U.P.B. Sci. Bull., Series A, Vol. 73, Iss. 2, 2011 ISSN 1223-7027

PREDA-MITITELU DUALITY FOR MULTIOBJECTIVE VARIATIONAL PROBLEMS

Ştefan MITITELU[1], Mihai POSTOLACHE[2]

Folosind condiţii de eficienţă normală, în această lucrare, introducem un dual tip Preda-Mititelu pentru o problemă variaţională multiobiectiv. Apoi, folosind ipoteze de (ρ, b)-quasiinvexitate, enunţăm şi demonstrăm teoreme de dualitate slabă, directă şi reciprocă.

Based on the normal efficiency conditions for a multiobjective variational problem, in this work we consider a Preda-Mititelu type dual, and under some assumptions of (ρ, b)-quasiinvexity, weak, direct and converse duality theorems are introduced and proved.

Keywords: Multiobjective variational problem, efficient solution, quasiinvexity.
MSC2000: Primary 65K10; Secondary 90C29.

1. Introduction and problem statement

The first result on the necessity of optimal solutions of scalar variational problems was established by Valentine [18] in 1937. The papers of Mond and Hanson [10], Mond, Chandra and Husain [11], Mond and Husain [12], Preda [16] developed the duality of the scalar variational problems involving convex and generalized convex functions. Mukherjee and Purnachandra [13], established weak efficiency conditions and developed different types of dualities for multiobjective variational problems under various types of generalized convex functions. Kim and Kim [2] used the efficiency property of the nondifferentiable multiobjective variational problems in duality theory.

In this work, we use the notion of normal efficient solution introduced by Mititelu [5] and establish certain new results of Preda-Mititelu duality type [3], [15] for multiobjective variational problems using (ρ, b)-quasiinvexity assumptions.

For related but different results obtained by other authors on this topic, we address the reader to [14] by Ariana Pitea, C. Udrişte and Şt. Mititelu.

In $\mathbb{R}^n$, the n-dimensional Euclidean space, consider the vectors $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. We recall that the relations $v = w$, $v < w$, $v \leqq w$, $v \leq w$ are defined as follows:

$$v = w \Leftrightarrow v_i = w_i, \quad i = \overline{1, n}; \qquad v < w \Leftrightarrow v_i < w_i, \quad i = \overline{1, n};$$
$$v \leqq w \Leftrightarrow v_i \leqq w_i, \quad i = \overline{1, n}; \qquad v \leq w \Leftrightarrow v \leqq w \text{ and } v \neq w.$$

[1]Professor, Technical University of Civil Engineering, 124 Bd. Lacul Tei, 020396 Bucharest, Romania, E-mail: st_mititelu@yahoo.com

[2] Professor, Department of Mathematics and Informatics I, Faculty of Applied Sciences, University "POLITEHNICA" of Bucharest, Romania, E-mail: mihai@mathem.pub.ro

Let $I=[a,b]$ be a real interval and $f=(f_1,\dots,f_p)\colon I\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^p$, $g=(g_1,\dots,g_m)\colon I\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^m$ and $h=(h_1,\dots,h_q)\colon I\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^q$ be two times differentiable functions.

Consider a vector-valued function $f(t,x,\dot{x})$, where $t\in I$, $x\colon I\to\mathbb{R}^n$ and $\dot{x}=\dfrac{\mathrm{d}x}{\mathrm{d}t}$. Denote by f_x and $f_{\dot{x}}$ the $p\times n$ matrices of first-order partial derivatives with respect to x and $\dot{x}$ respectively, that is $f_x=(f_{1x},f_{2x},\dots,f_{px})'$ and $f_{\dot{x}}=(f_{1\dot{x}},f_{2\dot{x}},\dots,f_{p\dot{x}})'$, with

$$f_{ix}=\left(\frac{\partial f_i}{\partial x_1},\dots,\frac{\partial f_i}{\partial x_n}\right)\quad\text{and}\quad f_{i\dot{x}}=\left(\frac{\partial f_i}{\partial \dot{x}_1},\dots,\frac{\partial f_i}{\partial \dot{x}_n}\right),\quad i=1,2,\dots,p.$$

By analogy, g_x, h_x and $g_{\dot{x}}$, $h_{\dot{x}}$ denote the $p\times n$, $n\times n$, $q\times n$ matrices of the first order partial derivatives of g and h respectively, with respect to x and $\dot{x}$.

Let X denote the space of piecewise smooth (continuously differentiable) functions x with the norm $\|x\|=\|x\|_\infty+\|Dx\|_\infty$, where the differential operator $D=\dfrac{\mathrm{d}}{\mathrm{d}t}$ is given by $u=Dx\Leftrightarrow x(t)=x(a)+\displaystyle\int_a^t u(s)\mathrm{d}s$, excepting the discontinuities, where $x(a)$ is a given boundary value.

