\{c_n\}_{n\in\mathbb{N}}$ be a sequence with non negative real numbers and $c_n \to c$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \int_0^{c_n} \varphi(t)dt = \int_0^{c} \varphi(t)dt.$$

Lemma 2.3. [23] *Let $\varphi \in \Phi_2$ and $\{c_n\}_{n\in\mathbb{N}}$ be a sequence, which is non negative with $c_n \to c$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \int_0^{c_n} \varphi(t)dt = 0 \ \ iff \ \ \lim_{n\to\infty} c_n = 0.$$

Lemma 2.4. [23] *Let $\phi \in \Phi_1$. Then $\phi(t) > 0 \Leftrightarrow t > 0$.*

Definition 2.7. [1] *A modular ω on X is said to satisfy the Δ_2-condition if $\lim_{n\to\infty} \omega_\nu(x_n, x) = 0$, for some $\nu > 0$ implies that $\lim_{n\to\infty} \omega_\nu(x_n, x) = 0$, for all $\nu > 0$.*

Note that in this paper, we suppose that the modular ω on X satisfies the Δ_2-condition on X.

3. Common fixed point theorems for contractive mappings of integral type in modular metric spaces.

Theorem 3.1. *Let X_ω be a complete modular metric space. Let $S, T : X_\omega \to X_\omega$ be self compatible mappings which satisfy*

(1)
$$S(X_\omega) \subseteq T(X_\omega), \tag{1}$$

(2)
$$\int_0^{\omega_\nu(Sx,Sy)} \varphi(t)dt \leq \alpha(\omega_\nu(Tx, Ty)) \int_0^{\omega_\nu(Tx,Ty)} \varphi(t)dt - \phi\Big(\int_0^{\omega_\nu(Tx,Ty)} \varphi(t)dt\Big) \tag{2}$$

$\forall x, y \in X_\omega$, where $(\phi, \varphi) \in (\Phi_1, \Phi_2)$ and $\alpha : \mathbb{R}^+ \to [0, 1)$ is a function with

$$\limsup_{s\to t} \alpha(s) < 1, \ \ \forall t > 0. \tag{3}$$

Then S and T have a unique common fixed point $u \in X_\omega$.

Proof. Let x_0 be an arbitrary point in X_ω. Since $S(X_\omega) \subseteq T(X_\omega)$, we choose $x_1 \in X_\omega$ such that $Sx_0 = Tx_1$. Continuing this process, we construct a sequence (x_{n+1}) such that $y_n = Tx_{n+1} = Sx_n$, for $n = 0, 1, 2..$
Taking $x = x_n$ and $y = x_{n+1}$.Then (2) implies

$$\int_0^{\omega_\nu(Sx_n,Sx_{n+1})} \varphi(t)dt = \int_0^{\omega_\nu(y_n,y_{n+1})} \varphi(t)dt$$

$$\leq \alpha(\omega_\nu(Tx_n, Tx_{n+1})) \int_0^{\omega_\nu(Tx_n,Tx_{n+1})} \varphi(t)dt - \phi\Big(\int_0^{\omega_\nu(Tx_n,Tx_{n+1})} \varphi(t)dt\Big)$$

$$= \alpha(\omega_\nu(y_{n-1}, y_n))\int_0^{\omega_\nu(y_{n-1},y_n)} \varphi(t)dt - \phi\Big(\int_0^{\omega_\nu(y_{n-1},y_n)} \varphi(t)dt\Big) \tag{4}$$
$$\leq \int_0^{\omega_\nu(y_{n-1},y_n)} \varphi(t)dt, \ \ \forall n \in \mathbb{N}.$$

Thus

$$\int_0^{\omega_\nu(y_n,y_{n+1})} \varphi(t)dt \leq \int_0^{\omega_\nu(y_{n-1},y_n)} \varphi(t)dt.$$

Now, we will prove that

$$\omega_\nu(y_n, y_{n+1}) \leq \omega_\nu(y_{n-1}, y_n), \ \ \forall n \in \mathbb{N}. \tag{5}$$

