U.P.B. Sci. Bull., Series A, Vol. 75, Iss. 1, 2013 ISSN 1223-7027

FACTORIZATION PROPERTIES AND TOPOLOGICAL CENTERS OF MODULE ACTIONS AND $*$-INVOLUTION ALGEBRAS

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In this paper, we extend some problems of Arens regularity and factorizations properties of Banach algebras for general structures and we establish the relationships between topological centers and factorization properties of left module actions with some conclusions in the Arens regularity of Banach algebras. To a Banach algebra A, we extend the definition of $* - involution$ algebra to a Banach $A - bimodule$ B with some results in the factorizations properties of B^*. We have some applications in group algebras.

MSC2010: 46L06; 46L07; 46L10; 47L25

Keywords: Arens regularity, bilinear mappings, topological center, second dual, module action, factorization, $* - involution$ algebra

1. Introduction and Preliminaries

For a Banach algebra A, Arens [1] showed that the second dual of A, A^{**}, has two multiplication each extending the multiplication on A. The constructions of two Arens multiplications in A^{**} lead us to definition of topological centers for A^{**} with respect to both Arens multiplications. The topological centers of Banach algebras, module actions and applications of them were introduced and discussed in [6, 8, 13, 14, 15, 16, 17, 19, 20, 21]. Let X, Y, Z be normed spaces and $m : X \times Y \to Z$ be a bounded bilinear mapping. Arens offers two natural extensions m^{***} and m^{t***t} of m from $X^{**} \times Y^{**}$ into Z^{**} as follows:
1. $m^* : Z^* \times X \to Y^*$, given by $\langle m^*(z', x), y\rangle = \langle z', m(x, y)\rangle$ where $x \in X$, $y \in Y$, $z' \in Z^*$,
2. $m^{**} : Y^{**} \times Z^* \to X^*$, given by $\langle m^{**}(y'', z'), x\rangle = \langle y'', m^*(z', x)\rangle$ where $x \in X$, $y'' \in Y^{**}$, $z' \in Z^*$,
3. $m^{***} : X^{**} \times Y^{**} \to Z^{**}$, given by $\langle m^{***}(x'', y''), z'\rangle = \langle x'', m^{**}(y'', z')\rangle$ where $x'' \in X^{**}$, $y'' \in Y^{**}$, $z' \in Z^*$.
The mapping m^{***} is the unique extension of m such that $x'' \to m^{***}(x'', y'')$ from X^{**} into Z^{**} is $weak^*{-}to{-}weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $y'' \to m^{***}(x'', y'')$ is not in general $weak^* - to - weak^*$ continuous from Y^{**} into Z^{**} unless $x'' \in X$. Hence the first topological center of m may be defined as following

$$Z_1(m) = \{x'' \in X^{**} : \ y'' \to m^{***}(x'', y'') \ is \ weak^* - to - weak^* - continuous\}.$$

Let $m^t : Y \times X \to Z$ be the transpose of m defined by $m^t(y, x) = m(x, y)$ for every $x \in X$ and $y \in Y$. Then m^t is a continuous bilinear map from $Y \times X$ to Z, and so it may be extended as above to $m^{t***} : Y^{**} \times X^{**} \to Z^{**}$. The mapping $m^{t***t} : X^{**} \times Y^{**} \to Z^{**}$ in general is not equal to m^{***}, see [1], if $m^{***} = m^{t***t}$, then m is called Arens regular. The mapping $y'' \to m^{t***t}(x'', y'')$ is $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$, but

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the mapping $x'' \to m^{t***t}(x'', y'')$ from X^{**} into Z^{**} is not in general $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$. So we define the second topological center of m as

$$Z_2(m) = \{y'' \in Y^{**} : \ x'' \to m^{t***t}(x'', y'') \ is \ weak^* - to - weak^* - continuous\}.$$

It is clear that m is Arens regular if and only if $Z_1(m) = X^{**}$ or $Z_2(m) = Y^{**}$. Arens regularity of m is equivalent to the following

$$\lim_i \lim_j \langle z', m(x_i, y_j)\rangle = \lim_j \lim_i \langle z', m(x_i, y_j)\rangle,$$

whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$, $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [6, 18].

The regularity of a normed algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Let a'' and b'' be elements of A^{**}, the second dual of A. By *Goldstin's* Theorem [7], P.424-425, there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in A such that $a'' = weak^* - \lim_\alpha a_\alpha$ and $b'' = weak^* - \lim_\beta b_\beta$. So it is easy to see that for all $a' \in A^*$, we have

$$\lim_\alpha \lim_\beta \langle a', m(a_\alpha, b_\beta)\rangle = \langle a''b'', a'\rangle$$

and

$$\lim_\beta \lim_\alpha \langle a', m(a_\alpha, b_\beta)\rangle = \langle a''ob'', a'\rangle,$$

where $a''b''$ and $a''ob''$ are the first and second Arens products of A^{**}, respectively, see [6, 14, 18].

The mapping m is left strongly Arens irregular if $Z_1(m) = X$ and m is right strongly Arens irregular if $Z_2(m) = Y$.

This paper is organized as follows.

a) In section two, for a left Banach $A - module$ B, we study some relationships between factorization properties and topological centers of left module action.

(1) Let $(e_\alpha)_\alpha \subseteq A$ be a $BLAI$ for B. Then the following assertions hold.

i) For each $b' \in B^*$, $\pi_\ell^*(b', e_\alpha) \overset{w^*}{\to} b'$.
ii) B^* factors on the left with respect to A if and only if B^{**} has a W^*BLAI $(e_\alpha)_\alpha \subseteq A$.
iii) B^{**} has a W^*BLAI $(e_\alpha)_\alpha \subseteq A$ if and only if B^{**} has a left unit element $e'' \in A^{**}$ such that $e_\alpha \overset{w^*}{\to} e''$.

