CONSEQUENCES OF APPLYING KLEIN-GORDON EQUATION ON AN ASSOCIATED WAVE-TRAIN

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By applying Klein-Gordon equation upon a wave-function described by a product of an envelope function R and a phase factor (a complex number) corresponding to an alternating function with almost constant frequency and wavelength and using formula connecting energy, linear momentum p and rest mass from special relativity, it results for the envelope function the three dimensional wave equation for a wave with light speed c. This aspect is similar to the result obtained using the Hamiltonian formalism for electron speed. For didactical purposes, this implies the necessity of adding an internal dynamics to the particle/wave-function..

Keywords: Klein-Gordon equation, wave equation, internal dynamics

1. Introduction

The Klein–Gordon equation is second-order differential equation in space-time. It is a differential equation version of the relativistic energy–momentum relation, obtained through substitution of energy and momentum with corresponding energy and momentum operators within relativistic formula connecting rest mass connecting rest mass m_0, energy E and linear momentum $\boldsymbol{p}$.

However, certain problems regarding its interpretation have been encountered very soon. Using the Hamiltonian formalism

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi \tag{1}$$

and the Ehrenfest theorem

$$\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\left[\hat{H},\hat{A}\right] \tag{2}$$

for the expectation $\langle A\rangle$ value of an operator $\hat{A}$ not depending explicitly of time, for the case when $\hat{A}$ is substituted with $\hat{x}$ (the position operator) and $\hat{H}$ with

$$\hat{H} = c\sum_i \alpha_i\hat{p_i} + \beta m_0 c^2 \tag{3}$$

(the relativistic Hamiltonian written in the cuadridimensional form, $\hat{p_i}$ being the momentum operator, c -light speed, m_0 - rest mass and

$\alpha_i = \begin{bmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{bmatrix}, \beta = \begin{bmatrix} I & 0\\ 0 & -I\end{bmatrix}$, σ_i the Pauli matrixes, I the 2x2unit matrix), it results that

$$\dot{x} = \frac{1}{i\hbar}\left[x,\hat{H}\right] = c\alpha_x \tag{4}$$

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This could suggest that the speed of an electron along Ox axis equals the speed of light c. Similar results can be obtained for $\dot{y}, \dot{z}$. There are numerous and recent attempts in trying to improve the understanding of Klein-Gordon equation, see [1]. Internal dynamics (Zitterbewegung) and time derivatives of α matrixes were also taking into consideration, being briefly exposed in [2]. The model of an electron as a massless charge spinning at light speed has been recently proposed (see [3]). An oscillatory velocity has been also suggested. Its maximum value could be equal to the speed of light, yet it does not contribute to the electronic momentum (see [4].

This study will show that a similar result can be obtained by applying Klein-Gordon equation to an associated wave supposed to have (in a limit case) constant frequency and wavelength, implying the necessity of an inner dynamics as alternating coordinate-momentum representation (see [5]).

2. Klein-Gordon equation and the associated wave function for a free-moving particle

According to special relativity theory, the formula connecting energy E, linear momentum $\boldsymbol{p}$ and rest mass m_0 is

$$E^2 = p^2c^2 + (m_0c^2)^2 \tag{5}$$

Substituting energy and momentum with quantum mechanics operators

$$\hat{E} = i\hbar\frac{\partial}{\partial t}, \hat{p} = -i\hbar\nabla \tag{6}$$

and using $\nabla^2 = \Delta$, which means

$$\hat{p}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 = (-i\hbar)^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \tag{7}$$

and

$$\hat{E}^2 = (i\hbar)^2\frac{\partial^2}{\partial t^2} \tag{8}$$

it results

$$-\hbar^2\frac{\partial^2}{\partial t^2} = -\hbar^2c^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + m_0^2c^4 \tag{9}$$

(the last term corresponding to a multiplication of a certain wave function Ψ written to the right). By dividing all terms to $\hbar^2c^2$ and by moving all terms Right-Hand-Side it results the Klein-Gordon equation (with partial derivatives)

$$0 = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - \left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right) + \frac{m_0^2c^2}{\hbar^2}\Psi \tag{10}$$

For a free-moving particle it is admitted that the associated wave Ψ should be described by an amplitude factor $R(x, y, z, t)$ multiplying a phase factor (written as a complex number) $e^{iS/\hbar}$. The function $S = S(x, y, z, t)$ is considered as action within Feynman theory upon path integral. Yet without taking into account Feynman interpretation, for a free-moving particle along Ox axis with almost constant energy and linear momentum it is accepted that

$$S = p_x x - Et \tag{11}$$

implying the eigenvalues p_x for $\hat{p}$ and E for $\hat{E}$ (the measured values) if function R is set to a constant quantity (usually determined using the normalization condition).

