U.P.B. Sci. Bull., Series A, Vol. 72, Iss. 4, 2010 ISSN 1223-7027

BIDIMENSIONAL BROUWERIAN ALGEBRAS

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Noţiunea de D-algebră Brouweriană (BrD-algebră) este introdusă printr-o algebră de tip $\tau_D = (2,2,2,2,0,0)$ *cu patru operaţii binare* $\vee, \wedge, \dot{\rightarrow}, \dot{-}$ *şi două constante 0, 1. O BrD-algebră este o algebră* $\mathbf{A} = (A, \vee, \wedge, \dot{\rightarrow}\, \dot{-}, 0, 1)$ *de tip* τ_D *dintr-o varietate specială* $\mathcal{D}$*. Introducem o definiţie ecuaţională şi un principiu al dualităţii pentru varietatea* $\mathcal{D}$*. Se prezintă proprietăţi esenţiale pentru* $\mathcal{D}$*. Orice BrD-algebră* **A** *este izomorfă cu un produs subdirect al unei perechi de algebre* $(\bar{\boldsymbol{H}}, \bar{\boldsymbol{B}})$ *de tip* τ_D *termen echivalente respectiv cu o algebră Heyting* **H** *şi cu o algebră Heyting duală* **B***. Clasa* $\mathcal{D}$ *este o varietate minimală de algebre incluzând algebrele Heyting şi algebrele Heyting duale. Varietatea* $\mathcal{D}$ *a BrD-algebrelor este aritmetică şi obiectele sale injective sunt algebrele Boole complete.*

The notion of Brouwerian D-algebra (BrD-algebra) is introduced by an algebra of type $\tau_D = (2,2,2,2,0,0)$ *with four binary operations* $\vee, \wedge, \dot{\rightarrow}, \dot{-}$ *and two constants 0, 1. A BrD-algebra is an algebra* $\mathbf{A} = (A, \vee, \wedge, \dot{\rightarrow}\, \dot{-}, 0, 1)$ *of type* τ_D *from a special variety* $\mathcal{D}$*. We introduce an equational definition and a duality principle for the variety* $\mathcal{D}$*. Basic algebraic properties of* $\mathcal{D}$ *are presented. Every BrD-algebra* **A** *is isomorphic to a subdirect product of a pair of algebras* $(\bar{\boldsymbol{H}}, \bar{\boldsymbol{B}})$ *of type* τ_D *termwise definitionally equivalent respectively to a Heyting algebra* **H** *and a dual Heyting algebra* **B***. The class* $\mathcal{D}$ *is a minimal variety of algebras including Heyting algebras and dual Heyting algebras. The variety* $\mathcal{D}$ *of all BrD-algebras is arithmetical and its injective objects are complete Boolean algebras.*

Keywords: Boolean algebra, Heyting algebra, intuitionistic logic, modal operator, arithmetical variety.

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This paper is near the talk ”A minimal variety including bounded Brouwerian algebras” in BLAST 2009 Conference, New Mexico State University, USA [http://subsessile.nmsu.edu/blast].

1. Introduction

The notion of D-algebra is introduced in [37, 38] to express properties of a Kolmogorov calculus of problems [23] with two components. This structure will be called a *Brouwerian D-algebra* (*BrD-algebra*). Any BrD-algebra is a bidimensional Brouwerian algebra in the sense that it is an algebra isomorphic to a subdirect product of a couple of structures associated respectively with a Heyting algebra and a dual Heyting algebra. In this paper some basic properties of BrD-algebras are presented.

We will use notions of universal algebra and lattice theory [3,5,7,12,13,22, 24,31-36]. Let K be the class of algebras with a finite sequence of binary operations, operators and constants. A type τ of algebras in K is a decreasing finite sequence of integer numbers of the set $\{0, 1, 2\}$. Let t(K) be the set of all types τ of algebras in K. The class of all algebras of type τ is denoted by $K(\tau)$, for all $\tau \in t(K)$. For any algebra $\boldsymbol{A} \in K(\tau)$, we denote by $Con(\boldsymbol{A})$ the set of all congruences of $\boldsymbol{A}$. We present now basic definitions and notations.

Definition 1.1. A *Boolean algebra* is an algebra $\boldsymbol{B} = (B, \vee, \wedge, ^c, 0, 1)$ of type $\tau_B = (2,2,1,0,0)$ such that the reduct $L(\boldsymbol{B}) = (B, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and for all $x \in B, x \wedge x^c = 0$ and $x \vee x^c = 1$.

Definition 1.2. A *Heyting algebra* is an algebra $\boldsymbol{H} = (H, \vee, \wedge, \rightarrow, 0, 1)$ of type $\tau_H = (2,2,2,0,0)$ such that the reduct $L(\boldsymbol{H}) = (H, \vee, \wedge, 0, 1)$ is a bounded lattice and the following condition holds, for all $x, y, z \in H$:

(H) $z \leq x \rightarrow y$ if and only if $z \wedge x \leq y$.

Definition 1.3. A *dual Heyting algebra* (*Brouwer algebra*) is an algebra $\boldsymbol{B} = (B, \vee, \wedge, -, 0, 1)$ of type τ_H such that the reduct $L(\boldsymbol{B}) = (B, \vee, \wedge, 0, 1)$ is a bounded lattice and the following condition holds, for all $x, y, z \in B$:

(H^o) $x - y \leq z$ if and only if $x \leq y \vee z$.

Let $\mathcal{B}$, $\mathcal{H}$ and $\mathcal{B}r$ be respectively the varieties of Boolean algebras, Heyting algebras and Brouwer algebras. For any type τ and every class $\mathcal{C} \subseteq K(\tau)$, let $\langle \mathcal{C} \rangle$ be the variety generated by $\mathcal{C}$.

2. An equational definition

Let $\tau_D = (2,2,2,2,0,0)$ be the type of algebras with binary operations $\vee, \wedge, \dot{\rightarrow}, \dot{-}$ and constants 0, 1. We introduce a variety $\mathcal{D} \subseteq K(\tau_D)$ of algebras called *Brouwerian D-algebras* (*BrD-algebras*).