(2) Suppose that $b' \in wap_\ell(B)$. Let $a'' \in A^{**}$ and $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \overset{w^*}{\to} a''$ in A^{**}. Then we have

$$\pi_\ell^*(b', a_\alpha) \overset{w}{\to} \pi_\ell^{****}(b', a'').$$

(3) Let B^* factors on the left with respect to A. If $AA^{**} \subseteq Z_{B^{**}}(A^{**})$, then $Z_{B^{**}}(A^{**}) = A^{**}$.

(4) If B^{**} has a $BLAI$ with respect to A^{**}, then B^{**} has a left unit with respect to A^{**}.

(5) Let B be a left Banach $A - module$ and A has a $BRAI$ $(e_\alpha)_\alpha \subset A$. Then, B^* factors on the left if and only if for each $b' \in B^*$, we have $\pi_\ell^*(b', e_\alpha) \overset{w}{\to} b'$ in B^*.

(6) Let A has a $BLAI$ $(e_\alpha)_\alpha \subseteq A$ such that $e_\alpha \overset{w^*}{\to} e''$ in A^{**} where e'' is a left unit for A^{**}. Suppose that $Z^t_{e''}(B^{**}) = B^{**}$. Then, B factors on the right with respect to A if and only if e'' is a left unit for B^{**}.

b) In section three, for a Banach $A - bimodule$ B we study $* - involution$ algebra on B^{**} with respect to first Arens product with some results in the factorization of B^*, that is, suppose that $(e_\alpha)_\alpha \subseteq A$ is a BAI for B and B^{**} is a Banach $* - involution$ algebra as A^{**}-module, then B^{**} is unital and B^* factors on the both side.

2. **Factorization properties and topological centers of left module actions**

Let B be a Banach $A-bimodule$, and let

$$\pi_\ell :\ A\times B\rightarrow B\ and\ \pi_r :\ B\times A\rightarrow B.$$

be the left and right module actions of A on B. Then B^{**} is a Banach $A^{**}-bimodule$ with module actions

$$\pi_\ell^{***} :\ A^{**}\times B^{**}\rightarrow B^{**}\ and\ \pi_r^{***} :\ B^{**}\times A^{**}\rightarrow B^{**}.$$

Similarly, B^{**} is a Banach $A^{**}-bimodule$ with module actions

$$\pi_\ell^{t***t} :\ A^{**}\times B^{**}\rightarrow B^{**}\ and\ \pi_r^{t***t} :\ B^{**}\times A^{**}\rightarrow B^{**}.$$

We may therefore define the topological centers of the left and right module actions of A on B as follows:

$$Z_{A^{**}}(B^{**})=Z(\pi_r)=\{b''\in B^{**} :\ the\ map\ a''\rightarrow\pi_r^{***}(b'',a'')\ :\ A^{**}\rightarrow B^{**}$$
$$is\ weak^*-to-weak^*\ continuous\}$$
$$Z_{B^{**}}(A^{**})=Z(\pi_\ell)=\{a''\in A^{**} :\ the\ map\ b''\rightarrow\pi_\ell^{***}(a'',b'')\ :\ B^{**}\rightarrow B^{**}$$
$$is\ weak^*-to-weak^*\ continuous\}$$
$$Z^t_{A^{**}}(B^{**})=Z(\pi_\ell^t)=\{b''\in B^{**} :\ the\ map\ a''\rightarrow\pi_\ell^{t***}(b'',a'')\ :\ A^{**}\rightarrow B^{**}$$
$$is\ weak^*-to-weak^*\ continuous\}$$
$$Z^t_{B^{**}}(A^{**})=Z(\pi_r^t)=\{a''\in A^{**} :\ the\ map\ b''\rightarrow\pi_r^{t***}(a'',b'')\ :\ B^{**}\rightarrow B^{**}$$
$$is\ weak^*-to-weak^*\ continuous\}$$

We note that if B is a left(resp. right) Banach $A-module$ and $\pi_\ell :\ A\times B\rightarrow B$ (resp. $\pi_r :\ B\times A\rightarrow B$) is left (resp. right) module action of A on B, then B^* is a right (resp. left) Banach $A-module$.

We write $ab=\pi_\ell(a,b)$, $ba=\pi_r(b,a)$, $\pi_\ell(a_1a_2,b)=\pi_\ell(a_1,a_2b)$, $\pi_r(b,a_1a_2)=\pi_r(ba_1,a_2)$, $\pi_\ell^*(a_1b',a_2)=\pi_\ell^*(b',a_2a_1)$, $\pi_r^*(b'a,b)=\pi_r^*(b',ab)$, for all $a_1,a_2,a\in A$, $b\in B$ and $b'\in B^*$ when there is no confusion.

Now we introduce some notations and definitions that we use throughout this paper.

Let A be a Banach algebra. We say that a net $(e_\alpha)_{\alpha\in I}$ in A is a left approximate identity $(=LAI)$ [resp. right approximate identity $(=RAI)$] if, for each $a\in A$, $e_\alpha a\longrightarrow a$ [resp. $ae_\alpha\longrightarrow a$]. For $a\in A$ and $a'\in A^*$, we denote by $a'a$ and aa' respectively, the functionals on A^* defined by $\langle a'a,b\rangle=\langle a',ab\rangle=a'(ab)$ and $\langle aa',b\rangle=\langle a',ba\rangle=a'(ba)$ for all $b\in A$. The Banach algebra A is embedded in its second dual via the identification $\langle a,a'\rangle$ - $\langle a',a\rangle$ for every $a\in A$ and $a'\in A^*$. We denote the set $\{a'a :\ a\in A\ and\ a'\in A^*\}$ and $\{aa' :\ a\in A\ and\ a'\in A^*\}$ by A^*A and AA^*, respectively, clearly these two sets are subsets of A^*. Let A has a BAI. If the equality $A^*A=A^*$, $(AA^*=A^*)$ holds, then we say that A^* factors on the left (right). If both equalities $A^*A=AA^*=A^*$ hold, then we say that A^* factors on both sides. We say that a Banach algebra A has $BLAI$, if A has bounded LAI. The definition of notations $BRAI$ and BAI for a Banach algebra A are similar.

