TWO SIMPLE FORMULAE FOR HALL-GEOMETRY FACTOR OF HALL-PLATES WITH 90° SYMMETRY

Udo AUSSERLECHNER[1]

Symmetrical Hall-plates with equal input and output resistance are considered. The Hall-geometry factor accounts for the loss of output voltage due to finite size of contacts. In the limit of low magnetic field it depends only on the effective number of squares (i.e. the ratio of internal resistance over sheet resistance). For the first time two simple formulae are given which express the Hall-geometry factor as a function of effective number of squares. These approximations are accurate up to 2% and 0.02%, respectively, for arbitrary contact size. They also suggest a symmetry property not mentioned in the literature. The results hold for all shapes of Hall-plates with four electrodes which are invariant to rotations of 90°. Thus, they can be used for the optimization of spinning current Hall probes by electronic circuit design engineers.

Keywords: Hall effect, Hall-geometry factor, Hall voltage.

1. Introduction

Hall plates are widely used in today's industry to measure linear or rotational position or electric current with galvanic isolation. There the circuit design engineer needs to optimize the signal over noise ratio SNR at low magnetic field while keeping the power drain low. This calls for Hall-effect devices with medium sized contacts, where the influence of the contacts cannot be neglected as assumed in most prior approaches [1-3]. However, for medium sized input and output contacts there are no closed analytical formulae for the Hall-geometry factor, which describes the loss in signal due to short-circuiting effects of the electrodes. There are indeed numerical solutions, yet they are not ready to use in daily practice [4-6]. Besides, modern Hall plates have a 90° symmetry, where input and output contacts are periodically swapped to implement the spinning current scheme, which drastically reduces the offset (i.e. the zero point error) [7, 8]. (In industrial practice this low offset is still one of the main advantages over magneto-resistors.) No one seems to have used this symmetry so far to provide simple electrical design formulae.

In contrast to a physical layout engineer a circuit design engineer is interested in the Hall-geometry factor versus internal resistance and not versus contact size or geometric shape. As known from conformal mapping theory there

[1] Infineon Technologies AG, Austria, email: udo.ausserlechner@infineon.com

is a unique relation between the effective number of squares of internal resistance and the Hall-geometry factor in the limit of low magnetic field. This applies for all kinds of geometry such as disks, squares, rectangles, crosses, octagons, clover leaves a.s.o., as long as a conformal mapping exists which transforms one into the other. In other words, if e.g. an octagonal Hall plate and a disk shaped Hall plate have the same number of squares, then they also have the same Hall-geometry factor, the same thermal noise, and the same magnetic sensitivity. They are equivalent within the scope of linear electrostatic theory. This led us to the goal of looking for a simple formula, which expresses the Hall-geometry factor as a function of effective number of squares. Such a formula has not yet been reported in literature. However, it would be of significant practical interest, because it is much simpler to first compute the effective number of squares and then get the Hall-geometry factor thereof with the simple formula, than to compute the Hall geometry factor directly. Another benefit is that the effective number of squares can be measured easily on wafer level by Van-der-Pauw's method without need to apply a magnetic field.

2. Definitions

The output voltage V_{out} of Hall-plates depends on input current I_{in} or voltage V_{in} and perpendicular magnetic flux density $B_{\perp}$.

$$V_{\text{out}} = B_{\perp} S_I I_{\text{in}} = B_{\perp} S_V V_{\text{in}} \tag{1}$$