Definition 2.1. For any $\boldsymbol{B} = (B, \vee, \wedge, ^c, 0, 1) \in \mathcal{B}$, define

$$\overline{\boldsymbol{B}} = (B, \vee, \wedge, \dot{\rightarrow}, \dot{-}, 0, 1) \in K(\tau_D),$$

where for all $x, y \in B, x \dot{\rightarrow} y = x^c \vee y$ and $x \dot{-} y = x \wedge y^c$.

Definition 2.2. For any $\boldsymbol{H} = (H, \vee, \wedge, \rightarrow, 0, 1) \in \mathcal{H}$, define

$$\overline{\boldsymbol{H}} = (H, \vee, \wedge, \dot{\rightarrow}, \dot{-}, 0, 1) \in K(\tau_D),$$

where for all $x, y \in H, x \dot{\rightarrow} y = x \rightarrow y$ and $x \dot{-} y = (x \rightarrow y) \rightarrow 0$.

Definition 2.3. For any $\boldsymbol{B} = (B, \vee, \wedge, \rightarrow, 0, 1) \in \mathcal{B}r$, define

$$\overline{\boldsymbol{B}} = (B, \vee, \wedge, \dot{\rightarrow}, \dot{-}, 0, 1) \in K(\tau_D),$$

where for all $x, y \in B, x \dot{\rightarrow} y = 1 - (x - y)$ and $x \dot{-} y = x - y$.

Remarks 2.4. *Define the classes*:

$$\overline{\mathcal{B}} = \left\{ \overline{\boldsymbol{B}} / \boldsymbol{B} \in \mathcal{B} \right\}, \overline{\mathcal{H}} = \left\{ \overline{\boldsymbol{H}} / \boldsymbol{H} \in \mathcal{H} \right\} \text{ and } \overline{\mathcal{B}r} = \left\{ \overline{\boldsymbol{B}} / \boldsymbol{B} \in \mathcal{B}r \right\}.$$

Then the following conditions hold:

(i) $\overline{\mathcal{B}} = \overline{\mathcal{H}} \cap \overline{\mathcal{B}r}$.

(ii) *The three classes* $\overline{\mathcal{B}}, \overline{\mathcal{H}}$ *and* $\overline{\mathcal{B}r}$ *are varieties of algebras from* $K(\tau_D)$ *termwise definitionally equivalent respectively to* $\mathcal{B}, \mathcal{H}$ *and* $\mathcal{B}r$.

Notations 2.5. For any $\boldsymbol{A} = (A, \vee, \wedge, \dot{\rightarrow}, \dot{-}, 0, 1) \in K(\tau_D)$, we introduce the following notations, for all $x, y \in A$:

(1) $\lambda(x) = 1 \dot{\rightarrow} x$, (2) $\rho(x) = x \dot{-} 0$, (3) $\dot{\neg} x = x \dot{\rightarrow} 0$,

(4) $x \wedge^o y = \lambda(x \wedge y)$, (5) $x \mathbin{\overline{\triangledown}} y = \rho(x \vee y)$, (6) $\lambda(A) = \left\{ \lambda(x) / x \in A \right\}$,

(7) $\rho(A) = \left\{ \rho(x) / x \in A \right\}$.

An equational definition of $\mathcal{D}$ is given. Then different rules of computation are derived in order to prove that $\mathcal{D}$ is the variety of algebras in $K(\tau_D)$ generated by the class $\overline{\mathcal{H}} \cup \overline{\mathcal{B}r}$.

Definition 2.6. Let $\mathcal{D}$ be the variety of all algebras **A** of type τ_D satisfying the following conditions:

(i) The reduct $L(\boldsymbol{A}) = (A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice with zero 0 and one 1.

(ii) The following equations hold, for all $x, y \in A$:

(d1) $x \dot{\rightarrow} (x \vee y) = 1$; (d2) $\lambda(x \vee y) = \lambda(x) \vee \lambda(y)$

(d3) $\lambda(x \wedge (x \dot{\rightarrow} y)) = \lambda(x \wedge y)$; (d4) $x \dot{\rightarrow} (y \dot{\rightarrow} z) = (x \wedge y) \dot{\rightarrow} z$

(d5) $x \dot{\to} y = \lambda(x) \dot{\to} y$; (d6) $x \dot{\to} (y \dot{-} z) = 1 \dot{-} (x \wedge (y \dot{\to} z))$
(d7) $x = \lambda(x) \vee (\rho(x) \wedge \dot{\neg} x)$; (d1°) $(x \wedge y) \dot{-} y = 0$

(d2°) $\rho(x \wedge y) = \rho(x) \wedge \rho(y)$; (d3°) $\rho((x \dot{-} y) \vee y) = \rho(x \vee y)$
(d4°) $(x \dot{-} y) \dot{-} z = x \dot{-} (y \vee z)$; (d5°) $x \dot{-} y = x \dot{-} \rho(y)$
(d6°) $(x \dot{\to} y) \dot{-} z = \dot{\neg}((x \dot{-} y) \vee z)$;(d7°) $x = \rho(x) \wedge (\lambda(x) \vee (1 \dot{-} x))$

The equations (d1°)-(d7°) are called dual of (d1) − (d7).

Definition 2.7. (i) A *Brouwerian D-algebra* (*BrD-algebra*) is an algebra $\boldsymbol{A} = (A, \vee, \wedge, \dot{\to}, \dot{-}, 0, 1) \in \mathcal{D}$.

(ii) For any $\boldsymbol{A} \in K(\tau_D)$, let $\tilde{\boldsymbol{A}} = (A, \tilde{\vee}, \tilde{\wedge}, \tilde{\dot{\to}}, \tilde{\dot{-}}, \tilde{0}, \tilde{1}) \in K(\tau_D)$ such that for all $x, y \in A$,

$$x \tilde{\vee} y = x \wedge y, x \tilde{\wedge} y = x \vee y, x \tilde{\dot{\to}} y = y \dot{-} x, x \tilde{\dot{-}} y = y \dot{\to} x, \tilde{0} = 1, \tilde{1} = 0 .$$

(iii) For any $\boldsymbol{A} \in \mathcal{D}$, let $\leq$ be the order relation on A corresponding to the lattice reduct $L(\boldsymbol{A})$.

The next lemma follows from Definition 2.6. A duality principle for the variety $\mathcal{D}$ is derived. Then the exact relation between BrD-algebras and the two kind of structures of Heyting algebra and Brouwer algebra is established.