However, if R is set to a constant value and p_x and E are also considered as constant quantities, Ψ would simply correspond to an infinitely extended (in space and time) propagating wave with constant frequency and wavelength. Yet this aspect is in disagreement with the interpretation of particles as wave-packets, according to wave-particle dualism. For this reason the amplitude of these oscillations should be significant only on limited space-time intervals. A possible solution consists in considering Ψ as a sum of oscillations centered around k_x (wave vector projection along propagating direction Ox) and ω, with unity amplitude and with the same differences Δk and $\Delta\omega$ when passing from central frequency/wavelength to higher frequencies/lower wavelengths. In this case function R is represented by $R = sin(Y)/Y$, where $Y = (\Delta k_x)x - (\Delta\omega)t$ (a direct wave-packet moving with group velocity $v_g = \Delta\omega/\Delta k$). A major advantage consists in the fact that R varies from unity to zero when Y varies from 0 to $\pm\pi$. However, when $|Y|$ exceeds π, function R will increase again (in disagreement with standard model of a decreasing amplitude when we "move far" from the center of the wave-train).

3. Applying Klein-Gordon equation upon a wave-train with almost constant frequency/wavelength

Let us consider the function Ψ as

$$\Psi = R(x,y,z,t)e^{iS(x,y,z,t)/\hbar} \tag{12}$$

It results

$$\frac{\partial\Psi}{\partial x} = \left(\frac{\partial R}{\partial x}e^{iS/\hbar}\right) + \left(R\frac{i}{\hbar}\frac{\partial S}{\partial x}e^{iS/\hbar}\right) \tag{13}$$

$$\frac{\partial^2\Psi}{\partial x^2} = \left(\frac{\partial^2 R}{\partial x^2}e^{iS/\hbar} + \frac{i}{\hbar}\frac{\partial R}{\partial x}\frac{\partial S}{\partial x}e^{iS/\hbar}\right) + \left(\frac{i}{\hbar}\frac{\partial R}{\partial x}\frac{\partial S}{\partial x}e^{iS/\hbar} + \frac{i}{\hbar}\left(\frac{\partial S}{\partial x}\right)^2 Re^{iS/\hbar} + \frac{i}{\hbar}\frac{\partial^2 S}{\partial x^2}Re^{iS/\hbar}\right)$$

$$= \left\{\frac{\partial^2 R}{\partial x^2}e^{iS/\hbar} - \frac{R}{\hbar^2}\left(\frac{\partial S}{\partial x}\right)^2 e^{iS/\hbar}\right\} + i\left\{\frac{2}{\hbar}\frac{\partial R}{\partial x}\frac{\partial S}{\partial x}e^{iS/\hbar} + \frac{R}{\hbar}\frac{\partial^2 S}{\partial x^2}e^{iS/\hbar}\right\} \tag{14}$$

and (in a similar manner)

$$\frac{\partial^2\Psi}{\partial y^2} = \left\{\frac{\partial^2 R}{\partial y^2}e^{iS/\hbar} - \frac{R}{\hbar^2}\left(\frac{\partial S}{\partial y}\right)^2 e^{iS/\hbar}\right\} + i\left\{\frac{2}{\hbar}\frac{\partial R}{\partial y}\frac{\partial S}{\partial y}e^{iS/\hbar} + \frac{R}{\hbar}\frac{\partial^2 S}{\partial y^2}e^{iS/\hbar}\right\}$$