Important note. To simplify the presentation, in our subsequent theory, we shall set

$$\pi_x(t)=(t,x(t),\dot{x}(t)),\quad \pi_{x^0}(t)=(t,x^0(t),\dot{x}^0(t)),\quad \pi_y(t)=(t,y(t),\dot{y}(t)).$$

Introduce the following multiobjective variational problem

$$\text{(MP)}\begin{cases}\min\displaystyle\int_a^b f(\pi_x(t))\mathrm{d}t=\left(\int_a^b f_1(\pi_x(t))\mathrm{d}t,\dots,\int_a^b f_p(\pi_x(t))\mathrm{d}t\right)\\ \text{subject to}\\ \quad x(a)=a_0,\quad x(b)=b_0,\\ \quad g(\pi_x(t))\leqq 0,\quad h(\pi_x(t))=0,\quad t\in I,\end{cases}$$

and denote

$$\mathcal{D}=\{x\in X\mid x(a)=a_0,\ x(b)=b_0,\ g(\pi_x(t))\leqq 0,\ h(\pi_x(t))=0,\ \forall t\in I\}$$

the set of all feasible solutions of problem (MP).

2. Previous results

In this section, we recall some definitions and auxiliary results that will be needed later in our discussion of efficiency conditions and Preda-Mititelu duality [3], [15] for problem (MP).

Definition 2.1. ([1]) A feasible solution $x^0\in\mathcal{D}$ is an *efficient solution* of problem (MP) if there is no $x\in\mathcal{D}$, $x\neq x^0$, such that

$$\int_a^b f(\pi_x(t))\mathrm{d}t\leq\int_a^b f(\pi_{x^0}(t))\mathrm{d}t.$$

For problem (MP) we quote the following result of efficiency

Theorem 2.1. (NECESSARY EFFICIENCY CONDITIONS FOR (MP)) ([3], [?]) *Let $x^0 \in \mathcal{D}$ be an efficient solution of problem* (MP). *Then there exist a vector $\lambda^0 \in \mathbb{R}^p$ and piecewise smooth functions $\mu^0 \colon I \to \mathbb{R}^m$ and $\nu^0 \colon I \to \mathbb{R}^q$ which satisfy*

$$(\mathrm{MV}) \begin{cases} \lambda^{0\,\prime} f_x(\pi_{x^0}(t)) + \mu^0(t)' g_x(\pi_{x^0}(t)) + \nu^0(t)' h_x(\pi_{x^0}(t)) \\ \quad = \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\lambda^{0\,\prime} f_{\dot{x}}(\pi_{x^0}(t)) + \mu^0(t)' g_{\dot{x}}(\pi_{x^0}(t)) + \nu^0(t)' h_{\dot{x}}(\pi_{x^0}(t))\right] \\ \mu^0(t)' g(\pi_{x^0}(t)) = 0, \quad \mu_i(t) \geqq 0, \quad \forall t \in I, \\ \lambda^0 \geqq 0. \end{cases}$$

Definition 2.2. $x^0 \in \mathcal{D}$ is called *normal efficient solution* of problem (MP) if $\lambda^0 \geq 0$, or equivalent if $e'\lambda^0 = 1$, where $e = (1, \ldots, 1) \in \mathbb{R}^p$.

Now, let us consider ρ be a real number and $b \colon X \times X \to [0, \infty)$ a functional. Denote

$$H(x) = \int_a^b h(\pi_x(t))\mathrm{d}t.$$

Definition 2.3. The function H is *(strictly)* (ρ, b)*-quasiinvex* at the point x^0 if there exist vector functions $\eta \colon I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^q$, with $\eta(\pi_x(t)) = 0$ for $x(t) = x^0(t)$, and $\theta \colon X \times X \to \mathbb{R}^n$ such that for any x ($x \neq x^0$),

$$\begin{aligned} H(x) \leqq H(x^0) \;\Rightarrow\; & b(x, x^0) \int_a^b \left[\eta' h_x(\pi_{x^0}(t)) + (D\eta)' h_{\dot{x}}(\pi_{x^0}(t))\right] \mathrm{d}t \\ & (<) \leqq -\rho b(x, x^0) \left\| \theta(x, x^0) \right\|^2. \end{aligned}$$

Definition 2.4. The function H is *monotonic* (ρ, b)*-quasiinvex* at the point x^0 if there exist vector functions $\eta \colon I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^q$ with $\eta(\pi_x(t)) = 0$ for $x(t) = x^0(t)$, and $\theta \colon X \times X \to \mathbb{R}^n$ such that for any x ($x \neq x^0$),

$$\begin{aligned} H(x) = H(x^0) \;\Rightarrow\; & b(x, x^0) \int_a^b \left[\eta' h_x(\pi_{x^0}(t)) + (D\eta)' h_{\dot{x}}(\pi_{x^0}(t))\right] \mathrm{d}t \\ & = -\rho b(x, x^0) \left\| \theta(x, x^0) \right\|^2. \end{aligned}$$