Let assume (5) is not true. Then there exists $n_0 \in \mathbb{N}$ such that

$$\omega_\nu(y_{n_0-1}, y_{n_0}) \leq \omega_\nu(y_{n_0}, y_{n_0+1}).$$

Therefore,

$$\int_0^{\omega_\nu(y_{n_0-1},y_{n_0})} \varphi(t)dt \leq \int_0^{\omega_\nu(y_{n_0},y_{n_0+1})} \varphi(t)dt$$
$$\leq \alpha(\omega_\nu(y_{n_0-1}, y_{n_0}))\int_0^{\omega_\nu(y_{n_0-1},y_{n_0})} \varphi(t)dt - \phi\Big(\int_0^{\omega_\nu(y_{n_0-1},y_{n_0})} \varphi(t)dt\Big)$$
$$< \int_0^{\omega_\nu(y_{n_0-1},y_{n_0})} \varphi(t)dt,$$

a contradiction. So, we have

$$\omega_\nu(y_n, y_{n+1}) \leq \omega_\nu(y_{n-1}, y_n), \ \ \forall n \in \mathbb{N}.$$

Hence, we deduce that $\{\omega_\nu(y_n, y_{n+1})\}$ is a non increasing sequence. Therefor, there exists a constant $a_0 \geq 0$ such that,

$$\lim_{n\to\infty} \omega_\nu(y_n, y_{n+1}) = a_0.$$

Suppose that $a_0 > 0$, taking limit sup in (4). Then (2) and Lemma (2.2) imply that

$$0 < \int_0^{a_0} \varphi(t)dt = \limsup_{n\to\infty} \int_0^{\omega_\nu(y_n,y_{n+1})} \varphi(t)dt$$
$$\leq \limsup_{n\to\infty} \Big[\alpha(\omega_\nu(y_{n-1}, y_n))\int_0^{\omega_\nu(y_{n-1},y_n)} \varphi(t)dt - \phi\Big(\int_0^{\omega_\nu(y_{n-1},y_n)} \varphi(t)dt\Big)\Big]$$
$$\leq \limsup_{n\to\infty} \alpha(\omega_\nu(y_{n-1}, y_n)) \limsup_{n\to\infty} \int_0^{\omega_\nu(y_{n-1},y_n)} \varphi(t)dt - \limsup_{n\to\infty} \phi\Big(\int_0^{\omega_\nu(y_{n-1},y_n)} \varphi(t)dt\Big)$$
$$< \int_0^{a_0} \varphi(t)dt,$$

which is impossible. Hence $a_0 = 0$; that is,

$$\lim_{n\to\infty} \omega_\nu(y_n, y_{n+1}) = 0. \tag{6}$$

Now, we will prove that $\{y_n\}$ is a Cauchy sequence. Assume not, then there exists $\epsilon > 0$ and two subsequences $\{y_{m(i)}\}$ and $\{y_{n(i)}\}$ such that let $n(i)$ is the smallest integer exceeding $m(i)$ with $n(i) > m(i) > i$ and

$$\omega_\nu(y_{n(i)}, y_{m(i)}) > \epsilon, \ \ \omega_\nu(y_{n(i)-1}, y_{m(i)}) \leq \epsilon, \ \ \forall i \in \mathbb{N}. \tag{7}$$

Note that $\forall i \in \mathbb{N}$, we have

$$\omega_\nu(y_{n(i)}, y_{m(i)}) \leq \omega_{\frac{\nu}{2}}(y_{n(i)}, y_{n(i)-1}) + \omega_{\frac{\nu}{2}}(y_{n(i)-1}, y_{m(i)});$$
$$\omega_\nu(y_{n(i)}, y_{m(i)+1}) \leq \omega_{\frac{\nu}{2}}(y_{n(i)}, y_{m(i)}) + \omega_{\frac{\nu}{2}}(y_{m(i)}, y_{m(i)+1});$$

$$\omega_\nu(y_{n(i)+1}, y_{m(i)+1}) \leq \omega_{\frac{\nu}{2}}(y_{n(i)+1}, y_{n(i)}) + \omega_{\frac{\nu}{2}}(y_{n(i)}, y_{m(i)+1});$$
$$\omega_\nu(y_{n(i)+1}, y_{m(i)+1}) \leq \omega_{\frac{\nu}{2}}(y_{m(i)+1}, y_{m(i)+2}) + \omega_{\frac{\nu}{2}}(y_{m(i)+2}, y_{n(i)+1}). \tag{8}$$

That give us:

$$\epsilon \leq \omega_\nu(y_{n(i)}, y_{m(i)}) \leq \omega_{\frac{\nu}{2}}(y_{n(i)}, y_{n(i)-1}) + \omega_{\frac{\nu}{2}}(y_{n(i)-1}, y_{m(i)});$$
$$|\omega_\nu(y_{n(i)}, y_{m(i)+1}) - \omega_{\frac{\nu}{2}}(y_{n(i)}, y_{m(i)})| \leq \omega_{\frac{\nu}{2}}(y_{m(i)}, y_{m(i)+1});$$
$$|\omega_\nu(y_{n(i)+1}, y_{m(i)+1}) - \omega_{\frac{\nu}{2}}(y_{n(i)}, y_{m(i)+1})| \leq \omega_{\frac{\nu}{2}}(y_{n(i)+1}, y_{n(i)});$$
$$|\omega_\nu(y_{n(i)+1}, y_{m(i)+1}) - \omega_{\frac{\nu}{2}}(y_{m(i)+2}, y_{n(i)+1})| \leq \omega_{\frac{\nu}{2}}(y_{m(i)+1}, y_{m(i)+2}). \tag{9}$$