Lemma 2.8. $\boldsymbol{A} \in \mathcal{D}$ implies $\tilde{\boldsymbol{A}} \in \mathcal{D}$.

A duality principle.

Let $\mathcal{L}_{\mathcal{D}}$ be the language of BrD-algebras $\vee, \wedge, \dot{\to}, \dot{-}, 0, 1$, V be the set of variables and $T[\mathcal{L}_{\mathcal{D}}]$ be the set of terms. The corresponding algebra of terms $\boldsymbol{T}[\mathcal{L}_{\mathcal{D}}] = (T[\mathcal{L}_{\mathcal{D}}], \vee, \wedge, \dot{\to}, \dot{-}, 0, 1)$ is an algebra in $K(\tau_D)$ with the free generating set V. Let $\tilde{\boldsymbol{T}}[\mathcal{L}_{\mathcal{D}}] = (T[\mathcal{L}_{\mathcal{D}}], \tilde{\vee}, \tilde{\wedge}, \tilde{\dot{\to}}, \tilde{\dot{-}}, \tilde{0}, \tilde{1})$ be the algebra of type τ_D defined using Definition 2.7 (ii). *Then there exists a unique extension of the identitity function $id_V : V \to V$ to an isomorphism $^{\circ} : \boldsymbol{T}[\mathcal{L}_{\mathcal{D}}] \to \tilde{\boldsymbol{T}}[\mathcal{L}_{\mathcal{D}}]$. The following conditions hold, for all $\sigma, \tau \in T[\mathcal{L}_{\mathcal{D}}]$:*

(1) σ° *coincides with* σ, *if* $\sigma \in V$,

(2) 0° (1°) *is the constant* 1 (0),

(3) $(\sigma \vee \tau)^{\circ}$, $(\sigma \wedge \tau)^{\circ}$, $(\sigma \dot{\to} \tau)^{\circ}$ *and* $(\sigma \dot{-} \tau)^{\circ}$ *are respectively the terms* $\sigma^{\circ} \wedge \tau^{\circ}$, $\sigma^{\circ} \vee \tau^{\circ}$, $\tau^{\circ} \dot{-} \sigma^{\circ}$ *and* $\tau^{\circ} \dot{\to} \sigma^{\circ}$.

A duality principle follows from Lemma 2.8: *for all terms $\sigma, \tau \in T[\mathcal{L}_{\mathcal{D}}]$, if the equation $\sigma = \tau$ is valid in $\mathcal{D}$ then the equation $\sigma^{\circ} = \tau^{\circ}$ is valid in $\mathcal{D}$.*

Lemma 2.9. *Let $\boldsymbol{A} \in \mathcal{D}$. Then, for all $x, y \in A$:*

(i) $x \le y$ *implies* $\lambda(x) \le \lambda(y)$, ($ii$) $\lambda(\lambda(x)) = \lambda(x) \le x$, ($iii$) $\lambda(0) = 0$,
(iv) $x \dot{\to} x = 1$, (v) $\lambda(1) = 1$,
(vi) $\lambda(x) \dot{\to} \lambda(y) = x \dot{\to} \lambda(y) = x \dot{\to} y = \lambda(x \dot{\to} y)$,
(vii) $\lambda(x \wedge y) = \lambda(\lambda(x) \wedge y) = \lambda(\lambda(x) \wedge \lambda(y))$,
($viii$) $x \dot{\to} 0 = \lambda(x) \dot{\to} 0 = 1 \dot{-} x$,
(ix) $\lambda(x) \dot{-} \lambda(y) = ((\lambda(x) \dot{\to} 0) \vee \lambda(y)) \dot{\to} 0$.

Proof. (i) If $x \le y$ then $x \vee y = y$, thus using equation (d2) one obtains $\lambda(x) \le \lambda(y)$.

(ii) From equation (d4) and (d7) one derives

$$\lambda(\lambda(x)) = 1 \dot{\to} (1 \dot{\to} x) = (1 \wedge 1) \dot{\to} x = 1 \dot{\to} x = \lambda(x) \le x.$$

(iii)-(v) From (ii) and equation (d1) setting y = 0 one derives (iii) and (iv). The condition (v) follows from (iv).

(vi) Equations (d4) and (d5) imply

$$\lambda(x) \dot{\to} \lambda(y) = x \dot{\to} \lambda(y) = x \dot{\to} (1 \dot{\to} y) =$$
$$(x \wedge 1) \dot{\to} y = x \dot{\to} y = 1 \dot{\to} (x \dot{\to} y) = \lambda(x \dot{\to} y).$$

(vii) Relations (vi), equation (d3) and Definition 2.6 (i) imply:

$$\lambda(x \wedge y) = \lambda(x \wedge (x \dot{\to} y)) = \lambda(x \wedge (x \dot{\to} \lambda(y))) =$$
$$\lambda(x \wedge \lambda(y)) = \lambda(\lambda(y) \wedge (\lambda(y) \dot{\to} x)) =$$
$$\lambda(\lambda(y) \wedge (\lambda(y) \dot{\to} \lambda(x))) = \lambda(\lambda(x) \wedge \lambda(y)).$$

(viii) Equation ($d1^o$) for y = x implies $x \dot{-} x = 0$. Then equations (d5), (d6) and (iv) imply

$$x \dot{\to} 0 = \lambda(x) \dot{\to} 0,$$
$$x \dot{\to} 0 = x \dot{\to} (x \dot{-} x) = 1 \dot{-} (x \wedge (x \dot{\to} x)) = 1 \dot{-} (x \wedge 1) = 1 \dot{-} x.$$

(ix) From equations ($d6^o$) and (viii) one derives:

$$\lambda(x) \dot{-} \lambda(y) = (1 \dot{\to} x) \dot{-} \lambda(y) = \dot{\neg}((1 \dot{-} x) \vee \lambda(y)) =$$
$$((\lambda(x) \dot{\to} 0) \vee \lambda(y)) \dot{\to} 0. \blacksquare$$

Consequence 2.10. *For any* $\boldsymbol{A} \in \mathcal{D}$, *the following conditions hold*:

(i) *The operator* λ *is an interior operator of the bounded poset* $(A, \le, 0, 1)$.