3. Multiobjective Preda-Mititelu duality for (MP)

Let $\{J_1, \ldots, J_r\}$ be a partition of the set $J = \{1, \ldots, m\}$ and $\{S_1, \ldots, S_r\}$ a partition of $S = \{1, \ldots, q\}$. Consider the functions $y \in X$ and the piecewise nonsmooth functions $\mu \colon I \to \mathbb{R}^m$ and $\nu \colon I \to \mathbb{R}^q$. The Lagrangian associated to (MP) is

$$L(\pi_y(t)) \;=\; f(\pi_y(t)) + \left[\mu(t)' g(\pi_y(t)) + \nu(t)' h(\pi_y(t))\right] e,$$

where $L = (L_1, \ldots, L_p)$ and for $i = \overline{1, p}$,

$$L_i(\pi_y(t)) \;=\; f_i(\pi_y(t)) + \mu(t)' g(\pi_y(t)) + \nu(t)' h(\pi_y(t)).$$

The multiobjective dual Preda-Mititelu problem [3], associated to (MP), is the next multiobjective variational problem:

$$
\text{(MPD)}\begin{cases}
\text{Maximize } \displaystyle\int_a^b L(\pi_y(t))\mathrm{d}t = \left(\int_a^b L_1(\pi_y(t)), \dots, \int_a^b L_p(\pi_y(t))\mathrm{d}t\right) \\
\text{subject to} \\
\quad y(a) = y_0, \quad y(b) = b_0, \\
\quad \lambda' f_y(\pi_y(t)) + \mu(t)' g_y(\pi_y(t)) + \nu(t)' h_y(\pi_y(t)) \\
\qquad = \dfrac{\mathrm{d}}{\mathrm{d}t}\left\{\lambda' f_{\dot y}(\pi_y(t)) + \mu(t)' g_{\dot y}(\pi_y(t)) + \nu(t)' h_{\dot y}(\pi_y(t))\right\} \\
\quad \mu_{J_\alpha}(t)' g_{J_\alpha}(\pi_y(t)) \geqq 0, \ \nu_{S_\alpha}(t) h_{S_\alpha}(\pi_y(t)) = 0, \ \alpha = \overline{1,r}, \ t \in I, \\
\quad \lambda \geq 0, \quad e'\lambda = 1, \ \mu(t) \geqq 0, \ t \in I.
\end{cases}
$$

We denote by $\varpi(x)$ the value of problem (MP) at $x \in \mathcal{D}$ and by $\delta(y, \lambda, \mu, \nu)$ the value of dual (MPD) at $(y, \lambda, \mu, \nu) \in \Delta$, where Δ is the domain of (MPD). We assume that the elements of Δ and $\mathcal{D}$ are corresponding.

Theorem 3.1. (WEAK DUALITY) *Let $x \in \mathcal{D}$ and $(y, \lambda, \mu, \nu) \in \Delta$. Assume satisfied the following conditions:*

a) *For each $i = \overline{1,p}$, the integral $\displaystyle\int_a^b L_i(\pi_x(t))\mathrm{d}t$ is strictly (ρ_i, b)-quasiinvex at y with respect to η and θ.*

b) $\displaystyle\sum_{i=1}^{p} \lambda_i \rho_i \geqq 0.$

Then $\varpi(x) \leq \delta(y, \lambda, \mu, \nu)$ is false.

Proof. We proceed by reductio ad absurdum. Suppose there exist points $x \in \mathcal{D}$ and $(y, \lambda, \mu, \nu) \in \Delta$ such that $\varpi(x) \leq \delta(y, \lambda, \mu, \nu)$. Hence,

$$\int_a^b f(\pi_x(t))\mathrm{d}t \leq \int_a^b L(\pi_y(t))\mathrm{d}t,$$

or componentwise

$$\int_a^b f_i(\pi_x(t))\mathrm{d}t \ \leq \ \int_a^b L_i(\pi_y(t))\mathrm{d}t, \quad i = \overline{1,p}. \tag{3.1}$$

But $\mu(t)' g(\pi_x(t)) \leqq 0$, and $\nu(t)' h(\pi_x(t)) = 0, \quad \forall t \in I$. Using (3.1), we obtain

$$\int_a^b [f_i(\pi_x(t)) + \mu(t)' g(\pi_x(t)) + \nu(t)' h(\pi_x(t))]\mathrm{d}t \leq \int_a^b L_i(\pi_y(t))\mathrm{d}t, \quad i = \overline{1,p},$$

that is

$$\int_a^b L_i(\pi_x(t))\mathrm{d}t \leq \int_a^b L_i(\pi_y(t))\mathrm{d}t, \quad i = \overline{1,p}. \tag{3.2}$$