Now by definition of Δ_2-condition, (6), (7) and (9), we obtain that

$$\epsilon = \lim_{i\to\infty} \omega_\nu(y_{n(i)}, y_{m(i)}) = \lim_{i\to\infty} \omega_\nu(y_{n(i)}, y_{m(i)+1})$$
$$= \lim_{i\to\infty} \omega_\nu(y_{n(i)+1}, y_{m(i)+1}) = \lim_{i\to\infty} \omega_\nu(y_{m(i)+2}, y_{n(i)+1}).$$

Then by (2), we have

$$\int_0^{\omega_\nu(y_{n(i)+1}, y_{m(i)+2})} \varphi(t)dt$$
$$\leq \alpha(\omega_\nu(y_{n(i)}, y_{m(i)+1})) \int_0^{\omega_\nu(y_{n(i)}, y_{m(i)+1})} \varphi(t)dt - \phi\Big(\int_0^{\omega_\nu(y_{n(i)}, y_{m(i)+1})} \varphi(t)dt\Big).$$

Taking limit sup in the above inequality, we obtain

$$0 < \int_0^{\epsilon} \varphi(t)dt = \limsup_{i\to\infty} \int_0^{\omega_\nu(y_{n(i)+1}, y_{m(i)+2})} \varphi(t)dt$$

$$\leq \limsup_{i\to\infty} \Big[\alpha(\omega_\nu(y_{n(i)}, y_{m(i)+1})) \int_0^{\omega_\nu(y_{n(i)}, y_{m(i)+1})} \varphi(t)dt - \phi\Big(\int_0^{\omega_\nu(y_{n(i)}, y_{m(i)+1})} \varphi(t)dt\Big)\Big]$$

$$\leq \limsup_{i\to\infty} \alpha(\omega_\nu(y_{n(i)}, y_{m(i)+1})) \limsup_{i\to\infty} \int_0^{\omega_\nu(y_{n(i)}, y_{m(i)+1})} \varphi(t)dt - \limsup_{i\to\infty} \phi\Big(\int_0^{\omega_\nu(y_{n(i)}, y_{m(i)+1})} \varphi(t)dt\Big)$$
$$\leq \limsup_{r\to\epsilon} \alpha(r) \int_0^{\epsilon} \varphi(t)dt$$
$$< \int_0^{\epsilon} \varphi(t)dt,$$

which is impossible. Hence $\{y_n\}$ is a Cauchy sequence.
Since X_ω is a complete modular metric space, therefore there exists $u \in X_\omega$ such that, $Sx_n \to u$ and $Tx_n \to u$ as $n \to \infty$. Thus we can take $k \in X_\omega$ such that $Tk = u$. Now

$$\int_0^{\omega_\nu(Sx_n, Sk)} \varphi(t)dt \leq \alpha(\omega_\nu(Tx_n, Tk)) \int_0^{\omega_\nu(Tx_n, Tk)} \varphi(t)dt - \phi\Big(\int_0^{\omega_\nu(Tx_n, Tk)} \varphi(t)dt\Big).$$

Letting $n \to \infty$, we obtain

$$\int_0^{\omega_\nu(u, Sk)} \varphi(t)dt = 0.$$

By Lemma (2.3), we have

$$\omega_\nu(u, Sk) = 0 \Rightarrow Sk = u.$$

Hence u is the point of coincidence of S and T.
Finally, we show that u is unique. Assume not, then there exists $v \neq u$ and there exists w

such that $S(w) = T(w) = v$.
By (2), we have

$$\int_0^{\omega_\nu(Tu,Tw)} \varphi(t)dt = \int_0^{\omega_\nu(Su,Sw)} \varphi(t)dt$$

$$\leq \alpha(\omega_\nu(Tu,Tw)) \int_0^{\omega_\nu(Tu,Tw)} \varphi(t)dt - \phi\Big(\int_0^{\omega_\nu(Tu,Tw)} \varphi(t)dt\Big).$$