(ii) *The set* $\lambda(A)$ *is a subalgebra of* $(A, \vee, \dot{\to}, 0, 1)$.

(iii) *The system* $\lambda(\boldsymbol{A}) = (\lambda(A), \vee, \wedge^o, \dot{\to}, 0, 1)$ *is a Heyting algebra*.

Proof. (i) This property is expressed by Lemma 2.9 (i), (ii) and (v).

(ii) This fact follows from axiom (d2) and Lemma 2.9 (vi), (iii) and (v).

(iii) From equation (d2) and Lemma 2.9 (iii), (v), (vii) it follows that $(\lambda(A), \vee, \wedge^o, 0, 1)$ is a bounded distributive lattice. Then equations (d3)-(d5) and Lemma 2.9 (vi) imply

$$\lambda(x) \wedge^o (\lambda(x) \dot{\rightarrow} \lambda(y)) = \lambda(x) \wedge^o \lambda(y),$$
$$\lambda(x) \dot{\rightarrow} (\lambda(y) \dot{\rightarrow} \lambda(z)) = (\lambda(x) \wedge^o \lambda(y)) \dot{\rightarrow} \lambda(z).$$

Thus $\lambda(\boldsymbol{A}) \in \mathcal{H}$. ■

Lemma 2.9^o. Let $\boldsymbol{A} \in \mathcal{D}$. Then, for all $x, y \in A$:
(*i*) $x \leq y$ *implies* $\rho(x) \leq \rho(y)$, (*ii*) $x \leq \rho(x) = \rho(\rho(x))$,
(*iii*) $\rho(1) = 1$, (*iv*) $x \dot{-} x = 0$, (*v*) $\rho(0) = 0$,
(*vi*) $\rho(x) \dot{-} \rho(y) = \rho(x) \dot{-} y = x \dot{-} y = \rho(x \dot{-} y)$,
(*vii*) $\rho(x \vee y) = \rho(\rho(x) \vee y) = \rho(\rho(x) \vee \rho(y))$,
(*viii*) $1 \dot{-} x = 1 \dot{-} \rho(x)$,
(*ix*) $\rho(x) \dot{\rightarrow} \rho(y) = 1 \dot{-} ((1 \dot{-} \rho(y)) \wedge \rho(x))$.

Consequence 2.10^o. *For any* $\boldsymbol{A} \in \mathcal{D}$, *the following conditions hold*:
(*i*) *The operator* ρ *is a closure operator of the bounded poset* $(A, \leq, 0, 1)$.
(*ii*) *The set* $\rho(A)$ *is a subalgebra of* $(A, \wedge, \dot{-}, 0, 1)$.
(*iii*) *The system* $\rho(\boldsymbol{A}) = (\rho(A), \overline{\vee}, \wedge, \dot{-}, 0, 1)$ *is a dual Heyting algebra*.

Lemma 2.9^o and Consequence 2.10^o follow from 2.9 and 2.10 by the duality principle. The following lemma presents the result that any BrD-algebra is isomorphic to a subalgebra of $\overline{\boldsymbol{H}} \times \overline{\boldsymbol{B}}$, where $(\boldsymbol{H}, \boldsymbol{B}) \in \mathcal{H} \times \mathcal{B}r$.

Lemma 2.11. *Let* $\boldsymbol{A} \in \mathcal{D}$ *and* $i : A \rightarrow \lambda(A) \times \rho(A)$ *be a function defined by* $i(x) = (\lambda(x), \rho(x))$, *for all* $x \in A$. *Let* $\lambda(\boldsymbol{A}) \in \mathcal{H}$ *and* $\rho(\boldsymbol{A}) \in \mathcal{B}r$ *defined by Consequences* 2.10 (*iii*) *and* 2.10^o (*iii*). *Then the following conditions hold*:
(*i*) *The algebra* $\overline{\lambda(\boldsymbol{A})} \in \overline{\mathcal{H}}$ *is defined by* $\overline{\lambda(\boldsymbol{A})} = (\lambda(A), \vee, \wedge^o, \dot{\rightarrow}, \dot{-}, 0, 1)$.
(*ii*) *The algebra* $\overline{\rho(\boldsymbol{A})} \in \overline{\mathcal{B}r}$ *is defined by* $\overline{\rho(\boldsymbol{A})} = (\rho(A), \overline{\vee}, \wedge, \dot{\rightarrow}, \dot{-}, 0, 1)$.
(*iii*) *The function i is an injective homomorphism from* **A** *into the direct product* $\overline{\lambda(\boldsymbol{A})} \times \overline{\rho(\boldsymbol{A})}$ *such that the image algebra* $i(\boldsymbol{A})$ *is a subdirect product of the pair* $\left(\overline{\lambda(\boldsymbol{A})}, \overline{\rho(\boldsymbol{A})}\right)$.

Proof. (i) Equations valid in $\lambda(\boldsymbol{A}) \in \mathcal{H}$ and Lemma 2.9 (ix) imply the following relations, for all $x, y \in A$,
(1) $\dot{\neg}(\lambda(x) \dot{\rightarrow} \lambda(y)) = \dot{\neg}(\dot{\neg}\lambda(x) \vee \lambda(y))$,
(2) $\lambda(x) \dot{-} \lambda(y) = \dot{\neg}(\dot{\neg}\lambda(x) \vee \lambda(y))$.
From (1) and (2) one derives
(3) $\lambda(x) \dot{-} \lambda(y) = \dot{\neg}(\lambda(x) \dot{\rightarrow} \lambda(y))$.
Equation (3), Definition 2.2 and Consequence 2.10 (iii) imply (i).

(ii) Using the duality principle, condition (ii) follows from (i).

(iii) Lemmas 2.9 and 2.9^{o} imply the fact that i is an homomorphism. Suppose that $i(x)=i(y)$. Definition of i implies $\lambda(x)=\lambda(y)$ and $\rho(x)=\rho(y)$, thus from Lemma 2.9 (viii) one derives $\dot{\neg}x=\dot{\neg}y$. Using equation (d7) this implies $x=\lambda(x)\vee(\rho(x)\wedge\dot{\neg}x)=\lambda(y)\vee(\rho(y)\wedge\dot{\neg}y)=y$. Thus, the function i is injective. Definition of i implies also $\pi_1(i(A))=\lambda(A)$ and $\pi_2(i(A))=\rho(A)$, where the functions $\pi_1:\lambda(A)\times\rho(A)\to\lambda(A)$ and $\pi_2:\lambda(A)\times\rho(A)\to\rho(A)$ are the canonical projections of the direct product $\lambda(A)\times\rho(A)$. Thus (iii) holds. ■

The following result is a model theoretical characterization of $\mathcal{D}$.