Let B be a left Banach $A-module$ and e be a left unit element of A. We say that e is a left unit (resp. weakly left unit) for B, if $\pi_\ell(e,b)=b$ (resp. $\langle b',\pi_\ell(e,b)\rangle=\langle b',b\rangle$ for all $b'\in B^*$) where $b\in B$. The definition of right unit (resp. weakly right unit) is similar. A Banach $A-bimodule$ B is unital, if it has the same left and right unit as $A-module$. Let B be a left Banach $A-module$ and $(e_\alpha)_\alpha\subseteq A$ be a LAI [resp. weakly left approximate identity(=WLAI)] for A. $(e_\alpha)_\alpha$ is left approximate identity $(=LAI)$[resp. weakly left

approximate identity (=$WLAI$)] for B, if for all $b \in B$, $\pi_\ell(e_\alpha, b) \to b$ (resp. $\pi_\ell(e_\alpha, b) \xrightarrow{w} b$). The definition of the right approximate identity (= RAI)[resp. weakly right approximate identity (= $WRAI$)] is similar. $(e_\alpha)_\alpha \subseteq A$ is called a approximate identity (= AI)[resp. weakly approximate identity (WAI)] for B, if B has the same left and right approximate identity [resp. weakly left and right approximate identity].
Let $(e_\alpha)_\alpha \subseteq A$ be $weak^*$ left approximate identity for A^{**}. Then $(e_\alpha)_\alpha$ is $weak^*$ left approximate identity as $A^{**}-module$ (= W^*LAI *as* $A^{**}-module$) for B^{**}, if for all $b'' \in B^{**}$, we have $\pi_\ell^{***}(e_\alpha, b'') \xrightarrow{w^*} b''$. The definition of the $weak^*$ right approximate identity (= W^*RAI) is similar. $(e_\alpha)_\alpha \subseteq A$ is called a $weak^*$ approximate identity (= W^*AI) for B^{**}, if B^{**} has the same $weak^*$ left and right approximate identity. We say that a left Banach $A-module$ B has $BLAI$, if B has bounded LAI. The definition of notations $BRAI$, BAI, $WBLAI$, $WBRAI$, $WBAI$, W^*BLAI, W^*BRAI and W^*BAI are similar.
Let B be a Banach $A-bimodule$. We say that B is a left [resp. right] factors with respect to A, if $BA = B$ [resp. $AB = B$].

Theorem 2.1. *Let A be a Banach algebra with a BAI $(e_\alpha)_\alpha$. Let B be a left Banach $A-module$ and $(e_\alpha)_\alpha \subseteq A$ be a $BLAI$ for B. Then the following assertions hold.*

(1) For each $b' \in B^*$, we have $\pi_\ell^*(b', e_\alpha) \xrightarrow{w^*} b'$.
(2) B^* factors on the left with respect to A if and only if B^{**} has a W^*BLAI $(e_\alpha)_\alpha \subseteq A$.
(3) B^{**} has a W^*BLAI $(e_\alpha)_\alpha \subseteq A$ if and only if B^{**} has a left unit $e'' \in A^{**}$ such that $e_\alpha \xrightarrow{w^*} e''$.

Proof. (1) For every $b \in B$, since $\pi_\ell(e_\alpha, b) \xrightarrow{\|\cdot\|} b$, $\pi_\ell(e_\alpha, b) \xrightarrow{w} b$. Take $b' \in B^*$. Then we have
$$\lim_\alpha \langle \pi_\ell^*(b', e_\alpha), b\rangle = \lim_\alpha \langle b', \pi_\ell(e_\alpha, b)\rangle = \langle b', b\rangle.$$
It follows that $\pi_\ell^*(b', e_\alpha) \xrightarrow{w^*} b'$.

(2) Let B^* factors on the left with respect to A. Then for every $b' \in B^*$, there are $x' \in B^*$ and $a \in A$ such that $b' = x'a$. Then for every $b'' \in B^{**}$, we have
$$\langle \pi_\ell^{***}(e_\alpha, b''), b'\rangle = \langle e_\alpha, \pi_\ell^{**}(b'', b')\rangle = \langle \pi_\ell^{**}(b'', b'), e_\alpha\rangle$$
$$= \langle b'', \pi_\ell^*(b', e_\alpha)\rangle = \langle b'', \pi_\ell^*(x'a, e_\alpha)\rangle = \langle b'', \pi_\ell^*(x', ae_\alpha)\rangle$$
$$= \langle \pi_\ell^{**}(b'', x'), ae_\alpha\rangle \to \langle \pi_\ell^{**}(b'', x'), a\rangle$$
$$= \langle b'', b'\rangle.$$
It follows that
$$\pi_\ell^{***}(e_\alpha, b'') \xrightarrow{w^*} b'',$$
consequently, B^{**} has W^*BLAI.
Conversely, let $b' \in B^*$. Then for every $b'' \in B^{**}$, we have
$$\langle b'', \pi_\ell^*(b', e_\alpha)\rangle = \langle \pi_\ell^{***}(e_\alpha, b''), b'\rangle \to \langle b'', b'\rangle.$$
It follows that
$$\pi_\ell^*(b', e_\alpha) \xrightarrow{w} b',$$
and so by Cohen factorization theorem, we are done.

