COMPOSITE ALGORITHMS FOR SPLIT EQUILIBRIUM PROBLEMS AND VARIATIONAL INEQUALITIES

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Equilibrium is a central concept in numerous disciplines including economics, management science/operations research, and engineering. In this paper, we investigate iterative methods for solving a split problem related to equilibrium problems and variational inequalities in Hilbert spaces. We construct a composite algorithm for approximating a special solution of the split equilibrium problems and variational inequalities. Convergence analysis is proved under some additional assumptions.

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1. Introduction

Equilibrium is a central concept in numerous disciplines including economics, management science/operations research, and engineering. Methodologies that have been applied to the formulation, qualitative analysis, and computation of equilibria have included: systems of equations, optimization theory, complementarity theory, fixed point theory, variational inequality, variational inclusion and so on. Some related background and works, please refer to [9, 13, 15, 18–26, 29, 31, 33, 34, 37–39].

Recently, the split problem has attracted so much attention, see [1, 7, 8, 12, 30, 32, 40–42]. The main purpose of this paper is to design iterative algorithms for solving split equilibrium problems and variational inequalities in Hilbert spaces. Next, we recall the background of issues. Recall that the SFP is to find a vector p^* such that

$$p^* \in C \text{ and } Ap^* \in Q, \tag{1}$$

where C and Q are two nonempty closed convex subsets of two real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. The SFP arises in the intensity-modulated radiation therapy [2]. A popular method to solve (1) is CQ method ([1]) which generates a sequence $\{x_n\}$ by

$$x_0 \in H_1,\ x_{n+1} = P_C(x_n - \tau A^*(I - P_Q)Ax_n), \quad n \geq 0, \tag{2}$$

where $P_C : H_1 \to C$ and $P_Q : H_2 \to Q$ are metric projection, $\tau > 0$ is a constant and A^* is the adjoint of A.

Further, if $C := \mathrm{Fix}(S)(:= \{x|S(x) = x\})$ and $Q := \mathrm{Fix}(T)$ are the fixed point sets of nonlinear operators S and T, then problem (1) is extended to the SCFPP [4] which aims to find a vector $p^* \in H_1$ such that

$$p^* \in \mathrm{Fix}(S) \text{ and } Ap^* \in \mathrm{Fix}(T). \tag{3}$$

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A remarkable method to solve (3) was proposed in [4] which defines an iterative sequence $\{x_n\}$ by

$$x_0 \in H_1,\ x_{n+1} = S(x_n - \tau A^*(I-T)Ax_n), \quad n \geq 0. \tag{4}$$

Variant forms of methods (2) and (4) were proposed and constructed to solve the SFP and the SCFPP, see [6, 10, 14, 16, 27]. Moreover, the split equilibrium problems and the split variational inequalities have been studied extensively in the literature, see [3, 11, 17, 35, 36].

Let D be a nonempty closed convex subset of a real Hilbert space H. Recall that the equilibrium problem is to find a vector $x^* \in D$ such that

$$\phi(x^*, x) \geq 0,\ \forall x \in D, \tag{5}$$

where $\phi : D \times D \to R$ is a bifunction. The solution set of (5) is denoted by $\mathrm{EP}(\phi)$. Recall also that the variational inequality is to find a vector $q \in D$ such that

$$\langle f(q), x - q\rangle \geq 0,\ \forall x \in D, \tag{6}$$

where $f : D \to H$ is an operator. The solution set of (6) is denoted by $\mathrm{VI}(D, f)$.

Let C and Q be two nonempty closed convex subsets of two real Hilbert spaces H_1 and H_2, respectively. Let $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ be two nonlinear operators. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $\psi : C \times C \to R$ and $\varphi : Q \times Q \to R$ be two bi-functions. In this paper, we consider the following split equilibrium problems and variational inequalities:

$$\text{Find a point } p^* \in \mathrm{VI}(C, f_1) \cap \mathrm{EP}(\psi) \text{ such that } \ Ap^* \in \mathrm{VI}(Q, f_2) \cap \mathrm{EP}(\varphi). \tag{7}$$

Motivated by the works in the literature, we construct a composite algorithm for approximating a special solution of the above split problem (7). We show that the presented algorithm converges strongly to a special solution of the split problem (7) under additional assumptions. Our method and techniques provide a unified pattern for approximating the solution of the split problem.