Theorem 2.12. $\mathcal{D}=\left\langle\overline{\mathcal{H}}\cup\overline{\mathcal{B}r}\right\rangle$.

Proof. Using Definitions 1.2 and 2.2, respectively 1.3 and 2.3, from Definition 2.6 it follows that $\overline{\mathcal{H}}\cup\overline{\mathcal{B}r}\subseteq\mathcal{D}$, thus $\left\langle\overline{\mathcal{H}}\cup\overline{\mathcal{B}r}\right\rangle\subseteq\mathcal{D}$. Let $\mathcal{V}\subseteq K(\tau_D)$ be any variety such that $\overline{\mathcal{H}}\cup\overline{\mathcal{B}r}\subseteq\mathcal{V}$. We prove that in this case $\mathcal{D}\subseteq\mathcal{V}$. Let $\boldsymbol{A}\in\mathcal{D}$. Lemma 2.11 implies that $\boldsymbol{A}$ is isomorphic to a subalgebra of $\overline{\lambda(\boldsymbol{A})}\times\overline{\rho(\boldsymbol{A})}$ with $\overline{\lambda(\boldsymbol{A})}\in\overline{\mathcal{H}}\subseteq\mathcal{V}$ and $\overline{\rho(\boldsymbol{A})}\in\overline{\mathcal{B}r}\subseteq\mathcal{V}$, thus $\boldsymbol{A}\in\mathcal{V}$. Therefore, the next inclusion also holds: $\mathcal{D}\subseteq\left\langle\overline{\mathcal{H}}\cup\overline{\mathcal{B}r}\right\rangle$. ■

3. Basic algebraic properties

Consequences 2.10 and 2.10^{o} express the properties that any BrD-algebra includes a subposet which is a Heyting algebra and a subposet which is a dual Heyting algebra. We present now a typical example of BrD-algebra.

Example 3.1. For any topological space U and $S\subseteq U$, let S^c be the complement of S, S^o be the interior of S and $\overline{S}$ be the closure of S in U. Let X, Y be topological spaces. Let $\overline{\boldsymbol{O}p(X)}$ be the algebra of $\overline{\mathcal{H}}$ associated with the Heyting algebra $\boldsymbol{O}p(X)$ of open sets of X and let $\overline{\boldsymbol{C}l(Y)}$ be the algebra of $\overline{\mathcal{B}r}$ associated with the dual Heyting algebra $\boldsymbol{C}l(Y)$ of closed sets of Y.

From Theorem 2.12 it follows that $\boldsymbol{A}(X,Y)=\overline{\boldsymbol{O}p(X)}\times\overline{\boldsymbol{C}l(Y)}\in\mathcal{D}$, where for all $(P,F),(Q,G)\in A(X,Y)=Op(X)\times Cl(Y)$,

$$(P,F)\vee(Q,G)=(P\cup Q,F\cup G),$$
$$(P,F)\wedge(Q,G)=(P\cap Q,F\cap G),$$
$$(P,F)\dot{\to}(Q,G)=\left((P^c\cup Q)^o,\overline{(F^c\cup G)^o}\right),$$

$$(P,F) \dot{-} (Q,G) = \left(\left(\overline{P \cap Q^c} \right)^o, \overline{F \cap G^c} \right),$$
$$0 = (\varnothing, \varnothing) \text{ and } 1 = (X, Y).$$

Theorem 3.2. *For every* $\boldsymbol{A} \in \mathcal{D}$, *there exist a pair of topological spaces* (X, Y) *and an injective homomorphism from* **A** *into the algebra* $\boldsymbol{A}(X,Y) \in \mathcal{D}$ *introduced in Example* 3.1.

Proof. Let $\boldsymbol{A} \in \mathcal{D}$. From Lemma 2.11 it follows that there exists an injective homomorphism i from **A** into $\overline{\lambda(\boldsymbol{A})} \times \overline{\rho(\boldsymbol{A})}$, where $\lambda(\boldsymbol{A}) \in \mathcal{H}$ and $\rho(\boldsymbol{A}) \in \mathcal{B}r$. The topological representation theorems of Heyting algebras and Brouwer algebras imply that there exist a pair of topological spaces (X, Y) and an injective homomorphism j from $\overline{\lambda(\boldsymbol{A})} \times \overline{\rho(\boldsymbol{A})}$ into $\overline{\boldsymbol{Op}(X)} \times \overline{\boldsymbol{Cl}(Y)} = \boldsymbol{A}(X,Y)$. Then the function $j \circ i$ is an embedding of **A** into $\boldsymbol{A}(X,Y)$. ■

The following lemma expresses the fact that any BrD-algebra includes a subposet which is a Boolean algebra. Then different additional properties of the variety $\mathcal{D}$ are obtained.

Lemma 3.3. *Let* $\boldsymbol{A} \in \mathcal{D}$ *and* $\dot{\neg} A = \{\dot{\neg} x / x \in A\}$. *Then*:
(i) $\dot{\neg} A = \lambda(A) \cap \rho(A)$.
(ii) *The subposet* $(\dot{\neg} A, \leq, 0, 1)$ *of* $(A, \leq, 0, 1)$ *is a Boolean lattice such that its corresponding Boolean algebra is defined by* $B(\boldsymbol{A}) = (\dot{\neg} A, \overline{\vee}, \wedge^o, ^c, 0, 1)$, *where for all* $u \in \dot{\neg} A$, *the complement* u^c *of* u *in* $B(\boldsymbol{A})$ *is* $u^c = \dot{\neg} u$.
(iii) *The algebra* $\overline{B(\boldsymbol{A})} \in \overline{\mathcal{B}}$ *associated with* $B(\boldsymbol{A}) \in \mathcal{B}$ *using Definition* 2.1 *is defined by* $\overline{B(\boldsymbol{A})} = (\dot{\neg} A, \overline{\vee}, \wedge^o, \dot{\rightarrow}, \dot{-}, 0, 1)$.

