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SCALAR-WEYL STRUCTURES IN GENERALIZED LAGRANGE GEOMETRIES AND COMPATIBLE CONNECTIONS

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Dedicated to Academician Radu Miron
on the occasion of his 90th birthday
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Scalar-Weyl structures and their compatible linear connections are introduced in the setting of generalized Lagrange geometry as a natural generalization of similar notions from semi-Riemannian and Finsler geometries. As an example, it is discussed a generalized Lagrange metric used by R. Miron in a geometric framework unifying gravitation and electromagnetism.

Keywords: generalized Lagrange manifold; scalar-Weyl structure; compatible linear connection; horizontal symmetric d-connection.
MSC2010: 53C 60.

1. Introduction

Hermann Weyl attempted in [12] an unification of gravitation and electromagnetism in a model of space-time geometry combining conformal and projective structures. More precisely, let $\mathcal{G}$ be a conformal structure on the smooth manifold M i.e. an equivalence class of (semi-)Riemannian metrics: $g \sim \overline{g}$ if there exists a smooth function $f \in C^{\infty}(M)$ such that $\overline{g} = e^{2f}g$. A (semi-Riemannian) Weyl structure is a map $W : \mathcal{G} \to \Omega^1(M)$ such that $W(\overline{g}) = W(g) + 2df$. In [6] it is proved that for a Weyl manifold $(M, \mathcal{G}, W)$ there exists an unique torsion-free linear connection ∇ on M such that for every $g \in \mathcal{G}$:

$$\nabla g = W(g) \otimes g. \tag{1.1}$$

Two natural extensions of Riemannian metrics are the Finsler and (generalized) Lagrange metrics. The last class of metrics, introduced by Radu Miron around 1983, are suitable in geometrical approaches of general relativity and gauge theory as it is pointed out in [9]. The Weyl structures were studied in Finslerian setting by T. Aikou in [1]-[3] and L. Kozma in [7]-[8] and in the generalized Lagrange geometry by the first author in [5].

There exists also a generalization of Weyl structures provided by a scalar field $u \in C^{\infty}(M)$ with $u \neq 0$. Namely, we call *scalar-Weyl structure* on $(M, \mathcal{G})$ a map: $W : \mathcal{G} \to \Omega^1(M)$ such that $W(\overline{g}) = W(g) + 2udf$. In [11, p. 32] it is proved that for a scalar-Weyl manifold $(M, \mathcal{G}, u, W)$ there exists an unique torsion-free linear connection ∇ on M such

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that for every $g \in \mathcal{G}$:

$$u\nabla g = W(g) \otimes g. \tag{1.2}$$

In this paper we extend the scalar-Weyl structures and their compatible connections (1.2) in the generalized Lagrangian framework. This setting is introduced in the first section. The compatibility between a distinguished connection and a scalar-Weyl structure is considered in the second section where the main result gives the existence and uniqueness of such a compatible connection under certain horizontal symmetries. When the considered scalar field u is equal to 1 we obtain the main result of [5] while for the above physical example a natural scalar field is the square norm of the 1-form defining the given Weyl structure.

2. Generalized Lagrange-Weyl manifolds

Let M be a smooth n-dimensional real manifold and $\pi : TM \to M$ the tangent bundle of M. A local chart $x = \left(x^i\right)_{1\leq i\leq n}$ of M lifts to a chart $(x, y) = \left(x^i, y^i\right)$ on TM. A tensor field of (r, s)-type on TM with law of change, at a change of charts on M, exactly as a tensor field of (r, s)-type on M, is called *d-tensor field of* (r, s)*-type*. The first main notion of this work is:

Definition 2.1([9]) A d-tensor field of $(0, 2)$-type $g = (g_{ij}(x, y))$ on TM is called *generalized Lagrange metric* (*GL-metric*, on short) if:
(i) is symmetric: $g_{ij} = g_{ji}$,
(ii) is non-degenerate: $\det(g_{ij}) \neq 0$,
(iii) the quadratic form $g_{ij}(x, y)\xi^i\xi^j$ has a constant signature, $\xi = \left(\xi^i\right) \in \mathbb{R}^n$.
The pair (M, g) is a *generalized Lagrange manifold.*

Definition 2.2 Two GL-metrics $g, \overline{g}$ are called *horizontally conformal equivalent* if there exists $f \in C^\infty(M)$ such that $\overline{g} = e^{2f}g$.