2. **Preliminaries**

Throughout this section, suppose that: (a) C is a nonempty closed convex subset of a real Hilbert space H; (b) $S : C \to H$ is a nonlinear operator; (c) $\phi : C \times C \to R$ is a bifunction.

Notations: We collect several related notations: (i) S is said to be

- L-Lipschitz if for some $L > 0$,

$$\|S(x) - S(y)\| \leq L\|x - y\|, \forall x, y \in C.$$

  S is nonexpansive when $L = 1$.
- firmly nonexpansive if

$$\|S(x) - S(y)\|^2 \leq \langle S(x) - S(y), x - y\rangle, \forall x, y \in C$$

- μ-ism if for some $\mu > 0$,

$$\langle S(x) - S(y), x - y\rangle \geq \mu\|S(x) - S(y)\|^2, \forall x, y \in C.$$

(ii) Recall that the metric projection P_C is the nearest point projection from H onto C, which satisfies

$$\|x^* - P_C(x^*)\| = \inf_{y\in C} \|y - x^*\|,\ x^* \in H$$

and

$$x^* \in H,\ y = P_C(x^*) \Leftrightarrow \langle x^* - y, x - y\rangle \leq 0,\ \forall x \in C.$$

Conditions: Assume that ϕ satisfies the following conditions:

(ep1): $\phi(u, u) = 0, \forall u \in C$;

(ep2): $\phi(u, v) + \phi(v, u) \leq 0, \forall u, v \in C$ (monotonicity);

(ep3): $\lim_{t\downarrow 0} \phi(tp + (1-t)u, v) \le \phi(u, v), \forall u, v, p \in C$;
(ep4): for each $y \in C$, $x \mapsto \phi(y, x)$ is convex and lower semicontinuous.

Tools: We gather some known results:
(i) If $S : C \to H$ is μ-ism, then $I - \eta S$ is nonexpansive when $\eta \in [0, 2\mu]$.
(ii) If $S : C \to C$ is nonexpansive, then $I - S$ is demiclosed at the origin.
(iii) In any Hilbert space H, there hold

$$\|x + y\|^2\| \le \|x\|^2 + 2\langle y, x + y\rangle$$

and

$$\|x + y\|^2 = \|x\| + 2\langle x, y\rangle + \|y\|^2$$

for all $x, y \in H$.

At the end of this section, we give two useful lemmas.

Lemma 2.1 ([5]). *If ϕ satisfies conditions (ep1)-(ep4), then for $\lambda > 0$ and $x \in C$, there exists $q^* \in C$ such that*

$$\phi(q^*, y) + \frac{1}{\lambda}\langle y - q^*, q^* - x\rangle \ge 0, \forall y \in C.$$

Set

$$J_\lambda^\phi(x) = \{q^* \in C : \phi(q^*, y) + \frac{1}{\lambda}\langle y - q^*, q^* - x\rangle \ge 0, \forall y \in C\}.$$

Then

(i) J_λ^ϕ is single-valued and J_λ^ϕ is firmly nonexpansive;
(ii) EP(ϕ) is closed and convex and EP(ϕ) = Fix(J_λ^ϕ).

Lemma 2.2 ([28]). *Let $\{a_n\} \subset (0, \infty)$, $\{b_n\} \subset (0, 1)$ and $\{c_n\}$ be three real numbers sequences. If the following assumptions hold:*
(i) $a_{n+1} \le (1 - b_n)a_n + c_n, n \ge 0$,
(ii) $\sum_{n=1}^\infty b_n = \infty$;
(iii) $\limsup_{n\to\infty} \frac{c_n}{b_n} \le 0$ or $\sum_{n=1}^\infty |c_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. **Main results**

Let C and Q be two nonempty closed convex subsets of two real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ be μ_1-ism and μ_2-ism, respectively. Let $\psi : C \times C \to R$ and $\varphi : Q \times Q \to R$ be two bi-functions satisfying conditions (ep1)-(ep4). Write

$$\Omega := \{q | q \in \mathrm{VI}(C, f_1) \cap \mathrm{EP}(\psi) \ and\ Aq \in \mathrm{VI}(Q, f_2) \cap \mathrm{EP}(\varphi)\}.$$

Next, we construct an iterative algorithm for solving the split problem (7).