Thus

$$\int_0^{\omega_\nu(Tu,Tw)} \varphi(t)dt < \int_0^{\omega_\nu(Tu,Tw)} \varphi(t)dt,$$

which is impossible. Hence $u = v$. Then Lemma (2.1) implies that S and T have a unique common fixed point. □

Theorem 3.2. *Let X_ω be a complete modular metric space. Let $S, T : X_\omega \to X_\omega$ be self compatible mappings satisfy*

(1)

$$S(X_\omega) \subseteq T(X_\omega). \tag{10}$$

(2)

$$\phi\Big(\int_0^{\omega_\nu(Sx,Sy)} \varphi(t)dt\Big) \leq \alpha(\omega_\nu(Tx,Ty))\psi\Big(\int_0^{\omega_\nu(Tx,Ty)} \varphi(t)dt\Big) + \beta(\omega_\nu(Tx,Ty))\phi\Big(\int_0^{\omega_\nu(Ty,Sy)} \varphi(t)dt\Big) \tag{11}$$

$\forall x, y \in X_\omega$, where $(\phi, \varphi, \psi) \in (\Phi_1, \Phi_2, \Phi_3)$, $\psi(l) \leq \phi(l)$, $\forall l \in \mathbb{R}^+$, $\alpha, \beta : \mathbb{R}^+ \to [0, 1)$ are functions with $\limsup_{s\to t} \beta(s) < 1$, $\limsup_{s\to t} \alpha(s) < 1$, $\beta(0) = 0$, $\forall t > 0$.

$$\limsup_{s\to t} \frac{\alpha(s)}{1 - \beta(s)} < 1, \ \ \forall t > 0. \tag{12}$$

Then S and T have a unique common fixed point $u \in X_\omega$.

Proof. Let x_0 be an arbitrary point in X_ω. Since $S(X_\omega) \subseteq T(X_\omega)$, we choose $x_1 \in X_\omega$ such that $Sx_0 = Tx_1$. Continuing this process, we construct a sequence (x_{n+1}) in X_ω such that $y_n = Tx_{n+1} = Sx_n$, for $n = 0, 1, 2..$
Taking $x = x_n$ and $y = x_{n+1}$ in (11), we have

$$\phi\Big(\int_0^{\omega_\nu(Sx_n,Sx_{n+1})} \varphi(t)\Big)dt = \phi\Big(\int_0^{\omega_\nu(y_n,y_{n+1})} \varphi(t)dt\Big)$$

$$\leq \alpha(\omega_\nu(y_{n-1}, y_n))\psi\Big(\int_0^{\omega_\nu(y_{n-1},y_n)} \varphi(t)dt\Big) + \beta(\omega_\nu(y_{n-1}, y_n))\phi\Big(\int_0^{\omega_\nu(y_n,y_{n+1})} \varphi(t)dt\Big).$$

Then

$$(1 - \beta(\omega_\nu(y_{n-1}, y_n)))\phi\Big(\int_0^{\omega_\nu(y_n,y_{n+1})} \varphi(t)dt\Big) \leq \alpha(\omega_\nu(y_{n-1}, y_n))\psi\Big(\int_0^{\omega_\nu(y_{n-1},y_n)} \varphi(t)dt\Big).$$

Hence

$$\phi\Big(\int_0^{\omega_\nu(y_n,y_{n+1})} \varphi(t)dt\Big) \leq \frac{\alpha(\omega_\nu(y_{n-1}, y_n))}{1 - \beta(\omega_\nu(y_{n-1}, y_n))}\psi\Big(\int_0^{\omega_\nu(y_{n-1},y_n)} \varphi(t)dt\Big). \tag{13}$$

Thus

$$\phi\Big(\int_0^{\omega_\nu(y_n,y_{n+1})} \varphi(t)dt\Big) \leq \phi\Big(\int_0^{\omega_\nu(y_{n-1},y_n)} \varphi(t)dt\Big).$$

Since ϕ is non decreasing, we have

$$\int_0^{\omega_\nu(y_n,y_{n+1})} \varphi(t)dt \leq \int_0^{\omega_\nu(y_{n-1},y_n)} \varphi(t)dt.$$

Using the same method given in Theorem (3.1), we get $\{\omega_\nu(y_n, y_{n+1})\}$ is a non increasing sequence and

$$\lim_{n\to\infty} \omega_\nu(y_n, y_{n+1}) = 0$$

Now, we will prove that $\{y_n\}$ is a Cauchy sequence. Assume not, then there exists $\epsilon > 0$ and two subsequences $\{y_{m(i)}\}$ and $\{y_{n(i)}\}$ such that for each $i \in \mathbb{N}$, let $n(i)$ be the smallest integer exceeding $m(i)$ such that

$$\omega_\nu(y_{n(i)}, y_{m(i)}) > \epsilon, \;\; \omega_\nu(y_{n(i)-1}, y_{m(i)}) \leq \epsilon, \;\; \forall i \in \mathbb{N}.$$