Proof. (i) From Lemmas 2.9 and 2.9^o it follows that for all $x \in A$:
(1) $\dot{\neg} x = 1 \dot{-} x$,
(2) $\lambda(\dot{\neg} x) = \dot{\neg} x = \dot{\neg} \lambda(x)$,
(3) $\rho(\dot{\neg} x) = \dot{\neg} x = \dot{\neg} \rho(x)$,
(4) $\lambda(\rho(x)) = \dot{\neg}\dot{\neg} x = \rho(\lambda(x))$,
(5) $x \in \lambda(A)$ if and only if $\lambda(x) = x$,
(6) $x \in \rho(A)$ if and only if $\rho(x) = x$.
The equality (i) follows from equations (1)-(6).
(ii) For all $x, y \in A$:
(7) $\dot{\neg} x \,\overline{\vee}\, \dot{\neg} y = \dot{\neg}\dot{\neg}(\dot{\neg} x \vee \dot{\neg} y)$,
(8) $\dot{\neg} x \wedge^o \dot{\neg} y = \dot{\neg}\dot{\neg}(\dot{\neg} x \wedge \dot{\neg} y)$,

(9) $\dot{\neg}x \,\overline{\vee}\, \dot{\neg}\dot{\neg}x = 1$,

(10) $\dot{\neg}x \wedge^{o} \dot{\neg}\dot{\neg}x = 0$.

Thus (ii) holds.

(iii) For all $x, y \in A$:

(11) $\dot{\neg}x \,\dot{\rightarrow}\, \dot{\neg}y = \dot{\neg}\dot{\neg}x \,\overline{\vee}\, \dot{\neg}y$,

(12) $\dot{\neg}x \,\dot{-}\, \dot{\neg}y = \dot{\neg}x \wedge^{o} \dot{\neg}\dot{\neg}y$.

Then (iii) follows from (11) and (12) using Definition 2.1 and (ii). ■

Remark 3.4. *Let* $\boldsymbol{A} \in \mathcal{D}$ *and* $a \in A$. *Then*:

(*i*) $a \wedge \dot{\neg}a = 0$ *if and only if* $\rho(a) \wedge \dot{\neg}a = 0$,

(*ii*) $a \wedge \dot{\neg}a = 0$ *implies* $\lambda(a) = a$,

(*iii*) $a \vee \dot{\neg}a = 1$ *if and only if* $\lambda(a) \vee \dot{\neg}a = 1$,

(*iv*) $a \vee \dot{\neg}a = 1$ *implies* $\rho(a) = a$,

(*v*) $\boldsymbol{A} \in \overline{\mathcal{H}}$ *if and only if* $x \wedge \dot{\neg}x = 0$, *for all* $x \in A$,

(*vi*) $\boldsymbol{A} \in \overline{\mathcal{B}r}$ *if and only if* $x \vee \dot{\neg}x = 1$, *for all* $x \in A$,

(*vii*) $\boldsymbol{A} \in \overline{\mathcal{B}}$ *if and only if* $x \wedge \dot{\neg}x = 0$ *and* $x \vee \dot{\neg}x = 1$, *for all* $x \in A$.

Notations 3.5. We will denote by $\mathfrak{C}$ the algebraic category defined by $\mathcal{C}$, for any $\mathcal{C} \subseteq K(\tau)$ with $\tau \in t(K)$. For any objects $\boldsymbol{A}$ and $\boldsymbol{B}$ of an algebraic category we denote by $[\boldsymbol{A}, \boldsymbol{B}]$ the set of all homomorphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$.

The following theorem presents the basic fact that the variety $\mathcal{D}$ is an arithmetical variety. The categorical relation of $\mathcal{D}$ with the three subvarieties $\overline{\mathcal{H}}$, $\overline{\mathcal{B}r}$ and $\overline{\mathcal{B}}$ is also derived.

Theorem 3.6. *The variety* $\mathcal{D}$ *is arithmetical i.e.* $\mathcal{D}$ *is congruence-distributive and congruence-permutable.*

Proof. We will use Burris and Sankappanavar theorem [7, Theorem II.12.5]: a variety $\mathcal{V}$ is arithmetical if there exists a ternary term t(x, y, z) such that the following equations are valid in $\mathcal{V}$,

$$(*)\; t(x, y, x) = t(x, y, y) = t(y, y, x) = x .$$

Define the following ternary terms associated in the language of $\mathcal{D}$ with the known ternary terms of the variety $\mathcal{B}$, $\mathcal{H}$ and $\mathcal{B}r$:

$$m_B(x, y, z) = (\dot{\neg}x \wedge^{o} \dot{\neg}z) \,\overline{\vee}\, (\dot{\neg}x \wedge^{o} \dot{\neg}\dot{\neg}y \wedge^{o} \dot{\neg}\dot{\neg}z) \,\overline{\vee}\, (\dot{\neg}\dot{\neg}x \wedge^{o} \dot{\neg}\dot{\neg}y \wedge^{o} \dot{\neg}z),$$

$$m_H(x, y, z) = ((x \,\dot{\rightarrow}\, y) \,\dot{\rightarrow}\, z) \wedge^{o} ((z \,\dot{\rightarrow}\, y) \,\dot{\rightarrow}\, x) \wedge^{o} (x \vee z),$$

$$m_{Br}(x, y, z) = (z \,\dot{-}\, (y \,\dot{-}\, x)) \,\overline{\vee}\, (x \,\dot{-}\, (y \,\dot{-}\, z)) \,\overline{\vee}\, (x \wedge z).$$

Using the above definitions, Lemmas 3.3 and 2.11 (i), (ii) imply the validity in $\mathcal{D}$ of the following relations:

$m_H(x,y,x) = m_H(x,y,y) = m_H(y,y,x) = \lambda(x)$,
$m_{Br}(x,y,x) = m_{Br}(x,y,y) = m_{Br}(y,y,x) = \rho(x)$,
$m_B(x,y,x) = m_B(x,y,y) = m_B(y,y,x) = \dot{\neg}x$.