Algorithm 3.1. *Fix $\hat{u} \in H_1$. For an initial guess $x_0 \in H_1$, let $\{x_n\}$ be a sequence generated by for all $n \ge 0$,*

$$x_{n+1} = P_C(I - \alpha_1 f_1)J_{\lambda_1}^\psi[\gamma_n \hat{u} + (1 - \gamma_n)(x_n - \beta A^*(Ax_n - P_Q(I - \alpha_2 f_2)J_{\lambda_2}^\varphi Ax_n))], \quad (8)$$

where $\alpha_1, \alpha_2, \lambda_1, \lambda_2, \beta$ are five constants and $\{\gamma_n\}$ is a real number sequence in $(0, 1)$ such that
(i) *$\alpha_1 \in (0, 2\mu_1)$ and $\alpha_2 \in (0, 2\mu_2)$;*
(ii) *$\lambda_1 \in (0, \infty)$ and $\lambda_2 \in (0, \infty)$;*
(iii) *$\beta \in (0, \frac{1}{\|A\|^2})$;*
(iv) *$\lim_{n\to\infty} \gamma_n = 0, \sum_{n=1}^\infty \gamma_n = \infty$ and $\lim_{n\to\infty} \frac{\gamma_n - \gamma_{n-1}}{\gamma_n} = 0$.*

Now, we demonstrate the convergence of the sequence $\{x_n\}$ generated by Algorithm 3.1.

Theorem 3.1. *If $\Omega \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $P_\Omega(\hat{u})$.*

Proof. Let q^* be any point in Ω. Hence, $q^* \in \mathrm{VI}(C, f_1) \cap \mathrm{EP}(\psi)$ and $Aq^* \in \mathrm{VI}(Q, f_2) \cap \mathrm{EP}(\varphi)$. It follows that $q^* = P_C(q^* - \alpha_1 f_1(q^*))$, $q^* = J^{\psi}_{\lambda_1} q^*$, $Aq^* = P_Q(Aq^* - \alpha_2 f_2(Aq^*))$ and $Aq^* = J^{\varphi}_{\lambda_2} Aq^*$.

Set $z_n = J^{\varphi}_{\lambda_2} Ax_n$, $v_n = P_Q(I - \alpha_2 f_2) z_n$, $y_n = \gamma_n \hat{u} + (1 - \gamma_n)(x_n - \beta A^*(Ax_n - v_n))$, and $u_n = J^{\psi}_{\lambda_1} y_n$ for all $n \geq 0$. Note that $P_Q(I - \alpha_2 f_2) J^{\varphi}_{\lambda_2}$ is nonexpansive and $J^{\psi}_{\lambda_1}$ and $J^{\varphi}_{\lambda_2}$ are firmly-nonexpansive. Then, we have

$$\|z_n - Aq^*\| = \|J^{\varphi}_{\lambda_2} Ax_n - J^{\varphi}_{\lambda_2} Aq^*\| \leq \|Ax_n - Aq^*\|, \tag{9}$$

$$\|u_n - q^*\| = \|J^{\psi}_{\lambda_1} y_n - J^{\psi}_{\lambda_1} q^*\| \leq \|y_n - q^*\|, \tag{10}$$

and

$$\begin{aligned} \|v_n - Aq^*\|^2 &= \|P_Q(I - \alpha_2 f_2) J^{\varphi}_{\lambda_2} Ax_n - P_Q(I - \alpha_2 f_2) J^{\varphi}_{\lambda_2} Aq^*\| \\ &\leq \|J^{\varphi}_{\lambda_2} Ax_n - J^{\varphi}_{\lambda_2} Aq^*\|^2 \\ &\leq \|Ax_n - Aq^*\|^2 - \|z_n - Ax_n\|^2. \end{aligned} \tag{11}$$

From (8) and (10), we have

$$\begin{aligned} \|x_{n+1} - q^*\|^2 &= \|P_C(I - \alpha_1 f_1) u_n - P_C(I - \alpha_1 f_1) q^*\|^2 \\ &\leq \|u_n - q^*\|^2 \\ &\leq \|y_n - q^*\|^2. \end{aligned} \tag{12}$$