Hence by the same method given in Theorem (3.1), we have

$$\epsilon = \lim_{i\to\infty} \omega_\nu(y_{n(i)}, y_{m(i)}) = \lim_{i\to\infty} \omega_\nu(y_{n(i)}, y_{m(i)+1})$$
$$= \lim_{i\to\infty} \omega_\nu(y_{n(i)+1}, y_{m(i)+1}) = \lim_{i\to\infty} \omega_\nu(y_{m(i)+2}, y_{n(i)+1}).$$

So in (11) implies that:

$$\phi\Big(\int_0^{\omega_\nu(y_{n(i)+1}, y_{m(i)+2})} \varphi(t)dt\Big) \leq \alpha(\omega_\nu(y_{n(i)}, y_{m(i)+1}))\psi\Big(\int_0^{\omega_\nu(y_{n(i)}, y_{m(i)+1})} \varphi(t)dt\Big)$$
$$+\beta(\omega_\nu(y_{n(i)}, y_{m(i)+1}))\phi\Big(\int_0^{\omega_\nu(y_{m(i)+1}, y_{m(i)+2})} \varphi(t)dt\Big).$$

Taking limit sup in the above inequality, we obtain

$$0 < \phi\Big(\int_0^{\epsilon} \varphi(t)dt\Big) = \phi\Big(\limsup_{i\to\infty} \int_0^{\omega_\nu(y_{n(i)+1}, y_{m(i)+2})} \varphi(t)dt\Big)$$
$$\leq \limsup_{i\to\infty} \Big[\alpha(\omega_\nu(y_{n(i)}, y_{m(i)+1}))\psi\Big(\int_0^{\omega_\nu(y_{n(i)}, y_{m(i)+1})} \varphi(t)dt\Big)$$
$$+\beta(\omega_\nu(y_{n(i)}, y_{m(i)+1}))\phi\Big(\int_0^{\omega_\nu(y_{m(i)+1}, y_{m(i)+2})} \varphi(t)dt\Big)\Big]$$
$$< \phi\Big(\int_0^{\epsilon} \varphi(t)dt\Big),$$

a contradiction. Hence $\{y_n\}$ is a Cauchy sequence.
Since X_ω is a complete modular metric space, then there exists $u \in X_\omega$ such that $Sx_n \to u$ and $Tx_n \to u$ as $n \to \infty$. Thus we can take $k \in X_\omega$ such that $Tk = u$.
By (11), we get

$$\phi\Big(\int_0^{\omega_\nu(Sx_n, Sk)} \varphi(t)dt\Big) \leq \alpha(\omega_\nu(Tx_n, Tk))\psi\Big(\int_0^{\omega_\nu(Tx_n, Tk)} \varphi(t)dt\Big) + \beta(\omega_\nu(Tx_n, Tk))\phi\Big(\int_0^{\omega_\nu(Tk, Sk)} \varphi(t)dt\Big).$$

Letting $n \to \infty$, we obtain

$$\phi\Big(\int_0^{\omega_\nu(u, Sk)} \varphi(t)dt\Big) = 0.$$

By Lemma (2.3), we have

$$\omega_\nu(u, Sk) = 0 \Rightarrow Sk = u.$$

Hence, u is the point of coincidence of S and T.
Finally, we show that u is unique. Assume not, then there exist $v \neq u$ and w such that $S(w) = T(w) = v$.
By (11), we have

$$\phi\Big(\int_0^{\omega_\nu(Tu, Tw)} \varphi(t)dt\Big) = \phi\Big(\int_0^{\omega_\nu(Su, Sw)} \varphi(t)dt\Big)$$
$$\leq \alpha(\omega_\nu(Tu, Tw))\psi\Big(\int_0^{\omega_\nu(Tu, Tw)} \varphi(t)dt\Big) + \beta(\omega_\nu(Tu, Tw))\phi\Big(\int_0^{\omega_\nu(Tw, Sw)} \varphi(t)dt\Big)$$

$$\leq \phi\Big(\int_0^{\omega_\nu(Tu,Sw)} \varphi(t)dt\Big).$$

Thus

$$\int_0^{\omega_\nu(Tu,Tw)} \varphi(t)dt < \int_0^{\omega_\nu(Tu,Tw)} \varphi(t)dt,$$

which is impossible, which give $u = v$. Thus Lemma (2.1) implies that S and T have a unique common fixed point.