$$\frac{\partial^2\Psi}{\partial z^2} = \left\{\frac{\partial^2 R}{\partial z^2}e^{iS/\hbar} - \frac{R}{\hbar^2}\left(\frac{\partial S}{\partial z}\right)^2 e^{iS/\hbar}\right\} + i\left\{\frac{2}{\hbar}\frac{\partial R}{\partial z}\frac{\partial S}{\partial z}e^{iS/\hbar} + \frac{R}{\hbar}\frac{\partial^2 S}{\partial z^2}e^{iS/\hbar}\right\} \tag{15}$$

$$\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = \frac{1}{c^2}\left\{\frac{\partial^2 R}{\partial t^2}e^{iS/\hbar} - \frac{R}{\hbar^2}\left(\frac{\partial S}{\partial t}\right)^2 e^{iS/\hbar}\right\} + i\frac{1}{c^2}\left\{\frac{2}{\hbar}\frac{\partial R}{\partial t}\frac{\partial S}{\partial t}e^{iS/\hbar} + \frac{R}{\hbar}\frac{\partial^2 S}{\partial t^2}e^{iS/\hbar}\right\} \tag{16}$$

Substituting in Klein-Gordon equation

$$0=\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}-\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}\right)+\frac{m_0^2c^2}{\hbar^2}\Psi \tag{17}$$

and grouping terms situated in the same position in the above brackets, it results

$$0=(T_A+T_B+T_C+T_D)+T_E \tag{18}$$

where

$$\frac{1}{c^2}\frac{\partial^2 R}{\partial t^2}e^{iS/\hbar}-\frac{\partial^2 R}{\partial x^2}e^{\frac{iS}{\hbar}}-\frac{\partial^2 R}{\partial y^2}e^{\frac{iS}{\hbar}}-\frac{\partial^2 R}{\partial x^2}e^{\frac{iS}{\hbar}}=T_A$$

$$-\frac{R}{c^2\hbar^2}\left(\frac{\partial S}{\partial t}\right)^2e^{\frac{iS}{\hbar}}+\frac{R}{\hbar^2}\left(\frac{\partial S}{\partial x}\right)^2e^{\frac{iS}{\hbar}}+\frac{R}{\hbar^2}\left(\frac{\partial S}{\partial y}\right)^2e^{\frac{iS}{\hbar}}+\frac{R}{\hbar^2}\left(\frac{\partial S}{\partial z}\right)^2e^{\frac{iS}{\hbar}}=T_B$$

$$\frac{2i}{\hbar c^2}\frac{\partial R}{\partial t}\frac{\partial S}{\partial t}e^{iS/\hbar}-\frac{2i}{\hbar}\frac{\partial R}{\partial x}\frac{\partial S}{\partial x}e^{\frac{iS}{\hbar}}-\frac{2i}{\hbar}\frac{\partial R}{\partial y}\frac{\partial S}{\partial y}e^{\frac{iS}{\hbar}}-\frac{2i}{\hbar}\frac{\partial R}{\partial z}\frac{\partial S}{\partial z}e^{\frac{iS}{\hbar}}=T_C$$

$$\frac{i}{\hbar c^2}\frac{\partial^2 S}{\partial t^2}Re^{iS/\hbar}-\frac{i}{\hbar}\frac{\partial^2 S}{\partial x^2}Re^{iS/\hbar}-\frac{i}{\hbar}\frac{\partial^2 S}{\partial y^2}Re^{\frac{iS}{\hbar}}-\frac{i}{\hbar}\frac{\partial^2 S}{\partial z^2}Re^{\frac{iS}{\hbar}}=T_D \tag{19}$$

and

$$\frac{m_0^2c^2}{\hbar^2}\Psi=\frac{m_0^2c^2}{\hbar^2}Re^{\frac{iS}{\hbar}}=T_E \tag{20}$$