Since

$$\begin{aligned} \langle Ax_n - Aq^*, v_n - Ax_n \rangle &= \langle v_n - Aq^*, v_n - Ax_n \rangle - \|v_n - Ax_n\|^2 \\ &= \frac{1}{2}(\|v_n - Aq^*\|^2 + \|v_n - Ax_n\|^2 - \|Ax_n - Aq^*\|^2) \\ &\quad - \|v_n - Ax_n\|^2, \end{aligned}$$

it follows from (11) that

$$\begin{aligned} \langle Ax_n - Aq^*, v_n - Ax_n \rangle &\leq \frac{1}{2}(\|Ax_n - Aq^*\|^2 - \|z_n - Ax_n\|^2 + \|v_n - Ax_n\|^2 \\ &\quad - \|Ax_n - Aq^*\|^2) - \|v_n - Ax_n\|^2 \\ &= -\frac{1}{2}\|z_n - Ax_n\|^2 - \frac{1}{2}\|v_n - Ax_n\|^2. \end{aligned} \tag{13}$$

Set $w_n = x_n - \beta A^*(Ax_n - v_n), n \geq 0$. From (13), we have

$$\begin{aligned} \|w_n - q^*\|^2 &= \|x_n - q^* - \beta A^*(Ax_n - v_n)\|^2 \\ &= \|x_n - q^*\| - 2\beta \langle Ax_n - Aq^*, Ax_n - v_n \rangle + \beta^2 \|A^*(Ax_n - v_n)\|^2 \\ &\leq \|x_n - q^*\| - \beta(\|z_n - Ax_n\|^2 + \|v_n - Ax_n\|^2) + \beta^2 \|A\|^2 \|Ax_n - v_n\|^2 \\ &= \|x_n - q^*\| - \beta \|z_n - Ax_n\|^2 + (\beta^2 \|A\|^2 - \beta) \|Ax_n - v_n\|^2 \\ &\leq \|x_n - q^*\|^2. \end{aligned} \tag{14}$$

Thanks to the definition of y_n, we have $y_n = \gamma_n \hat{u} + (1-\gamma_n)w_n, n \geq 0$. It follows from (14) that

$$\begin{aligned} \|y_n - q^*\|^2 &= \|\gamma_n(\hat{u} - q^*) + (1-\gamma_n)(w_n - q^*)\|^2 \\ &\leq (1-\gamma_n)\|w_n - q^*\|^2 + \gamma_n\|\hat{u} - q^*\|^2 \\ &\leq (1-\gamma_n)\|x_n - q^*\|^2 + \gamma_n\|\hat{u} - q^*\|^2. \end{aligned} \tag{15}$$

By (12) and (15), we have

$$\begin{aligned} \|x_{n+1} - q^*\|^2 &\leq (1-\gamma_n)\|x_n - q^*\|^2 + \gamma_n\|\hat{u} - q^*\|^2 \\ &\leq \max\{\|x_n - q^*\|^2, \|\hat{u} - q^*\|^2\} \\ &\vdots \\ &\leq \max\{\|x_0 - q^*\|^2, \|\hat{u} - q^*\|^2\}. \end{aligned}$$

Hence, the sequence $\{x_n\}$ is bounded. It is easily to check that the sequences $\{y_n\}$, $\{z_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are also bounded. By the definition of w_n, we have

$$\begin{aligned} \|w_{n+1} - w_n\|^2 &= \|x_{n+1} - x_n - \beta[A^*(Ax_{n+1} - v_{n+1}) - A^*(Ax_n - v_n)]\|^2 \\ &= \|x_{n+1} - x_n\|^2 - 2\beta\langle Ax_{n+1} - Ax_n, Ax_{n+1} - Ax_n - (v_{n+1} - v_n)\rangle \\ &+ \beta^2\|A^*(Ax_{n+1} - v_{n+1}) - A^*(Ax_n - v_n)\|^2 \\ &\leq \|x_{n+1} - x_n\|^2 + 2\beta\langle Ax_{n+1} - Ax_n, v_{n+1} - v_n - (Ax_{n+1} - Ax_n)\rangle \\ &+ \beta^2\|A\|^2\|v_{n+1} - v_n - (Ax_{n+1} - Ax_n)\|^2 \\ &= \|x_{n+1} - x_n\|^2 + \beta^2\|A\|^2\|v_{n+1} - v_n - (Ax_{n+1} - Ax_n)\|^2 \\ &+ \beta(\|v_{n+1} - v_n\|^2 + \|v_{n+1} - v_n - (Ax_{n+1} - Ax_n)\|^2 \\ &- \|Ax_{n+1} - Ax_n\|^2) - 2\beta\|v_{n+1} - v_n - (Ax_{n+1} - Ax_n)\|^2 \\ &= \|x_{n+1} - x_n\|^2 + (\beta^2\|A\|^2 - \beta)\|v_{n+1} - v_n - (Ax_{n+1} - Ax_n)\|^2 \\ &+ \beta(\|v_{n+1} - v_n\|^2 - \|Ax_{n+1} - Ax_n\|^2) \\ &\leq \|x_{n+1} - x_n\|^2 + \beta(\|v_{n+1} - v_n\|^2 - \|Ax_{n+1} - Ax_n\|^2). \end{aligned} \tag{16}$$