(3) Assume that B^{**} has a W^*BLAI $(e_\alpha)_\alpha \subseteq A$. Without loss generality, let $e'' \in A^{**}$ be a left unit for A^{**} with respect to the first Arens product such that $e_\alpha \stackrel{w^*}{\to} e''$. Then for each $b' \in B^*$, we have

$$\begin{aligned}\langle \pi_\ell^{***}(e'', b''), b'\rangle &= \langle e'', \pi_\ell^{**}(b'', b')\rangle \\ &= \lim_\alpha \langle e_\alpha, \pi_\ell^{**}(b'', b')\rangle = \lim_\alpha \langle \pi_\ell^{**}(b'', b'), e_\alpha\rangle \\ &= \lim_\alpha \langle b'', \pi_\ell^{*}(b', e_\alpha)\rangle = \lim_\alpha \langle \pi_\ell^{****}(b', e_\alpha), b''\rangle \\ &= \lim_\alpha \langle b', \pi_\ell^{***}(e_\alpha, b'')\rangle = \lim_\alpha \langle \pi_\ell^{***}(e_\alpha, b''), b'\rangle \\ &= \langle b'', b'\rangle.\end{aligned}$$

Thus $e'' \in A^{**}$ is a left unit for B^{**}.
Conversely, let $e'' \in A^{**}$ be a left unit for B^{**} and assume that $e_\alpha \stackrel{w^*}{\to} e''$ in A^{**}. Then for every $b'' \in B^{**}$ and $b' \in B^*$, we have

$$\begin{aligned}\langle \pi_\ell^{***}(e_\alpha, b''), b'\rangle &= \langle e_\alpha, \pi_\ell^{**}(b'', b')\rangle \\ &\to \langle e'', \pi_\ell^{**}(b'', b')\rangle = \langle \pi_\ell^{***}(e'', b''), b'\rangle \\ &= \langle b'', b'\rangle.\end{aligned}$$

It follows that $\pi_\ell^{***}(e_\alpha, b'') \stackrel{w^*}{\to} b''$.

□

Corollary 2.1. *Let B be a left Banach $A-module$ and A has a $BLAI$. If B^{**} has a W^*BLAI , then*

$$\{a'' \in A^{**} :\ Aa'' \subseteq A\} \subseteq Z_{B^{**}}(A^{**}).$$

Proof. By using the preceding theorem, since B^{**} has W^*BLAI, B^* factors on the left with respect to A. Suppose that $b' \in B^*$. Then there are $x' \in B^*$ and $a \in A$ such that $b' = x'a$. Assume that $a'' \in A^{**}$ such that $Aa'' \subseteq A$. Let $b'' \in B^{**}$ and $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \stackrel{w^*}{\to} b''$ in B^{**}. Then we have the following equality

$$\begin{aligned}\lim_\alpha \langle \pi_\ell^{***}(a'', b''_\alpha), b'\rangle &= \lim_\alpha \langle \pi_\ell^{***}(a'', b''_\alpha), x'a\rangle \\ = \lim_\alpha \langle a\pi_\ell^{***}(a'', b''_\alpha), x'\rangle &= \langle \pi_\ell^{***}(aa'', b''), x'\rangle = \langle \pi_\ell^{***}(a'', b''), b'\rangle.\end{aligned}$$

It follows that $a'' \in Z_{B^{**}}(A^{**})$. □

In the preceding corollary, if we take $B = A$, then we have the following conclusion

$$\{a'' \in A^{**} :\ Aa'' \subseteq A\} \subseteq Z_1(A^{**}).$$

Definition 2.1. *A functional a' in A^* is said to be wap (weakly almost periodic) on A if the mapping $a \to a'a$ from A into A^* is weakly compact.*

The preceding definition is equivalent to the following condition, see [6, 14, 18].
For any two net $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\{a \in A : \parallel a \parallel \leq 1\}$, we have

$$lim_\alpha lim_\beta \langle a', a_\alpha b_\beta\rangle = lim_\beta lim_\alpha \langle a', a_\alpha b_\beta\rangle,$$

whenever both iterated limits exist. The collection of all *wap* functionals on A is denoted by $wap(A)$. Also we have $a' \in wap(A)$ if and only if $\langle a''b'', a'\rangle = \langle a''ob'', a'\rangle$ for every $a'',\ b'' \in A^{**}$.
Let B be a left Banach $A-module$. Then, $b' \in B^*$ is said to be left weakly almost periodic functional if the set $\{\pi_\ell^*(b', a) :\ a \in A,\ \parallel a \parallel \leq 1\}$ is relatively weakly compact. We denote

by $wap_\ell(B)$ the closed subspace of B^* consisting of all the left weakly almost periodic functionals in B^*.
The definition of the right weakly almost periodic functional $(= wap_r(B))$ is the same.
By [18], the definition of $wap_\ell(B)$ is equivalent to the following

$$\langle \pi_\ell^{***}(a'',b''),b'\rangle = \langle \pi_\ell^{t****t}(a'',b''),b'\rangle$$

for all $a'' \in A^{**}$ and $b'' \in B^{**}$. Thus, we can write

$$wap_\ell(B) = \{b' \in B^* : \ \langle \pi_\ell^{***}(a'',b''),b'\rangle = \langle \pi_\ell^{t****t}(a'',b''),b'\rangle \\ for\ all\ a'' \in A^{**},\ b'' \in B^{**}\}.$$