□

If we put $\phi(x) = \psi(x) = x$ on Theorem (3.2), we get the following corollaries:

Corollary 3.1. *Let X_ω be a complete modular metric space. Let $S, T : X_\omega \to X_\omega$ be self compatible mappings satisfy*

(1) $S(X_\omega) \subseteq T(X_\omega)$.

(2)

$$\int_0^{\omega_\nu(Sx,Sy)} \varphi(t)dt \leq \alpha(\omega_\nu(Tx,Ty))\int_0^{\omega_\nu(Tx,Ty)} \varphi(t)dt + \beta(\omega_\nu(Tx,Ty))\int_0^{\omega_\nu(Ty,Sy)} \varphi(t)dt$$

$\forall x, y \in X_\omega$, *where* $\varphi \in \Phi_2$, $\alpha, \beta : \mathbb{R}^+ \to [0,1)$ *are functions with* $\limsup_{s\to t} \beta(s) < 1$, $\limsup_{s\to t} \alpha(s) < 1$, $\beta(0) = 0$, $\forall t > 0$.

$$\limsup_{s\to t} \frac{\alpha(s)}{1-\beta(s)} < 1, \ \ \forall t > 0.$$

Then S and T have a unique common fixed point $u \in X_\omega$.

Corollary 3.2. *Let X_ω be a complete modular metric space. Let $S, T : X_\omega \to X_\omega$ be self compatible mappings satisfy*

(1) $S(X_\omega) \subseteq T(X_\omega)$,

(2)

$$\int_0^{\omega_\nu(Sx,Sy)} \varphi(t)dt \leq \lambda(\omega_\nu(Tx,Ty))\Big(\int_0^{\omega_\nu(Tx,y)} \varphi(t)dt + \int_0^{\omega_\nu(Ty,Sy)} \varphi(t)dt\Big)$$

$\forall x, y \in X_\omega$, *where* $\varphi \in \Phi_2$, $\lambda : \mathbb{R}^+ \to [0, \frac{1}{2})$ *is a function with*

$$\limsup_{s\to t} \frac{\lambda(s)}{1-\lambda(s)} < 1, \ \ \forall t > 0.$$

Then S and T have a unique common fixed point $u \in X_\omega$.

In this example, we illustrate the equality for theorem (3.1).

Example 3.1. *Let $X_\omega = \mathbb{R}^+$ and $\omega_\nu(x,y) = \frac{|x-y|}{\nu}$. Let $S, T : \mathbb{R}^+ \to \mathbb{R}^+$ be two mappings, $\alpha : \mathbb{R}^+ \to [0,1)$ and $(\phi, \varphi) \in (\Phi_1, \Phi_2)$ defined by $S(x) = \frac{x}{1+x}$, $T(x) = \frac{2x}{1+x}$, $\forall x \in \mathbb{R}^+$, $\alpha(t) = \frac{1}{2}$, $\phi(t) = \frac{t}{4}$, and $\varphi(t) = 2t$, $\forall t \in \mathbb{R}^+$.*
By definition of T and S we can easily check that is T and S are self compatible mappings and $S(X_\omega) \subseteq T(X_\omega)$.
Now, we have:

$$\int_0^{\omega_\nu(Sx,Sy)} \varphi(t)dt = \int_0^{\frac{1}{\nu}|\frac{x}{1+x} - \frac{y}{1+y}|} 2tdt$$

$$= \frac{1}{\nu^2}\left(\frac{x-y}{(1+x)(1+y)}\right)^2$$

$$= \frac{2}{\nu^2}\frac{(x-y)^2}{(1+x)^2(1+y)^2} - \frac{1}{\nu^2}\frac{(x-y)^2}{(1+x)^2(1+y)^2}$$

$$= \frac{1}{2} \times \frac{4}{\nu^2} \frac{(x-y)^2}{(1+x)^2(1+y)^2} - \frac{1}{4} \times \frac{4}{\nu^2} \frac{(x-y)^2}{(1+x)^2(1+y)^2}$$

$$= \alpha(\omega_\nu(Tx,Ty)) \int_0^{\omega_\nu(Tx,Ty)} \varphi(t)dt - \phi\Big(\int_0^{\omega_\nu(Tx,Ty)} \varphi(t)dt\Big).$$

Then S and T have a unique common fixed point $0 \in X_\omega$.