S_I and S_V are current and voltage related magnetic sensitivity, respectively. With the input resistance $R_{\text{in}} = V_{\text{in}} / I_{\text{in}}$ it holds

$$S_I = S_V R_{\text{in}} = G_{\text{H}} \left| R_H \right| / t_{\text{H}} \tag{2}$$

with the plate thickness t_H and the Hall-geometry factor G_{H}. The Hall coefficient R_H depends solely on material properties. Due to symmetry the input resistance R_{in} equals the output resistance R_{out}. It is given by

$$R_{\text{in}} = R_{\text{out}} = \lambda R_{\text{sh}} \tag{3}$$

with the effective number of squares λ and the sheet resistance R_{sh}. Thermal noise is proportional to the square-root of the internal resistance and low-frequency noise is cancelled out by the spinning current scheme [9]. Combining this with the foregoing equations one obtains $SNR/I_{\text{in}} \propto G_{\text{H}} \lambda^{-1/2}$, $SNR/\sqrt{V_{\text{in}} I_{\text{in}}} \propto G_{\text{H}} \lambda^{-1}$, and $SNR/V_{\text{in}} \propto G_{\text{H}} \lambda^{-3/2}$, which need to be maximized depending on whether supply current, power, or voltage is limited [10]. For this kind of optimization we are interested in the low-field limit of the Hall-geometry factor only: $G_{\text{H}}^{(0)} = \lim_{B_{\perp} \to 0} G_{\text{H}}$. The magnetic field is considered to be low if

$\mu_h B_\perp << 1$ holds. For Hall-plates with 90° symmetry the field dependence of the Hall-geometry factor is approximated by [11]

$$G_{\mathrm{H}} \cong 1 - \left(1 - G_{\mathrm{H}}^{(0)}\right)\frac{\arctan\left(\mu_{\mathrm{H}} B_\perp\right)}{\mu_{\mathrm{H}} B_\perp} \tag{4}$$

3. An approximate fit formula

Let us consider disk shaped Hall-plates with four equal contacts, each one covering an aperture angle of 2θ [4]. First we compute the internal resistance and then the Hall-geometry factor.

Following the method of [12] we reduce the number of contacts from four to two: in the absence of magnetic fields the potential distribution is symmetric and so one needs only consider a quarter of the circular Hall plate. If e.g. 1V is applied at two diametrically opposite contacts of the original plate, the output contacts would be at 0.5V. Fig. 1(a) shows the lower right quarter of such a circular Hall plate. The half of the right output contact between Z3 and Z4 is at 0.5V and the right half of the ground contact between Z1 and Z2 is at 0V. Due to symmetry the potential along the line Z4-Z5 is also at 0.5V and so we can place a contact there and connect it to the output contact without changing potential and current distributions. Due to symmetry half of the current through the original device flows through Z1-Z2 and also between Z3-Z5. Hence, the total resistance of the plate is the ratio of voltage between Z1 and Z5 divided by the current through Z1-Z2. Next we map the interior of this quarter disk in fig. 1(a) to the interior of a rectangle in fig. 1(d) such that one edge corresponds to Z1-Z2 and the opposite edge corresponds to Z3-Z5. The ratio of the lengths of these edges $|Q_2Q_3|/|Q_3Q_5|$ is the number of squares of both rectangle and circular disk. The first transformation $w = (1-z^2)^2/(1+z^2)^2$ maps the interior of the quarter disk in the z-plane to the upper half-space of the w-plane in fig. 1(b) [13]. Then the contacts lie on the real axis in the w-plane: the ground contact extends from $w = -\infty$ at W1 to $w = -1/(\tan\theta)^2$ at W2 and the output contact extends from $w = -(\tan\theta)^2$ at W3 to $w = 1$ at W5. Thereby W1 corresponds to Z1, W2 corresponds to Z2, and so on. By a subsequent bilinear transformation $t = (A+Bw)/(C+Dw)$ these asymmetric points in the w-plane are mapped into symmetric position in the t-plane in fig. 1(c), such that $t = -1/k$ at T1, $t = -1$ at T2, $t = 1$ at T3, and $t = 1/k$ at T5 with $0 < k < 1$. We choose $B = -1$, from which follows $D = k$, $A = 1 - 2/\sin(2\theta)$, and $C = (\sin\theta \tan\theta - \cos\theta / \tan\theta)/ (\sin\theta + \cos\theta)$. For k we get a quadratic equation, from which we choose the solution with $k < 1$