Theorem 2.2. *Let B be a left Banach $A-module$ and suppose that $b' \in wap_\ell(B)$. Let $a'' \in A^{**}$ and $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \overset{w^*}{\to} a''$ in A^{**}. Then we have*

$$\pi_\ell^*(b',a_\alpha) \overset{w}{\to} \pi_\ell^{****}(b',a'').$$

Proof. Assume that $b'' \in B^{**}$. Then

$$\langle \pi_\ell^{****}(b',a''),b''\rangle = \langle \pi_\ell^{***}(a'',b''),b'\rangle = \lim_\alpha \langle \pi_\ell^{***}(a_\alpha,b''),b'\rangle \\ = \lim_\alpha \langle b'',\pi_\ell^*(b',a_\alpha)\rangle.$$

Now suppose that $(b''_\beta)_\beta \subseteq B^{**}$ such that $b''_\beta \overset{w^*}{\to} b''$. Since $b' \in wap_\ell(B)$, we have

$$\langle \pi_\ell^{****}(b',a''),b''_\beta\rangle = \langle \pi_\ell^{***}(a'',b''_\beta),b'\rangle \to \langle \pi_\ell^{***}(a'',b''),b'\rangle \\ = \langle \pi_\ell^{****}(b',a''),b''\rangle.$$

Thus $\pi_\ell^{****}(b',a'') \in (B^{**},weak^*)^* = B^*$. We conclude that

$$\pi_\ell^*(b',a_\alpha) \overset{w}{\to} \pi_\ell^{****}(b',a'')\ in\ B^{**}.$$

□

In the preceding corollary, if we take $B = A$, then we obtain the following result.
Suppose that $a' \in wap(A)$ and $a'' \in A^{**}$ such that $a_\alpha \overset{w^*}{\to} a''$ where $(a_\alpha)_\alpha \subseteq A$. Then we have $a'a_\alpha \overset{w}{\to} a'a''$.

Theorem 2.3. *Let B be a left Banach $A-module$ with $BLAI$ $(e_\alpha)_\alpha \subseteq A$. Suppose that $b' \in wap_\ell(B)$. Then*

$$\pi_\ell^*(b',e_\alpha) \overset{w}{\to} b'.$$

Proof. Let $b'' \in B^{**}$ and $(b_\beta)_\beta \subseteq B$ such that $b_\beta \overset{w^*}{\to} b''$ in B^{**}. Then for every $b' \in wap_\ell(B)$, we have the following equality

$$\lim_\alpha \langle b'',\pi_\ell^*(b',e_\alpha)\rangle = \lim_\alpha \langle \pi_\ell^{****}(b',e_\alpha),b''\rangle \\ = \lim_\alpha \langle b',\pi_\ell^{***}(e_\alpha,b'')\rangle = \lim_\alpha \langle \pi_\ell^{***}(e_\alpha,b''),b'\rangle \\ = \lim_\alpha \lim_\beta \langle \pi_\ell(e_\alpha,b_\beta),b'\rangle = \lim_\beta \lim_\alpha \langle \pi_\ell(e_\alpha,b_\beta),b'\rangle \\ = \lim_\beta \langle b_\beta,b'\rangle = \langle b'',b'\rangle.$$

It follows that

$$\pi_\ell^*(b',e_\alpha) \overset{w}{\to} b'.$$

□

Corollary 2.2. *Let B be a left Banach $A-$module with $BLAI$ $(e_\alpha)_\alpha \subseteq A$. Suppose that $wap_\ell(B) = B^*$. Then B^* factors on the left with respect to A.*

Corollary 2.3. *Let A be an Arens regular Banach algebra with $BLAI$. Then A^* factors on the left.*

Example 2.1. *i) Let G be finite group. Then we have the following equality*

$$M(G)^*L^1(G) = M(G)^* \text{ and } L^\infty(G)L^1(G) = L^\infty(G).$$

ii) Consider the Banach algebra $(\ell^1, .)$ which is Arens regular Banach algebra with unit element. Then we have $\ell^\infty.\ell^1 = \ell^\infty$.
iii) Let $\mathcal{K}(E)$ be the space of all compact operators from E into E. Let $E = \ell^p$ with $p \in (1, \infty)$. Then by using Theorem 2.6.23 of [6], $\mathcal{K}(\ell^p)$ is Arens regular. Since $\mathcal{K}(\ell^p)$ has a BAI, by preceding corollary, $\mathcal{K}(\ell^p)^*$ factors on the left.

Theorem 2.4. *Let B be a left Banach $A-$module and B^* factors on the left with respect to A. If $AA^{**} \subseteq Z_{B^{**}}(A^{**})$, then $Z_{B^{**}}(A^{**}) = A^{**}$.*

Proof. Let $b'' \in B^{**}$ and $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \overset{w^*}{\to} b''$ in B^{**}. Suppose that $a'' \in A^{**}$. Since B^* factors on the left, for every $b' \in B^*$, there are $x' \in B^*$ and $a \in A$ such that $b' = x'a$. Since $aa'' \in Z_{B^{**}}(A^{**})$, we have

$$\langle \pi_\ell^{***}(a'', b''_\alpha), b'\rangle = \langle \pi_\ell^{***}(a'', b''_\alpha), x'a\rangle$$
$$= \langle a\pi_\ell^{***}(a'', b''_\alpha), x'\rangle = \langle \pi_\ell^{***}(aa'', b''_\alpha), x'\rangle$$
$$\to \langle \pi_\ell^{***}(aa'', b''), x'\rangle = \langle \pi_\ell^{***}(a'', b''), b'\rangle.$$

It follows that

$$\pi_\ell^{***}(a'', b''_\alpha) \overset{w^*}{\to} \langle \pi_\ell^{***}(a'', b''),$$

and so $a'' \in Z_{B^{**}}(A^{**})$. □

Corollary 2.4. *Let A be a Banach algebra and A^* factors on the left. If $AA^{**} \subseteq Z_1(A^{**})$, then A is Arens regular.*