According to hypothesis a), for $i = \overline{1,p}$, (3.2) implies

$$b(x, y) \int_a^b \eta' \left[L_{iy}(\pi_y(t)) + D\eta' L_{i\dot y}(\pi_y(t))\right] dt < -\rho b(x, y) \|\theta(x, y)\|^2. \tag{3.3}$$

Multiplying (3.3) by $\lambda_i \geqq 0$, $e'\lambda = 1$ and summing after i, we obtain

$$b(x,y)\int_a^b \eta'\big[\lambda' L_y(\pi_y(t)) + (D\eta')\lambda' L_{\dot{y}}(\pi_y(t))\big]\mathrm{d}t$$
$$< -\Big(\sum_{i=1}^{p}\lambda_i\rho_i\Big) b(x,y)\|\theta(x,y)\|^2.$$

Therefore, $b(x,y) > 0$ and taking into account the first constraint of dual (MPD), this inequality becomes

$$0 < -\|\theta(x,y)\|^2\Big(\sum_{i=1}^{p}\lambda_i\rho_i\Big),$$

which gets $0 < 0$, that is false.

Corollary 3.1. (WEAK DUALITY) *Let $x \in \mathcal{D}$ and $(y,\lambda,\mu,\nu) \in \Delta$. Suppose:*

a) *the integral $\int_a^b \lambda' L(\pi_x(t))\mathrm{d}t$ is strictly (ρ,b)-quasiinvex at y with respect to η and θ.*

b) $\rho \geqq 0$.

Then $\varpi(x) \leq \delta(y,\lambda,\mu,\nu)$ is false.

Theorem 3.2. (WEAK DUALITY) *Let $x \in \mathcal{D}$ and $(y,\lambda,\mu,\nu) \in \Delta$. Assume satisfied the following conditions:*

a) *For each $i = \overline{1,p}$, and $t \in I$,*

$$f_i\big(\pi_x(t)\big) \leqq f_i\big(\pi_y(t)\big) \Rightarrow \int_a^b \big[\eta' f_{ix}\big(\pi_y(t)\big) + (D\eta)' f_{i\dot{x}}\big(\pi_y(t)\big)\big]\mathrm{d}t \leqq 0.$$

b) *For each $\alpha = \overline{1,r}$, either*

(b1) *all the integrals $\int_a^b \big[\mu_{J_\alpha}(t)' g_{J_\alpha}\big(\pi_x(t)\big) + \nu_{K_\alpha}(t)' h_{K_\alpha}\big(\pi_x(t)\big)\mathrm{d}t$ are (ρ_α,b)-quasiinvex (and one of them being strictly (ρ_α,b)-quasiinvex) at y with respect to η and θ;*

(c1) $\sum_{\alpha=1}^{r}\rho_\alpha \geqq 0$;

or

(b2) *all the integrals $\int_a^b \mu_{J_\alpha}(t)' g_{J_\alpha}\big(\pi_x(t)\big)\mathrm{d}t$ are $(\rho_{1\alpha},b)$-quasiinvex (one of them being strictly $(\rho_{1\alpha},b)$-quasiinvex) at y and integrals $\int_a^b \nu_{K_\alpha}(t)' h_{K_\alpha}\big(\pi_x(t)\big)\mathrm{d}t$ are monotonic $(\rho_{2\alpha},b)$-quasiinvex at y, all with respect to η and θ;*

(c2) $\sum_{\alpha=1}^{r}(\rho_{1\alpha}+\rho_{2\alpha}) \geqq 0$.

Then $\varpi(x) \leq \delta(y,\lambda,\mu,\nu)$ is false.

Proof. It is sufficient to prove the version with hypotheses (a)+(b1)+(c1). We proceed by contradiction. Suppose there exist $x \in \mathcal{D}$ and $(y,\lambda,\mu,\nu) \in \Delta$ such that

$\varpi(x) \le \delta(y,\lambda,\mu,\nu)$, therefore for all $i=\overline{1,p}$, we have

$$\int_a^b f\big(\pi_x(t)\big)\mathrm{d}t \le \int_a^b L\big(\pi_y(t)\big)\mathrm{d}t,$$

or componentwise

$$\int_a^b f_i\big(\pi_x(t)\big)\mathrm{d}t \le \int_a^b L_i\big(\pi_y(t)\big). \tag{3.4}$$

But $\mu(t)'g\big(\pi_x(t)\big) \le 0, \quad \nu(t)'h\big(\pi_x(t)\big)=0, \quad \forall t \in I$ and from (3.4) it follows

$$\int_a^b L_i\big(\pi_x(t)\big)\mathrm{d}t \leqq \int_a^b L_i\big(\pi_y(t)\big)\mathrm{d}t, \quad i=\overline{1,p}. \tag{3.5}$$