Meanwhile,

$$\begin{aligned} \|v_{n+1} - v_n\| &= \|P_Q(I - \alpha_2 f_2)J^{\varphi}_{\lambda_2}Ax_{n+1} - P_Q(I - \alpha_2 f_2)J^{\varphi}_{\lambda_2}Ax_n\| \\ &\leq \|Ax_{n+1} - Ax_n\|, \end{aligned}$$

which together with (16) implies that

$$\|w_{n+1} - w_n\| \leq \|x_{n+1} - x_n\|. \tag{17}$$

Next, we estimate $\|x_{n+1} - x_n\|$. In fact, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|P_C(I - \alpha_1 f_1)J^{\psi}_{\lambda_1}[\gamma_n\hat{u} + (1-\gamma_n)w_n] \\ &\quad - P_C(I - \alpha_1 f_1)J^{\psi}_{\lambda_1}[\gamma_{n-1}\hat{u} + (1-\gamma_{n-1})w_{n-1}]\| \\ &\leq \|\gamma_n\hat{u} + (1-\gamma_n)w_n - \gamma_{n-1}\hat{u} - (1-\gamma_{n-1})w_{n-1}]\| \\ &\leq |\gamma_n - \gamma_{n-1}|(\|\hat{u}\| + \|w_{n-1}\|) + (1-\gamma_n)\|w_n - w_{n-1}\| \\ &\leq (1-\gamma_n)\|x_n - x_{n-1}\| + \gamma_n\frac{|\gamma_n - \gamma_{n-1}|}{\gamma_n}M, \end{aligned} \tag{18}$$

where $M \geq \sup_n\{\|\hat{u}\| + \|w_{n-1}\|\}$. Applying Lemma 2.2, we deduce that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|x_n - P_C(I - \alpha_1 f_1)u_n\| = 0. \tag{19}$$

According to (12), we have

$$\begin{aligned}\|x_{n+1}-q^*\|^2 &\le \|u_n-q^*\|^2\\ &= \|J^{\psi}_{\lambda_1}y_n - J^{\psi}_{\lambda_1}q^*\|^2\\ &\le \|y_n-q^*\|^2-\|u_n-y_n\|^2\\ &\le (1-\gamma_n)\|x_n-q^*\|^2+\gamma_n\|\hat{u}-q^*\|^2-\|u_n-y_n\|^2,\end{aligned}$$

which results in that

$$\begin{aligned}\|u_n-y_n\|^2 &\le (1-\gamma_n)\|x_n-q^*\|^2+\gamma_n\|\hat{u}-q^*\|^2-\|x_{n+1}-q^*\|^2\\ &\le (\|x_n-q^*\|+\|x_{n+1}-q^*\|)\|x_{n+1}-x_n\|\\ &\quad+\gamma_n(\|\hat{u}-q^*\|^2-\|x_n-q^*\|^2).\end{aligned} \tag{20}$$

Note that $\gamma_n\to 0$ and the sequence $\{x_n\}$ is bounded. Take into account of (19) and (20), we have

$$\lim_{n\to\infty}\|u_n-y_n\| = \lim_{n\to\infty}\|J^{\psi}_{\lambda_1}y_n-y_n\|=0. \tag{21}$$