In a more compact manner, it can be written

$$\left(\frac{1}{c^2}\frac{\partial^2 R}{\partial t^2}-\frac{\partial^2 R}{\partial x^2}-\frac{\partial^2 R}{\partial y^2}-\frac{\partial^2 R}{\partial x^2}\right)e^{\frac{iS}{\hbar}}=T_A$$

$$\frac{1}{\hbar^2}\left(-\frac{R}{c^2}\left(\frac{\partial S}{\partial t}\right)^2+R\left(\frac{\partial S}{\partial x}\right)^2+R\left(\frac{\partial S}{\partial y}\right)^2+R\left(\frac{\partial S}{\partial z}\right)^2\right)e^{iS/\hbar}=T_B$$

$$\frac{2i}{\hbar}\left(\frac{1}{c^2}\frac{\partial R}{\partial t}\frac{\partial S}{\partial t}e^{iS/\hbar}-\frac{\partial R}{\partial x}\frac{\partial S}{\partial x}e^{\frac{iS}{\hbar}}-\frac{\partial R}{\partial y}\frac{\partial S}{\partial y}e^{\frac{iS}{\hbar}}-\frac{\partial R}{\partial z}\frac{\partial S}{\partial z}\right)e^{iS/\hbar}=T_C$$

$$\frac{i}{\hbar}\left(\frac{1}{c^2}\frac{\partial^2 S}{\partial t^2}-\frac{\partial^2 S}{\partial x^2}-\frac{\partial^2 S}{\partial y^2}-\frac{\partial^2 S}{\partial z^2}\right)Re^{\frac{iS}{\hbar}}=T_D$$

$$\frac{m_0^2c^2}{\hbar^2}Re^{\frac{iS}{\hbar}}=T_E \tag{21}$$

In the limit case when $p_x=const$, $p_y=p_z=0$, $E=const$ and $S=p_xx-Et$

$$\frac{\partial S}{\partial t}=-E,\frac{\partial^2 S}{\partial t^2}=0,\frac{\partial S}{\partial x}=p_x,\ \frac{\partial S}{\partial y}=\frac{\partial S}{\partial z}=0,\ \frac{\partial^2 S}{\partial x^2}=\ \frac{\partial^2 S}{\partial y^2}=\ \frac{\partial^2 S}{\partial z^2}=0 \tag{22}$$

Thus

$$T_A=\left(\frac{1}{c^2}\frac{\partial^2 R}{\partial t^2}-\frac{\partial^2 R}{\partial x^2}-\frac{\partial^2 R}{\partial y^2}-\frac{\partial^2 R}{\partial x^2}\right)e^{\frac{iS}{\hbar}} \tag{23}$$

(the same form)

$$T_B=\frac{1}{\hbar^2}\left\{-\frac{1}{c^2}(-E)^2+p_x^2+0+0\right\}Re^{iS/\hbar}=\frac{1}{\hbar^2}\left(p_x^2-\frac{E^2}{c^2}\right)Re^{iS/\hbar} \tag{24}$$

Yet (according to special relativity theory)