$$k = (1 - \sin(2\theta))/\cos(2\theta) \tag{5}$$

It was verified that the bilinear transformation maps the upper half-space of the w-plane onto the upper half-space of the t-plane. According to [12] the transformation $q = \int_0^t ((1-x^2)(1-k^2x^2))^{-1/2} dx$ maps the upper half space of the t-plane onto the interior of the rectangle in the q-plane in fig. 1(d). Obviously the ratio of the lengths of both edges of this rectangle is equal to its effective number of squares

$$\lambda = |Q_2Q_3| / |Q_3Q_5| = 2K(k)/K'(k) \tag{6}$$

with $K'(k) = K((1-k^2)^{1/2})$ and $K(k) = \int_0^1 (1-t^2)^{-1/2}(1-k^2t^2)^{-1/2} dt$ is the complete elliptical integral of the first kind. In the computation of $|Q_3Q_5|$ we used the identity $K'(k) = \int_{t=0}^1 (1-t^2)^{-1/2}(1-(1-k^2)t^2)^{-1/2} dt = \int_{\tau=1}^{1/k} (\tau^2-1)^{-1/2}(1-k^2\tau^2)^{-1/2} d\tau$ which is proven by the substitution $(1-k^2)t^2 = 1-k^2\tau^2$.

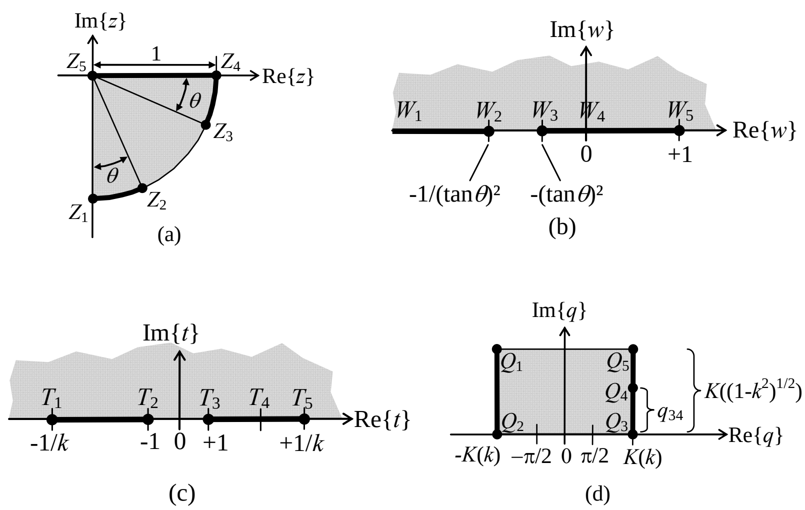

Fig. 1 (a-d). Sequence of conformal transformations that map the interior of a quarter circle onto the interior of a rectangle

The Hall-geometry factor for disk shaped Hall-plates with four equal contacts of medium size can be computed numerically according to [4]. To this end the relation between λ and θ according to (5) and (6) is used. Yet for too small or too large contacts numerical problems show up. In the limit of small contacts and small magnetic field one can derive an analytical formula with (27) in [4]

$$G_{\mathrm{H}}^{(0)} \cong 1 - 32\pi^{-2}\exp(-\pi\lambda/2) \tag{7}$$

The error of (7) is +0%/-0.32% for 1.83 < λ. In the limit of large contacts and small magnetic field one can derive from [4] (with appreciable effort)

$$G_{\mathrm{H}}^{(0)} \cong \frac{3}{2}\lambda^2\pi^2 \frac{1}{3\pi - 7\lambda\exp(-\pi/\lambda)} \frac{1}{\pi - 24\lambda(\ln 2)\exp(-\pi/\lambda)} \tag{8}$$

with an error bound of +1.4%/-0% for $\lambda < 0.51$.