In the following let $\mathcal{G}$ be a horizontal conformal structure i.e. an equivalence class of horizontal conformal equivalent GL-metrics and $u \in C^\infty(M)$ with $u \neq 0$. The second main notion of this paper is:

Definition 2.3 A *scalar-Weyl structure* (with respect to u) on the generalized Lagrange manifold (M, g) is a map $W : \mathcal{G} \to \Omega^1(M)$ such that for every $\overline{g} \in \mathcal{G}$ one has:

$$W(\overline{g}) = W(g) + 2udf. \tag{2.1}$$

The data $(M, \mathcal{G}, u, W)$ will be called *generalized Lagrange scalar-Weyl manifold.*

Recall that a vector field $X = X^i(x)\frac{\partial}{\partial x^i} \in \mathcal{X}(M)$ has a *vertical lift* $X^v \in \mathcal{X}(TM)$ given by $X^v = X^i\frac{\partial}{\partial y^i}$.

Because $\mathcal{G}$ implies the tangent bundle geometry it seems naturally the following definition: a linear connection ∇ on TM is vertical-compatible with the generalized Lagrange scalar-Weyl structure $(M, \mathcal{G}, u, W)$ if there exists $g \in \mathcal{G}$ such that for every $X \in \mathcal{X}(M)$:

$$u\nabla_{X^v}g = W(g)(X) \cdot g.$$

But this definition has a great fault: the fact that ∇ is vertical-compatible with a representative of $\mathcal{G}$ does not yields the vertical-compatibility with another representative of $\mathcal{G}$. Indeed, using (2.1), we have:

$$u\nabla_{X^v}\overline{g} = u\nabla_{X^v}\left(e^{2f}g\right) = u[X^v\left(e^{2f}\right)g + e^{2f}\nabla_{X^v}g] = e^{2f}W(g)(X)\,g = W(g)(X)\,\overline{g}$$
$$\neq W(\overline{g})(X)\,\overline{g}.$$

With this motivation we introduce the next notion, namely *nonlinear connections*, well-known in tangent bundle geometry.

3. **Compatibility with respect to a nonlinear connection**

Definition 3.1([9]) A distribution H on TM supplementary to the vertical distribution i.e. $TTM = H \oplus V(TM)$ is called a *nonlinear connection.*

An adapted basis for $V(TM)$ is $\left(\frac{\partial}{\partial y^i}\right)$ and an adapted basis for H has the form $\left(\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}\right)$. The functions $\left(N_i^j(x,y)\right)$ are called *the coefficients of the nonlinear connection* H. We obtain a new lift for vector fields; namely, to $X = X^i(x) \frac{\partial}{\partial x^i} \in \mathcal{X}(M)$ we associate the *horizontal lift* $X^h = X^i \frac{\delta}{\delta x^i} \in H$.

The nonlinear connection H yields a bundle denoted $H(TM)$ and called *horizontal.* The existence of a nonlinear connection is equivalent to the reduction of the standard almost tangent structure of TM to a $D(GL(n,\mathbb{R}))$-structure conform [3]; here:

$$D(GL(n,\mathbb{R}^n)) = \{C = \begin{pmatrix} A & O_n \\ O_n & B \end{pmatrix} \in GL(2n,\mathbb{R}); \quad A,B \in GL(n,\mathbb{R})\}.$$

The associated connections are given by:

Definition 3.2 A $D(GL(n,\mathbb{R}))$-connection on TM is called *d-connection* or *Finsler connection.*

A d-connection ∇ preserves by parallelism both the vertical and horizontal bundles. Hence, ∇ has a pair of Christoffel coefficients $\left(F_{jk}^i(x,y), C_{jk}^i(x,y)\right)$ defined by the relations:

$$\begin{cases} \nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^k} = F_{jk}^i \frac{\delta}{\delta x^i}, & \nabla_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^k} = F_{jk}^i \frac{\partial}{\partial y^i} \\ \nabla_{\frac{\partial}{\partial y^j}} \frac{\delta}{\delta x^k} = C_{jk}^i \frac{\delta}{\delta x^i}, & \nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^k} = C_{jk}^i \frac{\partial}{\partial y^i}. \end{cases} \tag{3.1}$$