Theorem 2.5. *Let B be a left Banach $A-$module and A has a $BRAI$ $(e_\alpha)_\alpha \subset A$. Then, B^* factors on the left if and only if for each $b' \in B^*$, we have $\pi_\ell^*(b', e_\alpha) \overset{w}{\to} b'$ in B^*.*

Proof. Assume that B^* factors on the left. Then for every $b' \in B^*$, there are $x' \in B^*$ and $a \in A$ such that $b' = x'a$. Then for every $b'' \in B^{**}$, we have

$$\langle b'', \pi_\ell^*(b', e_\alpha)\rangle = \langle b'', \pi_\ell^*(x'a, e_\alpha)\rangle = \langle b'', \pi_\ell^*(x', ae_\alpha)\rangle$$
$$= \langle \pi_\ell^{**}(b'', x'), ae_\alpha)\rangle \to \langle \pi_\ell^{**}(b'', x'), a\rangle$$
$$= \langle b'', b'\rangle.$$

It follows that $\pi_\ell^*(b', e_\alpha) \overset{w}{\to} b'$ in B^*.
By Cohen factorization theorem, the converse hold. □

In the preceding theorem, if we take $B = A$, we obtain Lemma 2.1 from [14].

Definition 2.2. *An element e'' of A^{**} is said to be a mixed unit if e'' is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. That is, e'' is a mixed unit if and only if for each $a'' \in A^{**}$, $a''e'' = e''oa'' = a''$.*

By [4], p.146, an element e'' of A^{**} is mixed unit if and only if it is a *weak** cluster point of some BAI $(e_\alpha)_{\alpha\in I}$ in A.
Let B be a Banach $A-bimodule$ and $a''\in A^{**}$. We define the locally topological center of the left and right module actions of a'' on B^{**}, respectively, as follows

$$Z^t_{a''}(B^{**})=Z^t_{a''}(\pi^t_\ell)=\{b''\in B^{**}:\ \pi_\ell^{t***t}(a'',b'')=\pi_\ell^{***}(a'',b'')\},$$
$$Z_{a''}(B^{**})=Z_{a''}(\pi^t_r)=\{b''\in B^{**}:\ \pi_r^{t***t}(b'',a'')=\pi_r^{***}(b'',a'')\}.$$

Thus we have

$$\bigcap_{a''\in A^{**}} Z^t_{a''}(B^{**})=Z^t_{A^{**}}(B^{**})=Z(\pi^t_r),$$

$$\bigcap_{a''\in A^{**}} Z_{a''}(B^{**})=Z_{A^{**}}(B^{**})=Z(\pi_r).$$

Theorem 2.6. *Let B be a left Banach $A-module$ and A has a BLAI $(e_\alpha)_\alpha\subseteq A$ such that $e_\alpha\xrightarrow{w^*}e''$ in A^{**}. Suppose that $Z^t_{e''}(B^{**})=B^{**}$. Then, B factors on the right if and only if e'' is a left unit for B^{**}.*

Proof. Assume that B factors on the right. Then for every $b\in B$, there are $x\in B$ and $a\in A$ such that $b=ax$. For every $b'\in B^*$, we have

$$\langle\pi^*_\ell(b',e_\alpha),b\rangle=\langle b',\pi_\ell(e_\alpha,b)\rangle=\langle\pi_\ell^{***}(e_\alpha,b),b'\rangle$$
$$=\langle\pi_\ell^{***}(e_\alpha,ax),b'\rangle=\langle\pi_\ell^{***}(e_\alpha a,x),b'\rangle$$
$$=\langle e_\alpha a,\pi_\ell^{**}(x,b')\rangle=\langle\pi_\ell^{**}(x,b'),e_\alpha a\rangle$$
$$\to\langle\pi_\ell^{**}(x,b'),a\rangle=\langle b',b\rangle.$$

It follows that $\pi^*_\ell(b',e_\alpha)\xrightarrow{w^*}b'$ in B^*. Let $b''\in B^{**}$ and $(b_\beta)_\beta\subseteq B$ such that $b_\beta\xrightarrow{w^*}b''$ in B^{**}. Since $Z^t_{e''}(B^{**})=B^{**}$, for every $b'\in B^*$, we have

$$\langle\pi_\ell^{***}(e'',b''),b'\rangle=\lim_\alpha\lim_\beta\langle b',\pi_\ell(e_\alpha,b_\beta)\rangle$$
$$=\lim_\beta\lim_\alpha\langle b',\pi_\ell(e_\alpha,b_\beta)\rangle=\lim_\beta\langle b',b_\beta\rangle$$
$$=\langle b'',b'\rangle.$$

It follows that $\pi_\ell^{***}(e'',b'')=b''$, and so e'' is a left unit for B^{**}.
Conversely, let e'' be a left unit for B^{**} and suppose that $b\in B$. Then for every $b'\in B^*$, we have

$$\langle b',\pi_\ell(e_\alpha,b)\rangle=\langle\pi_\ell^{***}(e_\alpha,b),b'\rangle=\langle e_\alpha,\pi_\ell^{**}(b,b')\rangle=\langle\pi_\ell^{**}(b,b'),e_\alpha\rangle$$
$$=\langle e'',\pi_\ell^{**}(b,b')\rangle=\langle\pi_\ell^{***}(e'',b),b'\rangle=\langle b',b\rangle.$$

Then $\pi^*_\ell(b',e_\alpha)\xrightarrow{w}b'$ in B^*, and so by Cohen factorization theorem we are done. □