From the constraints of $\mathcal{D}$ and Δ, we have

$$\begin{aligned}\int_a^b &\big[\mu_{J_\alpha}(t)'g_{J_\alpha}\big(\pi_x(t)\big)+\nu_{S_\alpha}(t)'h_{S_\alpha}\big(\pi_x(t)\big)\big]\mathrm{d}t\\ &\leqq \int_a^b \big[\mu_{J_\alpha}(t)'g_{J_\alpha}\big(\pi_y(t)\big)+\nu_{S_\alpha}(t)'h_{S_\alpha}\big(\pi_y(t)\big)\big]\mathrm{d}t\end{aligned} \tag{3.6}$$

and according to b), we obtain

$$\begin{aligned}&b(x,y)\int_a^b \eta\big[\mu_{J_\alpha}(t)'(g_{J_\alpha})_x\big(\pi_y(t)\big)+\nu_{S_\alpha}(t)'(h_{S_\alpha})_x\big(\pi_y(t)\big)\big]\mathrm{d}t\\ &\quad+b(x,y)\int_a^b D\eta'\big[\mu_{J_\alpha}(t)'(g_{J_\alpha})_x\big(\pi_y(t)\big)+\nu_{S_\alpha}(t)'(h_{S_\alpha})_x\big(\pi_y(t)\big)\big]\mathrm{d}t\\ &\leqq -\rho_\alpha b(x,y)\|\theta(x,y)\|.\end{aligned} \tag{3.7}$$

Summing side by side after $\alpha=\overline{1,r}$ in (3.6) and (3.7), and a), we obtain

$$\begin{aligned}\int_a^b &\big[f_i\big(\pi_x(t)\big)+\mu(t)'g\big(\pi_x(t)\big)+\nu(t)'h\big(\pi_x(t)\big)\big]\mathrm{d}t\\ &\leqq \int_a^b \big[f_i\big(\pi_y(t)\big)+\mu(t)'g\big(\pi_y(t)\big)+\nu(t)'h\big(\pi_y(t)\big)\big]\mathrm{d}t,\end{aligned}$$

(that is (3.5)), which implies

$$\begin{aligned}&b(x,y)\int_a^b \eta\big[f_{ix}\big(\pi_y(t)\big)+\mu(t)'g_x\big(\pi_y(t)\big)+\nu(t)'h_x\big(\pi_y(t)\big)\big]\mathrm{d}t\\ &\quad+b(x,y)\int_a^b D\eta'\big[f_{i\dot{x}}\big(\pi_y(t)\big)+\mu(t)'g_{\dot{x}}\big(\pi_y(t)\big)+\nu(t)'h_{\dot{x}}\big(\pi_y(t)\big)\big]\mathrm{d}t\\ &< -\Big(\sum_{\alpha=1}^r \rho_\alpha\Big)b(x,y)\|\theta(x,y)\|^2\end{aligned}$$

or, shortly,

$$\int_a^b \big\{\eta\big[L_{ix}\big(\pi_y(t)\big)+(D\eta')\big[\lambda' L_{i\dot{x}}\big(\pi_y(t)\big)\big]\big\}\,\mathrm{d}t < -\Big(\sum_{\alpha=1}^r \rho_\alpha\Big)b(x,y)\|\theta(x,y)\|^2, \tag{3.8}$$

where $b(x,y) > 0$. Therefore, from (3.5) with $x(t) \neq y(t)$ and (3.8) we see that integrals $\int_a^b L_i\big(t, x(t), \mu(t), \nu(t)\big)\mathrm{d}t$, $i = \overline{1,p}$, are strictly $\Big(\sum_{\alpha=1}^r \rho_\alpha, b\Big)$-quasiinvex at y with respect to η and θ.

Multiplying (3.8) by λ_i and summing after $i = \overline{1,p}$, we obtain

$$\int_a^b \{\eta\big[\lambda' L_x\big(\pi_y(t)\big) + (D\eta')\big[\lambda' L_{\dot{x}}\big(\pi_y(t)\big)\big]\big\}\,\mathrm{d}t < -\Big(\sum_{\alpha=1}^r \rho_\alpha\Big) b(x,y)\|\theta(x,y)\|^2.$$

Taking into account the first constraint of problem (MPD), the above relation becomes $0 < -\|\theta(x,y)\|^2\Big(\sum_{\alpha=1}^r \rho'_\alpha\Big)$ and with (c1), it follows $0 < 0$, that is false. According to Theorem 3.1, the supposition made at the beginning is false.