Using equation 2.6 (d7), the previous relations imply that for the new term
$t(x,y,z) = m_H(x,y,z) \vee (m_{Br}(x,y,z) \wedge m_B(x,y,z))$,
the following equations are valid in $\mathcal{D}$:
$t(x,y,x) = m_H(x,y,x) \vee (m_{Br}(x,y,x) \wedge m_B(x,y,x)) = \lambda(x) \vee (\rho(x) \wedge \dot{\neg}x) = x$
Thus for the ternary term t(x, y, z) the equations $(*)$ are valid in $\mathcal{D}$. ■

Theorem 3.7. *The three algebraic categories $\overline{\mathfrak{H}}, \overline{\mathfrak{B}r}$ and $\overline{\mathfrak{B}}$ are reflective full subcategories of $\mathfrak{D}$ such that their corresponding reflectors preserve the injective homomorphisms.*

Proof. For all $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{D}$ and $f \in [\boldsymbol{A}, \boldsymbol{B}]$, let
$f_H : \lambda(A) \to \lambda(B)$, $f_{Br} : \rho(A) \to \rho(B)$ and $f_B : \dot{\neg}A \to \dot{\neg}B$
be the functions such that for all $x \in A$,
$f_H(\lambda(x)) = \lambda(f(x))$, $f_{Br}(\rho(x)) = \rho(f(x))$ and $f_B(\dot{\neg}x) = \dot{\neg}f(x)$.
Lemmas 2.11 and 3.3 imply that there exists three functors
$\mathcal{R}_H : \mathfrak{D} \to \overline{\mathfrak{H}}$, $\mathcal{R}_{Br} : \mathfrak{D} \to \overline{\mathfrak{B}r}$ and $\mathcal{R}_B : \mathfrak{D} \to \overline{\mathfrak{B}}$
defined by the following conditions:
$\mathcal{R}_H(\boldsymbol{A}) = \overline{\lambda(\boldsymbol{A})} \in \overline{\mathcal{H}}$ and $\mathcal{R}_H(f) = f_H \in [\mathcal{R}_H(\boldsymbol{A}), \mathcal{R}_H(\boldsymbol{B})]$,
$\mathcal{R}_{Br}(\boldsymbol{A}) = \overline{\rho(\boldsymbol{A})} \in \overline{\mathcal{B}r}$ and $\mathcal{R}_{Br}(f) = f_{Br} \in [\mathcal{R}_{Br}(\boldsymbol{A}), \mathcal{R}_{Br}(\boldsymbol{B})]$,
$\mathcal{R}_B(\boldsymbol{A}) = \overline{B(\boldsymbol{A})} \in \overline{\mathcal{B}}$ and $\mathcal{R}_B(f) = f_B \in [\mathcal{R}_B(\boldsymbol{A}), \mathcal{R}_B(\boldsymbol{B})]$.
Then $\mathcal{R}_H$, $\mathcal{R}_{Br}$ and $\mathcal{R}_B$ are reflectors respectively of the full subcategories $\overline{\mathfrak{H}}, \overline{\mathfrak{B}r}$ and $\overline{\mathfrak{B}}$ of $\mathfrak{D}$ preserving injective homomorphisms. ■

It is known that injective Heyting (Brouwer) algebras are just complete Boolean algebras. The following result is that complete Boolean algebras of $\overline{\mathcal{B}}$ are also injective algebras of $\mathcal{D}$.

Theorem 3.8. *An algebra $\mathbf{A} \in \mathcal{D}$ is an injective object of $\mathcal{D}$ if and only if $\mathbf{A} \in \overline{\mathcal{B}}$ and the reduct $L(\mathbf{A})$ is a complete lattice.*

Proof. Suppose that $\boldsymbol{A} \in \overline{\mathcal{B}}$ and the reduct $L(\boldsymbol{A})$ is a complete lattice. From Sikorski theorem it follows that $\boldsymbol{A}$ is injective in $\overline{\mathcal{B}}$. This implies that A is injective in $\mathcal{D}$ because of the fact that the functor $\mathcal{R}_B : \mathfrak{D} \to \overline{\mathfrak{B}}$ introduced in the proof of Theorem 3.7 is a reflector preserving injective homomorphisms. Suppose now that $\boldsymbol{A} \in \mathcal{D}$ is an injective object of $\mathcal{D}$. From Lemma 2.11 (iii) it follows that there exists an injective homomorphism $i \in [\boldsymbol{A}, \overline{\lambda(\boldsymbol{A})} \times \overline{\rho(\boldsymbol{A})}]$. Any Heyting (Brouwer) algebra can be extended to a complete Heyting (Brouwer) algebra by a regular extension [18]. Thus, the relations $\overline{\lambda(\boldsymbol{A})} \in \overline{\mathcal{H}}$ and $\overline{\rho(\boldsymbol{A})} \in \overline{\mathcal{B}r}$ imply that there exist $\boldsymbol{A}_c \in \mathcal{D}$ and a regular embedding $i_c \in [\boldsymbol{A}, \boldsymbol{A}_c]$ such that $L(\boldsymbol{A}_c)$ is a complete lattice. From the condition that $\boldsymbol{A}$ is injective it follows that there exists $j \in [\boldsymbol{A}_c, \boldsymbol{A}]$ such that $j \circ i_c = id_{\boldsymbol{A}}$, where $id_{\boldsymbol{A}} \in [\boldsymbol{A}, \boldsymbol{A}]$ is the identity morphism of $\boldsymbol{A}$. This implies $j(\sup_{L(\boldsymbol{A}_c)}(i_c(X))) = \sup_{L(\boldsymbol{A})}(X)$, for any subset $X \subseteq A$. Thus, $L(\boldsymbol{A})$ is a complete lattice. Arguments similar to those used in the proof of Balbes-Horn theorem ([3], theorem IX.5.5) imply

$$\{x \in A / \rho(x) = 1\} = \{1\}.$$

By duality one derives also

$$\{x \in A / \lambda(x) = 0\} = \{0\}.$$

The previous relations together with the next equations $\lambda(x \wedge \dot{\neg} x) = 0$ and $\rho(x \vee \dot{\neg} x) = 1$ imply $x \wedge \dot{\neg} x = 0$ and $x \vee \dot{\neg} x = 1$, for all $x \in A$. Thus using Remark 3.4 (vii) one derives $\boldsymbol{A} \in \overline{\mathcal{B}}$. ■

The next result presents a factorization condition for BrD-algebras.