Thanks to (14) and (15), we have

$$\begin{aligned}\|x_{n+1}-q^*\|^2 &\le \|y_n-q^*\|^2\\ &\le \|w_n-q^*\|^2+\gamma_n\|\hat{u}-q^*\|^2\\ &\le \|x_n-q^*\|^2+(\beta^2\|A\|^2-\beta)\|v_n-Ax_n\|^2\\ &\quad-\beta\|z_n-Ax_n\|^2+\gamma_n\|\hat{u}-q^*\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}&0\le(\beta-\beta^2\|A\|^2)\|v_n-Ax_n\|^2+\beta\|z_n-Ax_n\|^2\\ &\le \|x_n-q^*\|^2-\|x_{n+1}-q^*\|^2+\gamma_n\|\hat{u}-q^*\|^2\\ &\le (\|x_n-q^*\|+\|x_{n+1}-q^*\|)\|x_{n+1}-x_n\|+\gamma_n\|\hat{u}-q^*\|^2\\ &\to 0.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\|v_n-Ax_n\|=\lim_{n\to\infty}\|z_n-Ax_n\|=\lim_{n\to\infty}\|J^{\varphi}_{\lambda_2}Ax_n-Ax_n\|=0, \tag{22}$$

and

$$\lim_{n\to\infty}\|v_n-z_n\|=\lim_{n\to\infty}\|P_Q(I-\alpha_2 f_2)z_n-z_n\|=0. \tag{23}$$

Noticing that $\|w_n-x_n\|\le\beta\|A\|\|Ax_n-v_n\|$, from (22), we conclude that

$$\lim_{n\to\infty}\|w_n-x_n\|=0. \tag{24}$$

Since $y_n=\gamma_n\hat{u}+(1-\gamma_n)w_n$, it follows that

$$\|y_n-x_n\|\le\gamma_n\|\hat{u}-x_n\|+(1-\gamma_n)\|w_n-x_n\|.$$

Thus,

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{25}$$

Combining (19), (21) and (25), we deduce

$$\lim_{n\to\infty}\|P_C(I-\alpha_1 f_1)x_n-x_n\|=0. \tag{26}$$

Write $x^*=P_\Omega(\hat{u})$. Next, we prove

$$\limsup_{n\to\infty}\langle\hat{u}-x^*,w_n-x^*\rangle\le 0.$$

Since $\{w_n\}$ is bounded, there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that

$$\limsup_{n\to\infty}\langle \hat{u}-x^*, w_n-x^*\rangle = \lim_{i\to\infty}\langle \hat{u}-x^*, w_{n_i}-x^*\rangle \tag{27}$$

and $w_{n_i} \rightharpoonup \hat{p}, i\to\infty$. Based on the facts, we have $x_{n_i}\rightharpoonup \hat{p}$, $Ax_{n_i}\rightharpoonup A\hat{p}$, $y_{n_i}\rightharpoonup \hat{p}$, $u_{n_i}\rightharpoonup \hat{p}$ and $z_{n_i}\rightharpoonup A\hat{p}$. Then,

$$\left.\begin{array}{r} x_{n_i}\rightharpoonup \hat{p} \\ \|P_C(I-\alpha_1 f_1)x_{n_i}-x_{n_i}\|\to 0 \end{array}\right\} \Rightarrow \hat{p}\in \mathrm{VI}(C,f_1),$$

and

$$\left.\begin{array}{r} z_{n_i}\rightharpoonup A\hat{p} \\ \|P_Q(I-\alpha_2 f_2)z_{n_i}-z_{n_i}\|\to 0 \end{array}\right\} \Rightarrow A\hat{p}\in \mathrm{VI}(Q,f_2).$$

On the other hand, since $u_n = J^{\psi}_{\lambda_1} y_n$, we have

$$\psi(u_n,p)+\frac{1}{\lambda_1}\langle p-u_n, u_n-y_n\rangle \geq 0, \forall p\in C.$$

Because ψ is monotone, we have

$$\frac{1}{\lambda_1}\langle p-u_n, u_n-y_n\rangle \geq \psi(p,u_n).$$

It follows that

$$\langle p-u_{n_i}, \frac{u_{n_i}-y_{n_i}}{\lambda_1}\rangle \geq \psi(p,u_{n_i}),\ \forall p\in C.$$

Since $\frac{\|u_{n_i}-y_{n_i}\|}{\lambda_1}\to 0, u_{n_i}\rightharpoonup \hat{p}$, we obtain $0\geq \psi(p,\hat{p})$.