Corollary 3.2. (WEAK DUALITY) *Let $x \in \mathcal{D}$ and $(y, \lambda, \mu, \nu) \in \Delta$ and assume satisfied the following conditions:*

a) *For all $t \in I$, we have*

$$\lambda' f\big(\pi_x(t)\big) \leqq \lambda' f\big(\pi_y(t)\big) \Rightarrow \int_a^b \big[\eta\lambda' f_x\big(\pi_y(t)\big) + (D\eta)'\lambda' f_{\dot{x}}\big(\pi_y(t)\big)\big]\mathrm{d}t \leqq 0.$$

b) *For each $\alpha = \overline{1,r}$, either*

(b1) *all the integrals* $\int_a^b \big[\mu_{J_\alpha}(t)' g_{J_\alpha}\big(\pi_x(t)\big) + \nu_{K_\alpha}(t)' h_{K_\alpha}\big(\pi_x(t)\big)\mathrm{d}t$ *are (ρ_α, b)-quasiinvex (and one of them being strictly (ρ_α, b)-quasiinvex) at y with respect to η and θ;*

(c1) $\sum_{\alpha=1}^r \rho_\alpha \geqq 0$;

or

(b2) *all the integrals* $\int_a^b \mu_{J_\alpha}(t)' g_{J_\alpha}\big(\pi_x(t)\big)\mathrm{d}t$ *are $(\rho_{1\alpha}, b)$-quasiinvex (one of them being strictly $(\rho_{1\alpha}, b)$-quasiinvex) at y and integrals* $\int_a^b \nu_{K_\alpha}(t)' h_{K_\alpha}\big(\pi_x(t)\big)\mathrm{d}t$ *are monotonic $(\rho_{2\alpha}, b)$-quasiinvex at y, all with respect to η and θ;*

(c2) $\sum_{\alpha=1}^r (\rho_{1\alpha} + \rho_{2\alpha}) \geqq 0$.

Then $\varpi(x) \leq \delta(y, \lambda, \mu, \nu)$ is false.

Proof. It follows from the proof of Theorem 3.2.

Corollary 3.3. (WEAK DUALITY) *Let $x \in \mathcal{D}$ and $(y, \lambda, \mu, \nu) \in \Delta'$ and assume satisfied the following conditions:*

a) *For $i = \overline{1,p}$, $\alpha = \overline{1,p}$, the implication holds:*

$$f_i\big(\pi_x(t)\big) \leqq f_i\big(\pi_y(t)\big) \Rightarrow$$
$$\int_a^b \Big\{\eta\big[\mu_\alpha(t)' g_{J_\alpha y}\big(\pi_y(t)\big) + \nu_{S_\alpha}(t)' h_{S_\alpha \dot{y}}\big(\pi_y(t)\big)\big]$$
$$+ (D\eta)'\big[\mu_\alpha(t)' g_{J_\alpha y}\big(\pi_y(t)\big) + \nu_{S_\alpha}(t)' h_{S_\alpha \dot{y}}\big(\pi_y(t)\big)\big]\Big\}\mathrm{d}t \geqq 0.$$

b) *For each* $\alpha = \overline{1, r}$*, either*

(b1) *all the integrals* $\int_a^b \big[\mu_{J_\alpha}(t)' g_{J_\alpha}\big(\pi_x(t)\big) + \nu_{K_\alpha}(t)' h_{K_\alpha}\big(\pi_x(t)\big) \mathrm{d}t$ *are* (ρ_α, b)*-quasiinvex (and one of them being strictly* (ρ_α, b)*-quasiinvex) at* y *with respect to* η *and* θ*;*

(c1) $\sum_{\alpha=1}^{r} \rho_\alpha \geqq 0$*;*

or

(b2) *all the integrals* $\int_a^b \mu_{J_\alpha}(t)' g_{J_\alpha}\big(\pi_x(t)\big) \mathrm{d}t$ *are* $(\rho_{1\alpha}, b)$*-quasiinvex (one of them being strictly* $(\rho_{1\alpha}, b)$*-quasiinvex) at* y *and integrals* $\int_a^b \nu_{K_\alpha}(t)' h_{K_\alpha}\big(\pi_x(t)\big) \mathrm{d}t$ *are monotonic* $(\rho_{2\alpha}, b)$*-quasiinvex at* y*, all with respect to* η *and* θ*;*

(c2) $\sum_{\alpha=1}^{r} (\rho_{1\alpha} + \rho_{2\alpha}) \geqq 0$.

Then $\varpi(x) \leq \delta(y, \lambda, \mu, \nu)$ *is false.*

Proof. The first constraint of (MPD) and Theorem 3.2 are used.

Theorem 3.3. (Direct duality) *Let* x^0 *be a normal efficient solution for* (MP) *and suppose satisfied the hypotheses of Theorem* 3.1. *Then there are* $\lambda^0 \in \mathbb{R}^p$ *and the piecewise smooth functions* $\mu^0 \colon I \to \mathbb{R}^m$ *and* $\nu^0 \colon I \to \mathbb{R}^q$ *such that* $(x^0, \lambda^0, \mu^0, \nu^0)$ *is an efficient solution to dual* (MPD) *and* $\varpi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Proof. According to Theorem 2.1 and Definition 2.3 conditions (MV) are satisfied. Also $\nu^0(t)' h\big(\pi_{x^0}(t)\big) = 0$. Then $(x^0, \lambda^0, \mu^0, \nu^0) \in \Delta$. Moreover,