Theorem 3.9. *Let* $\overline{\mathcal{H}} * \overline{\mathcal{B}r}$ *be the class of all algebras* $\boldsymbol{A} \in K(\tau_D)$ *such that there exist a pair* $(\boldsymbol{H}, \boldsymbol{B}) \in \mathcal{H} \times \mathcal{B}r$ *and an isomorphism* $f \in [\boldsymbol{A}, \overline{\boldsymbol{H}} \times \overline{\boldsymbol{B}}]$. *Let* $\boldsymbol{A} \in \mathcal{D}$. *The following conditions are equivalent*:

(i) $\boldsymbol{A} \in \overline{\mathcal{H}} * \overline{\mathcal{B}r}$.

(ii) *There exists an element* $\alpha \in A$ *such that for all* $x \in A$:

(1) $\lambda(x \wedge \alpha) = x \wedge \alpha$,

(2) $\rho(x \vee \alpha) = x \vee \alpha$.

Proof. $(i) \Rightarrow (ii)$. Suppose that the condition (i) holds. Thus, there exists an isomorphism $f \in [\boldsymbol{A}, \overline{\boldsymbol{H}} \times \overline{\boldsymbol{B}}]$ with $(\boldsymbol{H}, \boldsymbol{B}) \in \mathcal{H} \times \mathcal{B}r$. Let $\tilde{\alpha} = (1, 0) \in H \times B$. Then the element $\alpha = f^{-1}(\tilde{\alpha}) \in A$ satisfies both the conditions (1) and (2), for all $x \in A$, where $f^{-1} \in [\overline{\boldsymbol{H}} \times \overline{\boldsymbol{B}}, \boldsymbol{A}]$ is the inverse of *f*. Thus (ii) holds.

$(ii) \Rightarrow (i)$. Suppose that (ii) holds. Define the following subsets of A,

$$H^{\alpha} = [0, \alpha] = \{x \in A / x \leq \alpha\},$$

$$B_{\alpha} = [\alpha, 1] = \{x \in A / \alpha \leq x\}.$$

Let $f : A \to H^{\alpha} \times B_{\alpha}$ be a function such that for all $x \in A$,

$$f(x) = (x \wedge \alpha, x \vee \alpha).$$

We set $x = \dot{\neg}\alpha$ in (1) and (2). Thus $\alpha \wedge \dot{\neg}\alpha = 0$ and $\alpha \vee \dot{\neg}\alpha = 1$. Then one can verify that the following conditions hold:

(p1) $\boldsymbol{H}^{\alpha} = (H^{\alpha}, \vee, \wedge, \to^{\alpha}, 0, 1^{\alpha}) \in \mathcal{H}$, where $1^{\alpha} = \alpha$ and for all $x, y \in H^{\alpha}$, $x \to^{\alpha} y = (x \dot{\to} y) \wedge \alpha$;

(p2) $\boldsymbol{B}_{\alpha} = (B_{\alpha}, \vee, \wedge, -_{\alpha}, 0_{\alpha}, 1) \in \mathcal{B}r$, where $0_{\alpha} = \alpha$ and for all $x, y \in B_{\alpha}$, $x -_{\alpha} y = (x \dot{-} y) \vee \alpha$;

(p3) f is bijective and $f \in [\boldsymbol{A}, \overline{\boldsymbol{H}^{\alpha}} \times \overline{\boldsymbol{B}_{\alpha}}]$.

The properties (p1)-(p3) imply (i). ■

4. Concluding remarks

In this paper some basic properties of the variety $\mathcal{D}$ are presented. Then different properties regarding the study of congruences $Con(\boldsymbol{A})$ and the theory of normal filters and ideals for $\boldsymbol{A} \in \mathcal{D}$ can be established [38].

On this basis one can derive that the projective limits and the injective limits exist in $\mathcal{D}$. A construction of free objects in $\mathcal{D}$ can be established. One can obtain also that the variety $\mathcal{D}$ is a proper extension of $\overline{\mathcal{H}} * \overline{\mathcal{B}r}$.

The notion of *symmetric* (*involutive*) *Boolean algebra* is defined by a Boolean algebra together with an involutive (dual) automorphism and it is introduced in [25, 26] to express an algebra of electrical circuits with valves. Basic results of the theory of symmetric Boolean algebras are presented in [1, 40]. An important development of this theory is realized by the study of symmetric Heyting algebra in order to improve the algebraic semantics of Łukasiewicz logic [28,6,8-10,15,19-21] and many-valued modal logics [2].

A particular notion of bidimensional Brouwerian algebra called *involutive Brouwerian D-algebra* (*IBrD-algebra*) has been introduced in [39]. An IBrD-algebra is an algebra $\boldsymbol{A} = (A, \vee, \wedge, \dot{\to}, \dot{-}, {}^{d}, 0, 1)$ satisfying:

(1) $(A, \vee, \wedge, \dot{\to}, \dot{-}, 0, 1) \in \mathcal{D}$ (Definition 2.6).

(2) The operator ${}^{d}: A \to A$ is a De Morgan negation of the corresponding bounded lattice reduct $(A, \vee, \wedge, 0, 1)$.

(3) $x \dot{-} y = (y^{d} \dot{\to} x^{d})^{d}$, for all $x, y \in A$.

A basic example of IBrD-algebra is given by the direct product algebra $\boldsymbol{A}(X) = \overline{\boldsymbol{O}p(X)} \times \overline{\boldsymbol{C}l(X)} \in \mathcal{D}$ from Example 3.1 associated with a topological space X together with the operator ${}^{d}: Op(X) \times Cl(X) \to Op(X) \times Cl(X)$ such that $(P, F)^{d} = (F^{c}, P^{c})$, for all $(P, F) \in Op(X) \times Cl(X)$. Symmetric Boolean algebras are particular structures of IBrD-algebras. Then the study of IBrDalgebras in connection with symmetric Gödel algebras can be also considered in order to express an algebraic semantics for a logic of two criteria decision making.

An interesting special subject is the logic of two criteria optimization in fuzzy environment [41, 4, 11, 14, 16, 42, 29, 30, 17]. Then the case of multi-criteria optimization can be also considered.

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