For t with $0<t\leq 1$ and $p\in C$, write $p_t = tp+(1-t)\hat{p}\in C$. Then $\psi(p_t,\hat{p})\leq 0$ and

$$0=\psi(p_t,p_t)\leq t\psi(p_t,p)+(1-t)\psi(p_t,\hat{p})\leq t\psi(p_t,p),$$

which leads to $0\leq\psi(p_t,p)$. So, $0\leq \psi(\hat{p},p)$ and $\hat{p}\in \mathrm{EP}(\psi)$. Using similar techniques, we can conclude that $A\hat{p}\in\mathrm{EP}(\varphi)$. Thus, $\hat{p}\in \mathrm{VI}(C,f_1)\cap \mathrm{EP}(\psi), A\hat{p}\in \mathrm{VI}(Q,f_2)\cap\mathrm{EP}(\varphi)\Rightarrow \hat{p}\in\Omega$. In the light of (27), we have

$$\limsup_{n\to\infty}\langle \hat{u}-x^*, w_n-x^*\rangle = \langle \hat{u}-x^*, \hat{p}-x^*\rangle \leq 0. \tag{28}$$

Finally, we show $x_n\to x^*$. As a matter of fact, we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \|y_n-x^*\|^2\\ &= \|\gamma_n(\hat{u}-x^*)+(1-\gamma_n)(w_n-x^*)\|^2\\ &\leq (1-\gamma_n)\|w_n-x^*\|^2+2\gamma_n\langle \hat{u}-x^*, w_n-x^*\rangle\\ &\leq (1-\gamma_n)\|x_n-x^*\|^2+2\gamma_n\langle \hat{u}-x^*, w_n-x^*\rangle.\end{aligned} \tag{29}$$

Based on (28), (29) and Lemma 2.2, we deduce $x_n\to x^*$. This completes the proof. □

Algorithm 3.2. *Fix $\hat{u}\in H_1$. For an initial guess $x_0\in H_1$, let $\{x_n\}$ be a sequence generated by for all $n\geq 0$,*

$$x_{n+1} = J^{\psi}_{\lambda_1}[\gamma_n\hat{u}+(1-\gamma_n)(x_n-\beta A^*(Ax_n-J^{\varphi}_{\lambda_2}Ax_n))],$$

where $\lambda_1,\lambda_2,\beta$ are three constants and $\{\gamma_n\}$ is a real number sequence in $(0,1)$ such that

(ii) $\lambda_1\in(0,\infty)$ *and* $\lambda_2\in(0,\infty)$*;*

(iii) $\beta\in(0,\frac{1}{\|A\|^2})$*;*

(iv) $\lim_{n\to\infty}\gamma_n = 0, \sum_{n=1}^{\infty}\gamma_n=\infty$ *and* $\lim_{n\to\infty}\frac{\gamma_n-\gamma_{n-1}}{\gamma_n}=0$.

Corollary 3.1. *Suppose $\Omega_1 := \{q|q\in EP(\psi)$ and $Aq\in EP(\varphi)\}\neq\emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to $P_{\Omega_1}(\hat{u})$.*

Algorithm 3.3. *Fix $\hat{u} \in H_1$. For an initial guess $x_0 \in H_1$, let $\{x_n\}$ be a sequence generated by for all $n \geq 0$,*

$$x_{n+1} = P_C(I - \alpha_1 f_1)[\gamma_n \hat{u} + (1 - \gamma_n)(x_n - \beta A^*(Ax_n - P_Q(I - \alpha_2 f_2)Ax_n))],$$

where $\alpha_1, \alpha_2, \beta$ are three constants and $\{\gamma_n\}$ is a real number sequence in $(0, 1)$ such that

(i) $\alpha_1 \in (0, 2\mu_1)$ *and* $\alpha_2 \in (0, 2\mu_2)$*;*

(iii) $\beta \in (0, \frac{1}{\|A\|^2})$*;*

(iv) $\lim_{n\to\infty} \gamma_n = 0, \sum_{n=1}^{\infty} \gamma_n = \infty$ *and* $\lim_{n\to\infty} \frac{\gamma_n - \gamma_{n-1}}{\gamma_n} = 0$.

Corollary 3.2. *Suppose $\Omega_2 := \{q | q \in VI(C, f_1)$ and $Aq \in VI(Q, f_2)\} \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm 3.3 converges strongly to $P_{\Omega_2}(\hat{u})$.*

4. **Conclusion**

In this paper, we investigate iterative techniques for solving a split problem regarding equilibrium problems and variational inequalities in real Hilbert spaces. We design a composite iterative algorithm by using hybrid techniques for finding a special solution of this split problem. Under some mild assumptions, we show the constructed algorithm has strong convergence.

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