It follows that ∇ yields two algorithms of covariant derivation on d-tensor fields: a horizontal one, denoted $_|$, and a vertical one, denoted $|$. For example, on the d-tensor field $g = (g_{ij}(x,y))$ of $(0,2)$-type we have:

$$g_{jk|i} = \frac{\delta g_{jk}}{\delta x^i} - g_{ak}F_{ji}^a - g_{ja}F_{ki}^a, \quad g_{jk}|_i = \frac{\partial g_{jk}}{\partial y^i} - g_{ak}C_{ji}^a - g_{ja}C_{ki}^a. \tag{3.2}$$

Special classes of Finsler connections are provided by:

Definition 3.3 A d-connection is called:
(i) *horizontal* if all $C_{jk}^i = 0$,
(ii) *horizontal symmetric* (*h-symmetric* on short) if $F_{jk}^i = F_{kj}^i$ for all indices i,j,k,
(iii) *totally symmetric* if it is h-symmetric and vertical symmetric i.e. $C_{jk}^i = C_{kj}^i \neq 0$ for all i,j,k.

For example, if g is a Riemannian metric then the Levi-Civita connection is the unique d-connection both horizontal and h-symmetric; in this case F_{jk}^i does not depend of y since they are the usual Christoffel coefficients.

Returning to our setting it is natural to consider:

Definition 3.4 If $(M, \mathcal{G}, u, W)$ is a generalized Lagrange scalar-Weyl manifold then a d-connection ∇ is called *compatible* if there exists $g \in \mathcal{G}$ such that for every $X \in \mathcal{X}(M)$ we have:

$$u\nabla_{X^h} g = W(g)(X) \cdot g. \tag{3.3}$$

An important result concerning this type of compatibility (which can be called as *scalar horizontal-recurrence*) is:

Proposition 3.5 *If* (3.3) *holds for a given* $g \in \mathcal{G}$ *then* ∇ *is compatible with the whole class* $\mathcal{G}$.

Proof From (2.1) we get:

$$u\nabla_{X^h}\overline{g} = u\nabla_{X^h}\left(e^{2\sigma} \circ \pi\right) g = u[X^h\left(e^{2\sigma} \circ \pi\right) g + e^{2\sigma}\nabla_{X^h} g] =$$

$$= u[2d\sigma(X)\, e^{2\sigma} g] + e^{2\sigma} W(g)(X)\, g = e^{2\sigma} g\,[2ud\sigma + W(g)(X)] = W(\overline{g})(X) \cdot \overline{g}$$

which means the conclusion. □

The pair (g, H) yields four remarkable d-connections ([4]): Cartan, Berwald, Chern-Rund and Hashiguchi. For our aim, the Chern-Rund connection, denoted ∇^{CR}, is more convenient because it satisfies ([4]):
I) is horizontal-metrical: $\nabla^{CR}_{X^h} g = 0$ for every $X \in \mathcal{X}(M)$,
II) is totally symmetric with the vertical Christoffel coefficients:

$$\overset{CR}{C^i}_{jk} = \frac{1}{2} g^{ia}\left(\frac{\partial g_{ak}}{\partial y^j} + \frac{\partial g_{ja}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^a}\right). \tag{3.4}$$

The main result of this paper is the generalization of the results cited in Introduction:

Theorem 3.6 *In a generalized Lagrange scalar-Weyl manifold* $(M, \mathcal{G}, u, W)$ *there exists an unique compatible d-connection which is horizontal and h-symmetric.*