With the foregoing equations one can obtain numerical data for the Hall-geometry factor over a wide range of number of squares. A plot of the resulting curve is given in Fig. 2 in both (a) linear and (b) logarithmic scale. Figure 2(b) shows $G_{\mathrm{H}}\times(1-G_{\mathrm{H}})$ in order to magnify the curves at small and large λ. Numerical data for $0.4 < \lambda < 5$ was obtained from [4]. This curve can be approximated by the astonishingly simple fit formula

$$G_{\mathrm{H,1}} \cong \lambda^2 \Big/ \sqrt{\lambda^4 + \lambda^2/2 + 4} \tag{9}$$

Equation (9) is accurate for large and medium sized contacts, yet for small contacts (λ > 2.5) the error is visible in fig. 2(b). In general (9) slightly underestimates the Hall-geometry factor, as it is shown in Fig. 3. There as reference we took numerical data obtained from [4] in the range where they are reliable (notice first signs of round-off "noise" at $\lambda \approx 0.4$ and $\lambda \approx 5$). Outside this range we used the more accurate analytical limit formulas (7, 8). (9) has no error at $\lambda = 2^{1/2}$ and it has -1% error at λ = 0.355, 1, 2, 5.43. The curve is symmetric to $\lambda = 2^{1/2}$. An additional factor improves the accuracy by two orders in magnitude

$$G_{\mathrm{H,2}} \cong \left(1 + \Lambda^2 \exp\left(-c_0 - (c_2\Lambda)^2 + (c_4\Lambda)^4 - (c_6\Lambda)^6\right)\right) \times G_{\mathrm{H,1}} \tag{10}$$

with $\Lambda = \ln(\lambda/2^{1/2})$ and c_0 = 2.279, c_2 = 1.394, c_4 = 0.6699, c_6 = 0.4543. The error of (10) is between -6×10^{-5} and 2×10^{-4} for $0 < \lambda < \infty$ as shown in fig. 4. There the abscissa axis covers effective numbers of squares from 0.002 to 500 in logarithmic scale. Note again the striking symmetry of the curve with respect to $\lambda = 2^{1/2}$ where the approximation error vanishes.

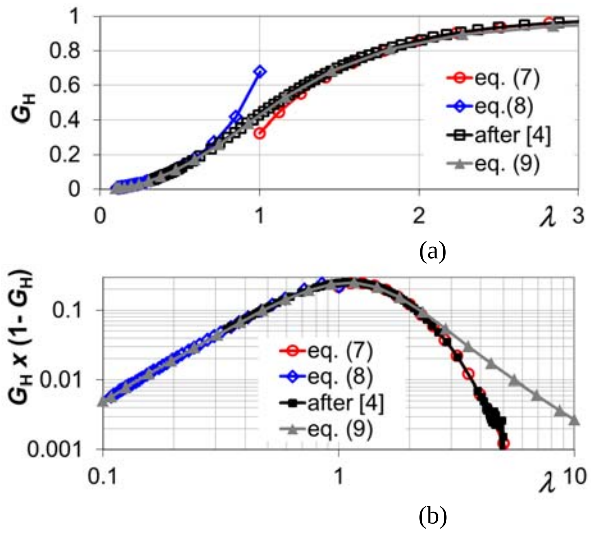

Fig. 2 (a-b). Hall-geometry factor G_H for symmetric Hall plates at small magnetic field versus effective number of squares λ: (a) linear scale, (b) logarithmic scale.

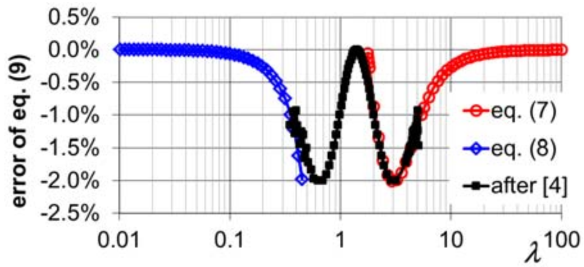

Fig. 3. The error of fit (9) for G_H versus λ.