Example 3.2. *Let $X_\omega = [1,7]$ be a modular metric space with $\omega_\nu(x,y) = \frac{|x-y|}{\nu}$.
Let $S,T : [1,7] \to [1,7]$ be two mappings, $(\phi,\varphi,\psi) \in (\Phi_1,\Phi_2,\Phi_3)$, and $\alpha,\beta : \mathbb{R}^+ \to [0,1)$, define by*

$$S(x) = \begin{cases} 1 & \text{for} \quad x \in [1,4) \\ \frac{x}{4} & \text{for} \quad x \in [4,7] \end{cases},$$

$$T(x) = \begin{cases} 1 & \text{for} \quad x \in [1,3) \\ \frac{x}{3} & \text{for} \quad x \in [3,7] \end{cases},$$

and $\varphi(t) = 2t$, $\phi(t) = \psi(t) = t$, $\alpha(t) = \frac{9}{16}$ and

$$\beta(t) = \begin{cases} 0 & \text{for} \quad t = 0 \\ \frac{6}{16} & \text{for} \quad t \neq 0 \end{cases}.$$

*By definition of T and S we can easily check that is T and S are self compatible mappings and we have:
$S(X_\omega) = \{1\} \cup [1, \frac{7}{4}] = [1, \frac{7}{4}] \subseteq [1, \frac{7}{3}] = T(X_\omega)$. So we obtain $S(X_\omega) \subseteq T(X_\omega)$. Also we have $\psi(l) \le \phi(l), \forall l \in \mathbb{R}^+$, $\limsup_{s\to t} \frac{\alpha(s)}{1-\beta(s)} < 1, \forall t > 0$.
To verify (11), we divided the example to the different case as follow:*

Case1: *$x,y \in [4,7]$ and $x \le y$. Note that*

$$\int_0^{\omega_\nu(Sx,Sy)} \varphi(t)dt = \int_0^{\frac{1}{\nu}|\frac{y}{4}-\frac{x}{4}|} 2tdt = \frac{1}{(4\nu)^2}(y-x)^2;$$

$$\int_0^{\omega_\nu(Tx,Ty)} \varphi(t)dt = \int_0^{\frac{1}{\nu}|\frac{y}{3}-\frac{x}{3}|} 2tdt = \frac{1}{(3\nu)^2}(y-x)^2;$$

$$\int_0^{\omega_\nu(Ty,Sy)} \varphi(t)dt = \int_0^{\frac{1}{\nu}|\frac{y}{3}-\frac{y}{4}|} 2tdt = \Big(\frac{1}{12\nu}y\Big)^2.$$

So, we have

$$\int_0^{\omega_\nu(Sx,Sy)} \varphi(t)dt = \frac{1}{(4\nu)^2}(y-x)^2 \le \frac{150}{256} \times \frac{1}{(3\nu)^2}(y-x)^2$$

$$= \frac{9\times 16 + 6}{16^2} \times \frac{1}{(3\nu)^2}(y-x)^2$$

$$= \frac{9}{16} \times \frac{1}{(3\nu)^2}(y-x)^2 + \frac{6}{16} \times \frac{1}{(12\nu)^2}(y-x)^2$$

$$\le \frac{9}{16} \times \frac{1}{(3\nu)^2}(y-x)^2 + \frac{6}{16} \times \Big(\frac{1}{12\nu}y\Big)^2$$

$$= \alpha(\omega_\nu(Tx,Ty))\psi\Big(\int_0^{\omega_\nu(Tx,Ty)} \varphi(t)dt\Big) + \beta(\omega_\nu(Tx,Ty))\phi\Big(\int_0^{\omega_\nu(Ty,Sy)} \varphi(t)dt\Big).$$

Case2: *$x,y \in [1,3)$. Notice that $T(x) = S(x) = T(y) = S(y) = 1$. It follows that*

$$\int_0^{\omega_\nu(Sx,Sy)} \varphi(t)dt = 0$$

$$\le \alpha(\omega_\nu(Tx,Ty))\psi\Big(\int_0^{\omega_\nu(Tx,Ty)} \varphi(t)dt\Big) + \beta(\omega_\nu(Tx,Ty))\phi\Big(\int_0^{\omega_\nu(Ty,Sy)} \varphi(t)dt\Big).$$

Case3: *$x, y \in [3,4)$. Notice that $S(x) = S(y) = 1$, $T(x) = \frac{x}{3}$ and $T(y) = \frac{y}{3}$. It follows that*

$$\int_0^{\omega_\nu(Sx,Sy)} \varphi(t)dt = 0$$

$$\leq \alpha(\omega_\nu(Tx,Ty))\psi\Big(\int_0^{\omega_\nu(Tx,Ty)} \varphi(t)dt\Big) + \beta(\omega_\nu(Tx,Ty))\phi\Big(\int_0^{\omega_\nu(Ty,Sy)} \varphi(t)dt\Big).$$