$$\int_a^b f\big(\pi_{x^0}(t)\big) \mathrm{d}t = \int_a^b \big[f\big(\pi_{x^0}(t)\big) + \mu^0(t)' g\big(\pi_{x^0}(t)\big) + \nu^0(t)' h\big(\pi_{x^0}(t)\big)\big] \mathrm{d}t = 0,$$

that is $\varpi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Corollary 3.4. (Direct duality) *Let* x^0 *a normal efficient solution for* (MP) *and suppose satisfied the hypotheses of Corollary* 3.1. *Then there are* $\lambda^0 \in \mathbb{R}^p$ *and the piecewise smooth functions* $\mu^0 \colon I \to \mathbb{R}^m$ *and* $\nu^0 \colon I \to \mathbb{R}^q$ *such that* $(x^0, \lambda^0, \mu^0, \nu^0)$ *is an efficient solution to dual* (MPD) *and* $\varpi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Theorem 3.4. (Direct duality) *Let* x^0 *a normal efficient solution for* (MP) *and suppose satisfied the hypotheses of Theorem* 3.2. *Then there are* $\lambda^0 \in \mathbb{R}^p$ *and the piecewise smooth functions* $\mu^0 \colon I \to \mathbb{R}^m$ *and* $\nu^0 \colon I \to \mathbb{R}^q$ *such that* $(x^0, \lambda^0, \mu^0, \nu^0)$ *is an efficient solution to dual* (MPD) *and* $\varpi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Corollary 3.5. (Direct duality) *Let* x^0 *a normal efficient solution for* (MP) *and suppose satisfied the hypotheses of Corollary* 3.2. *Then there are* $\lambda^0 \in \mathbb{R}^p$ *and the piecewise smooth functions* $\mu^0 \colon I \to \mathbb{R}^m$ *and* $\nu^0 \colon I \to \mathbb{R}^q$ *such that* $(x^0, \lambda^0, \mu^0, \nu^0)$ *is an efficient solution to dual* (MPD) *and* $\varpi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Corollary 3.6. (Direct duality) *Let* x^0 *a normal efficient solution for* (MP) *and suppose satisfied the hypotheses of Corollary* 3.3. *Then there are* $\lambda^0 \in \mathbb{R}^p$ *and the piecewise smooth functions* $\mu^0 \colon I \to \mathbb{R}^m$ *and* $\nu^0 \colon I \to \mathbb{R}^q$ *such that* $(x^0, \lambda^0, \mu^0, \nu^0)$ *is an efficient solution to dual* (MPD) *and* $\varpi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Theorem 3.5. (CONVERSE DUALITY) *Let* $(x^0, \lambda^0, \mu^0, \nu^0)$ *be an efficient solution of the dual* (MPD) *and suppose satisfied the following conditions:*

i) $x^0 \in \mathcal{D}$.

ii) *For each* $i = \overline{1,p}$ *integrals* $\displaystyle\int_a^b L_i\big(\pi_x(t)\big)\mathrm{d}t$ *are strictly* (ρ_i, b)*-quasiinvex at* x^0 *with respect to* η *and* θ.

Then x^0 *is an efficient solution to* (MP). *Moreover,* $\varpi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Proof. On the contrary, suppose that x^0 is not an efficient solution to (MP) and then we shall find a contradiction. Then, there exists $x \in \mathcal{D}$ such that, $\varpi(x) \leq \delta(x^0, \lambda^0, \mu^0, \nu^0)$. Following the proof of Theorem 3.1 with $(x^0, \lambda^0, \mu^0, \nu^0)$ instead of (y, λ, μ, ν), we obtain

$$0 < -\Big(\sum_{i=1}^{p} \lambda_i \rho_i\Big)\big\|\theta(x, x^0)\big\|^2,$$

which yelds $0 < 0$. Consequently, supposition above made is false.

Corollary 3.7. (CONVERSE DUALITY) *Let* $(x^0, \lambda^0, \mu^0, \nu^0)$ *be an efficient solution of the dual* (MPD) *and suppose satisfied the following conditions:*

i) $x^0 \in \mathcal{D}$.

ii) *Integral* $\displaystyle\int_a^b \lambda' L\big(\pi_x(t)\big)\mathrm{d}t$ *is strictly* (ρ, b)*-quasiinvex at* x^0 *with respect to* η *and* θ *and* $\rho \geqq 0$.

Then x^0 *is an efficient solution to* (MP). *Moreover,* $\varpi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Theorem 3.6. (CONVERSE DUALITY) *Let* $(x^0, \lambda^0, \mu^0, \nu^0)$ *be an efficient solution of the dual* (MPD) *and suppose satisfied the following conditions:*

i) $x^0 \in \mathcal{D}$.

ii) *The hypotheses* a)-b) *of Theorem* 3.2 *hold for* $(y, \lambda, \mu, \nu) = (x^0, \lambda^0, \mu^0, \nu^0)$.