Corollary 2.5. *Let B be a left Banach $A-module$ and A has a BLAI $(e_\alpha)_\alpha\subseteq A$ such that $e_\alpha\xrightarrow{w^*}e''$ in A^{**}. Suppose that $Z^t_{e''}(B^{**})=B^{**}$. Then $\pi^*_\ell(b',e_\alpha)\xrightarrow{w}b'$ in B^* if and only if e'' is a left unit for B^{**}.*

For a Banach algebra A, we recall that a bounded linear operator $T : A \rightarrow A$ is said to be a left (resp. right) multiplier if, for all $a, b \in A$, $T(ab) = T(a)b$ (resp. $T(ab) = aT(b)$). We denote by $LM(A)$ (resp. $RM(A)$) the set of all left (resp. right) multipliers of A. The set $LM(A)$ (resp. $RM(A)$) is normed subalgebra of the algebra $L(A)$ of bounded linear operator on A.
Let B be a Banach left [resp. right] $A-module$ and $T \in \mathbf{B}(A, B)$. Then T is called left [resp. right] multiplier if $T(a_1a_2) = \pi_r(T(a_1), a_2)$
$[resp.\ T(a_1a_2) = \pi_\ell(a_1, T(a_2))]$ for all $a_1, a_2 \in A$.
We show by $LM(A, B)$ $[resp.\ RM(A, B)]$ all of the Left [resp. right] multiplier from A into B.
Lau and Ülger in [14] showed that for Banach algebra A with a BAI, $RM(A)$ is isometrically isomorphic with $\widetilde{M_1} = \{\mu \in (A^*A)^* :\ A\mu \subseteq A\}$. Now, in the following we study it for left and right module actions.

Theorem 2.7. *The following assertions hold.*
i) Let B be a left Banach $A-module$ and A has a $BRAI$. Then

(1) *There is injective linear mapping from $LM(A, B)$ into B^{**}.*
(2) *$RM(A, B)$ is isometric with $\widetilde{M_1} = \{\mu \in (B^*A)^* :\ A\mu \subseteq B\}$.*

ii) Let B be a right Banach $A-module$ and A has a $BLAI$. Then

(1) *There is injective linear mapping from $RM(A, B)$ into B^{**}.*
(2) *$LM(A, B)$ is isometric with $\widetilde{M_2} = \{\mu \in (AB^*)^* :\ \mu A \subseteq B\}$.*

Proof. We prove (i-1); The proof of (ii-1) is similar to the proof of (i-1) and the proofs of (i-2) and (ii-2) have similar arguments of Theorem 4.4, from [14].
Assume that $e'' \in A^{**}$ is a weak*-cluster point in A^{**} of a $BLAI$ $(e_\alpha)_\alpha \subseteq A$ and without loss generality, let $e_\alpha \overset{w^*}{\rightarrow} e''$ in A^{**}. It is clear that for every $a \in A$, we have $e''a = a$. Let $T \in LM(A, B)$. Take the linear mapping $\phi : T \rightarrow T^{**}(e'')$ from $LM(A, B)$ into B^{**}. Therefore for every $T, S \in LM(A, B)$, if $T^{**}(e'') = S^{**}(e'')$, then we have $T^{**}(e'')a = S^{**}(e'')a$ where $a \in A$. Consequently, $T(a) = S(a)$. Thus ϕ is injective. □

With notice to preceding theorem for locally compact group G, if we take $A = B = L^1(G)$, then by Theorem 4.2 of [14], we have $\widetilde{M_1} = M(G)$ and by Corollary 4.5 of [14], $RM(L^1(G))$ is isometrically isomorphic to $M(G)$.
Problem. Let B be a Banach $A-bimodule$. Then
i) If B factors on the both side, when B^* factors?
ii) If B is separable, dose B necessarily factor on the one side?
iii) By notice to Theorem 2-15, for locally compact group G, if we take $A = B = L^1(G)$, what can say for $\widetilde{M_2}$ and $LM(L^1(G)$?

3. Involution $*-$algebra and Arens regularity of module actions

Definition 3.1. *Let B be a left Banach $A-module$ and let $(a_\alpha)_\alpha \subseteq A$ has weak* limit in A^{**}. We say that $(a_\alpha)_\alpha$ is left regular with respect to B, if for every $(b_\beta)_\beta \subseteq B$, we have*

$$w^* - \lim_\alpha w^* - \lim_\beta a_\alpha b_\beta = w^* - \lim_\beta w^* - \lim_\alpha a_\alpha b_\beta,$$

where $(b_\beta)_\beta$ has weak* limit in B^{**}.

The definition of the right regular is similar. For a Banach $A-bimodule$ B, if $(a_\alpha)_\alpha \subseteq A$ is left and right regular with respect to B, then $(a_\alpha)_\alpha$ is called regular with respect to B. If $(a_\alpha)_\alpha$ is left or right regular with respect to A, we write $(a_\alpha)_\alpha$ is left or right regular, respectively.

Example 3.1. *i) Let B be a right Banach $A-module$ and $a''\in A^{**}$. Suppose that $Z^r_{a''}(B^{**})=B^{**}$ and $w^*-\lim_\alpha a_\alpha=a''$ where $(a_\alpha)_\alpha\subseteq A$. Then $(a_\alpha)_\alpha$ is right regular with respect to B.*
ii) Let A be a Banach algebra and $(a_\alpha)_\alpha\subseteq A$ weak* convergence to some point of $Z_1(A^{**})$. It is clear that $(a_\alpha)_\alpha$ is regular.

Theorem 3.1. *i) Let B be a left Banach (resp. right) $A-module$ with $BLAI$ $(e_\alpha)_\alpha\subseteq A$. If $(e_\alpha)_\alpha\subseteq A$ is left (resp. right) regular with respect to B, then B^* factors on the left (resp. right).*
ii) Let B be a left (resp. right) Banach $A-module$ and suppose that B^{**} has a left (resp. right) unit as A^{**}-module. If B^*A (resp. AB^*) is closed subspace of B^*, then B^* factors on the left (resp. right).