Case4: *$x \in [1,3)$ and $y \in [4,7]$. Note that*

$$\int_0^{\omega_\nu(Sx,Sy)} \varphi(t)dt = \int_0^{\frac{1}{\nu}|\frac{y}{4}-1|} 2tdt = \frac{1}{(4\nu)^2}(y-4)^2;$$

$$\int_0^{\omega_\nu(Tx,Ty)} \varphi(t)dt = \int_0^{\frac{1}{\nu}|\frac{y}{3}-1|} 2tdt = \frac{1}{(3\nu)^2}(y-3)^2;$$

$$\int_0^{\omega_\nu(Ty,Sy)} \varphi(t)dt = \int_0^{\frac{1}{\nu}|\frac{y}{3}-\frac{y}{4}|} 2tdt = \Big(\frac{1}{12\nu}y\Big)^2.$$

So, we have

$$\begin{aligned}\int_0^{\omega_\nu(Sx,Sy)} \varphi(t)dt &= \frac{1}{(4\nu)^2}(y-4)^2 \leq \frac{1}{(4\nu)^2}(y-3)^2 \\ &\leq \frac{150}{256} \times \frac{1}{(3\nu)^2}(y-3)^2 \\ &= \frac{9\times 16+6}{16^2} \times \frac{1}{(3\nu)^2}(y-3)^2 \\ &= \frac{9}{16} \times \frac{1}{(3\nu)^2}(y-3)^2 + \frac{6}{16} \times \frac{1}{(12\nu)^2}(y-3)^2 \\ &\leq \frac{9}{16} \times \frac{1}{(3\nu)^2}(y-3)^2 + \frac{6}{16} \times \Big(\frac{1}{12\nu}y\Big)^2\end{aligned}$$

$$= \alpha(\omega_\nu(Tx,Ty))\psi\Big(\int_0^{\omega_\nu(Tx,Ty)} \varphi(t)dt\Big) + \beta(\omega_\nu(Tx,Ty))\phi\Big(\int_0^{\omega_\nu(Ty,Sy)} \varphi(t)dt\Big).$$

Case5: *$x \in [3,4)$ and $y \in [4,7]$. Note that*

$$\int_0^{\omega_\nu(Sx,Sy)} \varphi(t)dt == \int_0^{\frac{1}{\nu}|\frac{y}{4}-1|} 2tdt = \frac{1}{(4\nu)^2}(y-4)^2;$$

$$\int_0^{\omega_\nu(Tx,Ty)} \varphi(t)dt = \int_0^{\frac{1}{\nu}|\frac{y}{3}-\frac{x}{3}|} 2tdt = \frac{1}{(3\nu)^2}(y-x)^2;$$

$$\int_0^{\omega_\nu(Ty,Sy)} \varphi(t)dt = \int_0^{\frac{1}{\nu}|\frac{y}{3}-\frac{y}{4}|} 2t = \Big(\frac{1}{12\nu}y\Big)^2.$$

So, we have

$$\begin{aligned}\int_0^{\omega_\nu(Sx,Sy)} \varphi(t)dt &= \frac{1}{(4\nu)^2}(y-4)^2 \leq \frac{1}{(4\nu)^2}(y-x)^2 \\ &\leq \frac{150}{256} \times \frac{1}{(3\nu)^2}(y-x)^2 \\ &= \frac{9\times 16+6}{16^2} \times \frac{1}{(3\nu)^2}(y-x)^2 \\ &= \frac{9}{16} \times \frac{1}{(3\nu)^2}(y-x)^2 + \frac{6}{16} \times \frac{1}{(12\nu)^2}(y-x)^2 \\ &\leq \frac{9}{16} \times \frac{1}{(3\nu)^2}(y-x)^2 + \frac{6}{16} \times \Big(\frac{1}{12\nu}y\Big)^2\end{aligned}$$

$$= \alpha(\omega_\nu(Tx, Ty))\psi\Big(\int_0^{\omega_\nu(Tx,Ty)} \varphi(t)dt\Big) + \beta(\omega_\nu(Tx, Ty))\phi\Big(\int_0^{\omega_\nu(Ty,Sy)} \varphi(t)dt\Big).$$

For $y \leq x$ we use the same method.
Then S and T have a unique common fixed point $1 \in X_\omega$.

Conclusion: In this paper, we formulated and proved many common fixed point results for mappings satisfying contractive conditions of integral forms over modular metric spaces. Some examples have been constructed to show the validity of our results.

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