Then x^0 *is an efficient solution to* (MP). *Moreover,* $\varpi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Corollary 3.8. (CONVERSE DUALITY) *Let* $(x^0, \lambda^0, \mu^0, \nu^0)$ *be an efficient solution of the dual* (MPD) *and suppose satisfied the following conditions:*

i) $x^0 \in \mathcal{D}$.

ii) *The hypotheses* a)-b) *of Corollary* 3.2 *hold for* $(y, \lambda, \mu, \nu) = (x^0, \lambda^0, \mu^0, \nu^0)$.

Then x^0 *is an efficient solution to* (MP). *Moreover,* $\varpi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Corollary 3.9. (CONVERSE DUALITY) *Let* $(x^0, \lambda^0, \mu^0, \nu^0)$ *be an efficient solution of the dual* (MPD) *and suppose satisfied the following conditions:*

i) $x^0 \in \mathcal{D}$.

ii) *The hypotheses* a)-b) *of Corollary* 3.3 *hold for* $(y, \lambda, \mu, \nu) = (x^0, \lambda^0, \mu^0, \nu^0)$.

Then x^0 *is an efficient solution to* (MPD) *and* $\varpi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

4. **Conclusion**

Based on the normal efficiency conditions for a multiobjective variational problem, we introduced a Preda-Mititelu type dual, and under some assumptions of (ρ, b)-quasiinvexity, weak, direct and converse duality theorems are introduced and proved. The present study completes several results included in [3] and [14]. For other advances on this subject, the reader is encouraged to study [1]÷[18].

REFERENCES

[1] *M. A. Geoffrion*, Proper efficiency and the theory of vector minimization, J. Math. Anal. Appl., **149**(1968), No. 6, 618-630.

[2] *D. S. Kim and A. L. Kim*, Optimality and duality for nondifferentiable multiobjective variational problems, J. Math. Appl., **274**(2002), 255-278.

[3] *Şt. Mititelu*, Kuhn-Tucker conditions and Preda duality for multiobjective mathematical programs with n-set functions, Sci. Bull. UPB, Ser. A: Appl. Math. Phys., **64**(2002), No. 4, 3-16.

[4] *Şt. Mititelu*, Invex functions, Rev. Roumaine Math. Pures Appl, **49**(2004), No. 5-6, 529-544.

[5] *Şt. Mititelu*, Efficiency and duality for multiobjective fractional problems in optimal control with ρ-quasiinvexity, Conf. Proc. ICTCAM 2007, June 20-23, 2007, Bucharest.

[6] *Şt. Mititelu*, Extensions in invex theory, J. Adv. Math. Stud., **1**(2008), No. 1-2, 63-70.

[7] *Şt. Mititelu*, Optimality and duality for invex multi-time control problems with mixed constraints, J. Adv. Math. Stud., **2**(2009), No. 1, 25-34.

[8] *Şt. Mititelu*, Invex sets and preinvex functions, J. Adv. Math. Stud., **2**(2009), No. 2, 41-52.

[9] *Şt. Mititelu and M. Postolache*, Mond-Weir dualities with Lagrangians for multiobjective fractional and non-fractional variational problems, J. Adv. Math. Stud., **3**(2010), No. 1, 41-58.

[10] *B. Mond and M. A. Hanson*, Duality for variational problems, J. Math. Anal. Appl., **18**(1967), 355-364.

[11] *B. Mond, S. Chandra and I. Husain*, Duality for variational problems with invexity, J. Math. Anal. Appl., **134**(1988), No. 2, 322-328.

[12] *B. Mond and I. Husain*, Sufficient optimality criteria and duality for variational problems with generalized invexity, J. Aust. Math. Soc., Ser. B, **31**(1989), No. 1, 108-121.

[13] *R. N. Mukherjee and Ch. Purnachandra Rao*, Mixed type duality for multiobjective variational problems, J. Math. Anal. Appl., **225**(2000), 571-586.

[14] *Ariana Pitea, C. Udrişte and Şt. Mititelu*, New type dualities in PDI and PDE constrained optimization problems, J. Adv. Math. Stud., **2**(2009), No. 1, 81-90.

[15] *V. Preda*, Duality for multiobjective mathematical programming involving n-set functions, Scr. Sci. Mat., **1**(1999), No. 2, 339-354.

[16] *V. Preda*, On Mond-Weir duality for variational problems, Rev. Roumanie Math. Pures Appl., **28**(1993), No. 2, 155-164.

[17] *V. Preda and Sorina Gramatovici*, Some sufficient optimality conditions for class of multiobjective variational problems, An. Univ. Bucur, Ser. Mat. Inf., **61**(2002), No. 1, 33-43.

[18] *F. A. Valentine*, The problem of Lagrange with differentiable inequality as added side conditions, 407-448, in "Contributions to the Calculus of Variations", 1933-37, Univ. of Chicago Press, 1937.