Proof. i) Let $e''\in A^{**}$ and $e_\alpha\overset{w^*}{\to}e''$. Assume that $(b_\beta)_\beta\subseteq B$ such that $b_\beta\overset{w^*}{\to}b''$ in B^{**}. Then

$$e''b''=w^*-\lim_\alpha e_\alpha b''=w^*-\lim_\alpha w^*-\lim_\beta e_\alpha b_\beta=w^*-\lim_\beta w^*-\lim_\alpha e_\alpha b_\beta$$
$$=w^*-\lim_\beta b_\beta=b''.$$

Thus for every $b'\in B^*$, we have

$$\langle b'',b'\rangle=\langle e''b'',b'\rangle=\lim_\alpha\langle e_\alpha b'',b'\rangle=\lim_\alpha\langle b'',b'e_\alpha\rangle.$$

It follows that $b'e_\alpha\overset{w}{\to}b'$, and so by Cohen factorization theorem, we are done.
ii) Assume that $B^*A\neq B^*$. Let $e''\in A^{**}$ be a left unit element for B^{**} and suppose that there is a net $(e_\alpha)_\alpha\subseteq A$ such that $e_\alpha\overset{w^*}{\to}e''$. By Hahn Banach theorem take $0\neq b''\in B^{**}$ such that $\langle b'',B^*A\rangle=0$. Then for every $b'\in B^*$, we have

$$\langle b'',b'\rangle=\langle e''b'',b'\rangle=\lim_\alpha\langle e_\alpha b'',b'\rangle=\lim_\alpha\langle b'',b'e_\alpha\rangle=0.$$

This is contradiction, so $B^*A=B^*$.
Proof of the next part is similar. □

Suppose A is a Banach algebra with a continuous involution $\sim$. Then by Lemma 2.1 of [10] , $\sim$ has an extension to a continuous conjugate linear mapping on A^{**}, denote by the same symbol $\sim$, such that $(a''b'')^\sim=(b'')^\sim o(a'')^\sim$ for all $a'',b''\in A^{**}$.
In the following, we define $\sim$ -involution on a Banach $A-bimodule$ B, and we extend also it on B^{**} as $A^{**}-bimodule$, denote it by the same symbol $\sim$.

Definition 3.2. *Let B be a Banach $A-bimodule$ and suppose that A is a Banach $*-$ involution algebra. We say that B is a Banach $\sim-$involution algebra as $A-module$, if the mapping $b\to b^\sim$ from B into B satisfies in the following conditions*

$$(ab)^\sim=b^\sim a^\sim,\ (ba)^\sim=a^\sim b^\sim,\ (\lambda b)^\sim=\bar\lambda b^\sim,\ (b^\sim)^\sim=b\ ,\ \parallel b^\sim\parallel=\parallel b\parallel,$$

for all $a\in A$, $b\in B$ and $\lambda\in\mathbb{C}$.

Theorem 3.2. *Assume that B is a Banach $\sim-$involution $A-bimodule$. Then we have the following assertions.*
i) Let $\sim$ be extend to B^{**}. Then $\sim$ is an involution on B^{**} if and only if $Z^\ell_{A^{**}}(B^{**})=B^{**}$.
ii) Suppose that $(e_\alpha)_\alpha\subseteq A$ is a $BLAI$ for B and it is left regular with respect to B. If B^{**}

is a Banach $\sim -$involution algebra as A^{**}-module, then B^{**} is unital and B^* factors on the both side.

Proof. i) The proof has similar arguments to Lemma 2.1 from [10].
ii) By using Theorem 3-3, we know that B^* factors on the left and without loss generally, suppose that there is a left unit for B^{**} as $e'' \in A^{**}$ such that $e_\alpha \overset{w^*}{\to} e''$. Let $b'' \in B^{**}$. Then

$$b''(e'')^\sim = (e''(b'')^\sim)^\sim = ((b'')^\sim)^\sim = b''.$$

Since $e'' = (e'')^\sim$, B^{**} is unital. Since $(e_\alpha)_\alpha \subseteq A$ is a left regular, $Z^\ell_{e''}(B^{**}) = B^{**}$. Therefore by using Theorem 6-3(ii) from [11], we are done. □

Corollary 3.1. *Assume that A is a $\sim -$involution algebra. Let B be a Banach $A-$bimodule and $(e_\alpha)_\alpha \subseteq A$ be a BAI for B. If B^{**} is Banach $\sim -$involution algebra as A^{**}-module, then B^{**} is unital and B^* factors on the both side.*

Example 3.2. [10] *Let G be a locally compact group. Then, on $L^1(G)$ there is a natural involution $\sim$ defined by $f^\sim(x) = \Delta(x^{-1})\overline{f(x^{-1})}$, where Δ is modular function and $x \in G$. By preceding theorem if $L^1(G)^{**}$ is a Banach $\sim -$involution algebra, then $L^1(G)^{**}$ is unital with respect to first Arens product, and so $LUC(G) = L^\infty(G)$. It follows that G is discrete.*

Problem . Let G be a locally compact group. We know that $M(G)$ by $\mu^\sim(f) = \overline{\int \overline{f(x^{-1})} d\mu}$ for all $f \in C_0(G)$ is $\sim -$*involution* algebra. Is there any extension of this $\sim -$*involution* algebra to $M(G)^{**}$ whenever $M(G)^{**}$ equipped first Arens product.

Acknowledgments

We would like to thank the referees for careful reading of our paper and many valuable suggestions.

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