Proof Let $g \in \mathcal{G}$ be fixed and the associated ∇^{CR}. For arbitrary $X, Y \in \mathcal{X}(M)$ let us define $\nabla_{X^v} Y^v = 0$ and:

$$u\nabla_{X^h} Y^h := \nabla^{CR}_{X^h} Y^h - \frac{1}{2} W(g)(X) \cdot Y^h - \frac{1}{2} W(g)(Y) \cdot X^h + \frac{1}{2} g\left(X^h, Y^h\right) \cdot B \tag{3.5}$$

where:

$$g\left(X^h, Y^h\right) = g_{ij} X^i Y^j, \quad X = X^i(x)\frac{\partial}{\partial x^i}, Y = Y^j(x)\frac{\partial}{\partial x^j}$$

and $B \in \mathcal{X}(TM)$ is $B = \left(B^i(x, y)\right)$ is the g-contravariant version of $W(g)$:

$$B^i = g^{ij} wj, \quad W(g) = 2w_i(x) dx^i.$$

Here $\left(g^{ij}\right)$ is the inverse of (g_{ij}). Then:

$$u\nabla_{X^h} g = \nabla^{CR}_{X^h} g + W(g)(X) \cdot g \overset{I)}{=} W(g)(X) \cdot g$$

i.e. ∇ is horizontal-compatible with g. Applying the previous result we have the conclusion.
□

Remarks 3.7 i) The case $u = 1$ of this theorem is the main result of [5] and let us call *classical compatible connection* this ∇. For general u and g a (semi-) Riemannian metric

it is obtained the connection of (1.2) from Introduction.
ii) The non-null coefficients of ∇ are:

$$F^i_{jk} = \frac{1}{u}\left[\overset{CR}{F^i}_{jk} - \delta^i_j w_k - \delta^i_k w_j + g_{jk}B^i\right], \tag{3.6}$$

where $\left(\overset{CR}{F}\right)$ are the horizontal Christoffel coefficients of ∇^{CR}:

$$\overset{CR}{F^i}_{jk} = \frac{1}{2}g^{ia}\left(\frac{\delta g_{ak}}{\delta x^j} + \frac{\delta g_{ja}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^a}\right). \tag{3.7}$$

Example 3.8 Suppose that $W(g) = 2du$; let us call *self-scalar Weyl structure* such a type of scalar-Weyl structure. Then (3.6) becomes:

$$F^i_{jk} = \frac{1}{u}\overset{CR}{F^i}_{jk} - \delta^i_j \frac{\partial(\ln u)}{\partial x^k} - \delta^i_k \frac{\partial(\ln u)}{\partial x^j} + g_{jk}g^{ia}\frac{\partial(\ln u)}{\partial x^a}. \tag{3.8}$$

The relation (2.1) reads: $W(\overline{g}) = 2ud(\ln u + f)$. □

Example 3.9 Suppose that M is endowed with a Riemannian metric $\gamma = (\gamma_{ij}(x))$ with the Christoffel symbols (Γ^i_{jk}). In [10] is given a geometrical setting for gravitation and electromagnetism based on the generalized Lagrange metric:

$$g_{ij} = \gamma_{ij} + \frac{1}{c^2}y_i y_j \tag{3.9}$$

with $c > 0$ a real constant considered as the velocity of light and $y_i = \gamma_{ij}y^j$. It is proved that g is not reducible to a Lagrange (particularly Finsler) metric and it has the associated nonlinear connection:

$$N^i_j = \Gamma^i_{jk}y^k. \tag{3.10}$$

Its Chern-Rund connection has the coefficients:

$$F^i_{jk} = \Gamma^i_{jk}, \quad C^i_{jk} = \frac{1}{c^2 a}\gamma_{jk}y^i \tag{3.11}$$

where:

$$a = 1 + \frac{\|y\|^2_\gamma}{c^2}, \quad \|y\|^2_\gamma = y^i y_i. \tag{3.12}$$

Let $(\overset{c}{F^i}_{jk})$ be the coefficients of the classical compatible connection for $(M, \{e^{2f}\gamma; f \in C^\infty(M)\})$. From (3.6) it results that the compatible connection for (M, g, u) is:

$$F^i_{jk} = \frac{1}{u}\overset{c}{F^i}_{ij} + \frac{1}{uc^2}y_j y_k B^i. \tag{3.13}$$

In order to obtain a natural scalar field for this setting we suppose that $W(g)$ is a 1-from without zeros; hence we can consider as remarkable u its square norm with respect to γ:

$$u = \|W(g)\|^2_\gamma = 4\gamma^{ij}w_i w_j. \tag{3.14}$$

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