$$E^2=p^2c^2+m_0^2c^4 \tag{25}$$

implying

$$m_0^2c^4=E^2-p^2c^2$$

$$m_0^2c^2=\frac{E^2}{c^2}-p^2$$

$$\frac{m_0^2c^2}{\hbar^2} = \frac{1}{\hbar^2}\left(\frac{E^2}{c^2} - p^2\right)$$
$$\frac{m_0^2c^2}{\hbar^2} Re^{iS/\hbar} = \frac{1}{\hbar^2}\left(\frac{E^2}{c^2} - p^2\right) Re^{iS/\hbar} \tag{26}$$
The Left-Hand-Side corresponds to T_E, the Right-Hand-Side corresponds to $(-T_B)$, according to previous formulae. It results
$$T_B = -T_E \tag{27}$$
For T_C and T_D it results by substituting with previously written partial derivatives in this limit case
$$T_C = \frac{2i}{\hbar}\left(\frac{1}{c^2}\frac{\partial R}{\partial t}(-E) - \frac{\partial R}{\partial x}p_x - 0 - 0\right) e^{iS/\hbar}$$
$$T_C = -\frac{2i}{\hbar}\left(\frac{\partial R}{\partial t}\frac{E}{c^2} + p_x\frac{\partial R}{\partial x}\right) e^{iS/\hbar}$$
$$T_D = \frac{i}{\hbar}\left(\frac{1}{c^2}.0 - 0 - 0 - 0\right) Re^{iS/\hbar} = 0 \tag{28}$$
Inserting the above expressions (for the limit case when $p_x = const,\ p_y = p_z = 0,\ E = const$) into
$$0 = (T_A + T_B + T_C + T_D) + T_E \tag{29}$$
(the Klein-Gordon equation applied upon $\Psi = Re^{iS/\hbar}$, with partial derivatives grouped into $T_A.T_{B,}, T_C, T_D$, the term T_E corresponding to $(m_0^2c^2/\hbar^2)\ Re^{\frac{iS}{\hbar}}$), and taking into account that $T_E = -T_B$, $T_D = 0$ it results
$$T_A + T_C = 0$$
$$\left(\frac{1}{c^2}\frac{\partial^2 R}{\partial t^2} - \frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial y^2} - \frac{\partial^2 R}{\partial x^2}\right) e^{\frac{iS}{\hbar}} - \frac{2i}{\hbar}\left(\frac{\partial R}{\partial t}\frac{E}{c^2} + p_x\frac{\partial R}{\partial x}\right) e^{\frac{iS}{\hbar}} = 0$$
$$\left\{\left(\frac{1}{c^2}\frac{\partial^2 R}{\partial t^2} - \frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial y^2} - \frac{\partial^2 R}{\partial x^2}\right) - \frac{2i}{\hbar}\left(\frac{\partial R}{\partial t}\frac{E}{c^2} + p_x\frac{\partial R}{\partial x}\right)\right\} e^{iS/\hbar} = 0 \tag{30}$$
Since $e^{iS/\hbar}$ is not a null function, the previous equation implies that
$$\left(\frac{1}{c^2}\frac{\partial^2 R}{\partial t^2} - \frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial y^2} - \frac{\partial^2 R}{\partial x^2}\right) - \frac{2i}{\hbar}\left(\frac{\partial R}{\partial t}\frac{E}{c^2} + p_x\frac{\partial R}{\partial x}\right) = 0 \tag{31}$$
This equality holds if both real and imaginary parts Left-Hand-Side are null. This implies
$$\frac{1}{c^2}\frac{\partial^2 R}{\partial t^2} - \frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial y^2} - \frac{\partial^2 R}{\partial x^2} = 0 \tag{32}$$
$$\frac{2}{\hbar}\left(\frac{\partial R}{\partial t}\frac{E}{c^2} + p_x\frac{\partial R}{\partial x}\right) = 0 \tag{33}$$
Briefly analyzing the first equation
$$\frac{1}{c^2}\frac{\partial^2 R}{\partial t^2} - \frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial y^2} - \frac{\partial^2 R}{\partial x^2} = 0 \tag{34}$$
it can be noticed that it corresponds to the three dimensional wave equation for a wave with light speed c. Yet this result is in contradiction with the assumption that the amplitude of the propagating wave-train (corresponding to a free-moving particle) should be centered in a point (where the amplitude presents a maximum) which moves with speed v (the speed of the particle) along a certain direction (let us suppose Ox axis).

4. Conclusions

This study has shown that by applying Klein-Gordon equation upon a wave-function described by a product of an envelope function R and a phase factor (a complex number) corresponding to an alternating function with almost constant frequency and wavelength and using formula connecting energy, linear momentum p and rest mass from special relativity, it results for the envelope function the three dimensional wave equation for a wave with light speed c. This aspect is similar to the result obtained using the Hamiltonian formalism for electron speed. There are two main possibilities to solve this contradiction:

a) the function $R(x, y, z, t)$ is propagating in zig-zag trajectories, with speed c along each linear segment (similar to light within optical fibers with a very narrow section), BUT the main speed (the average speed) of this function $R(x, y, z, t)$ is represented by v (the velocity of the particle along the propagating direction, supposed to be represented by Ox axis), similar to fractional vibrations connected to beams presented in [6] or to response at sinusoidal forces [7], [8]

b) there is ALWAYS a certain inner dynamics of the associated wave-train, implying that the space-time positions and moments when the wave is passing through zero are not equally spaced (otherway ANY function with equally spaced zeros can be written as an envelope function multiplying a space-time sinusoidal function).

R E F E R E N C E S

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