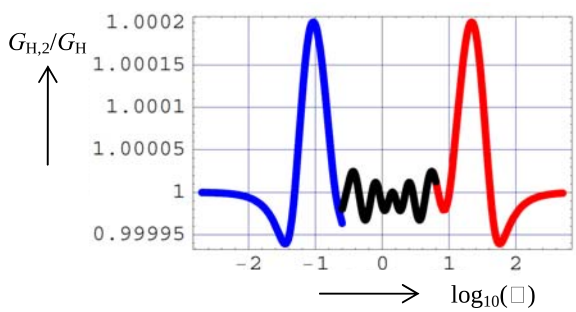

Fig. 4. Ratio of approximate Hall-geometry factor after (10) over the exact one ($G_{H,2}/G_H$).

4. Conclusions

For the first time the Hall-geometry factor of 90°-symmetric Hall-plates in the low magnetic field limit is given as a function of the effective number of squares λ. It is approximated by two simple formulae (9, 10) with 2% and 0.02% accuracy, respectively. They suggest an apparent symmetry between small and large contacts $G_H(\lambda = 2^{1/2}x) = x^2\, G_H(\lambda = 2^{1/2}/x)$ for $x > 0$. This symmetry has not yet been reported in the literature and we have no rigorous proof for it.

Equations (9, 10) are of practical relevance for concept engineers of Hall sensors systems, because they allow for an optimization of the electronic system in terms of a single degree of freedom (namely the effective number of squares λ) regardless of the specific geometry of the Hall plate.

Equations (9, 10) are also useful in the characterization of Hall-mobility. There one can measure the effective number of squares with Van-der-Pauw's method, determine the Hall-geometry factor with (9, 10) and use this result in the mobility measurement.

R E F E R E N C E S

[1]. *J. Raman, P. Rombouts, L. Weyten*, Method for electric field and potential calculations in Hall plates, Electronics Letters, 2013, **49**, (1), pp. 33–34.

[2]. *M. J. Moelter et al.*, Electric potential in the classical Hall effect: An unusual boundary-value problem, American Journal of Physics, 1998, **66**, (8), pp. 668–677.

[3]. *M. Rudan, et al.*, Determination of the Hall voltage in devices with arbitrary aspect ratio and probe position, European Solid-State Device Research, 2003. ESSDERC'03. 33rd Conference on. IEEE, 2003, pp. 355–358.

[4]. W. *Versnel*, Analysis of a circular Hall plate with equal finite contacts, Solid-State Electronics, 1981, **24**, (1), pp. 63–68.

[5]. *G. Zhang, et al.*, Geometry Optimization of Planar Hall Devices Under Voltage Biasing, Electron. Dev., IEEE Transactions on 61.12 (2014): 4216-4223.

[6]. *M.-A. Paun*, Single and Multi-phase Offset Numerical Estimation for CMOS Hall Effect Devices, Advances in Sensors and Interfaces (IWASI), 2015, 6th IEEE Intl' Workshop on. IEEE, 2015.

[7]. *P. J. A. Munter*, A low-offset spinning-current Hall plate, Sensors and Actuators A: Physical, 1989, **22** (1), pp. 743–746.

[8]. *U. Ausserlechner*, Limits of offset cancellation by the principle of spinning current Hall probe, Sensors, 2004. Proceedings of IEEE. IEEE, 2004, pp. 1117–1120.

[9]. *M. Demierre*, Improvements of CMOS Hall microsystems and application for absolute angular position measurements, PhD Thesis, Ecole Polytechnique Federale De Lausanne, 2003.

[10]. *M.-A. Paun, J.-M. Sallese, M. Kayal*, Geometry influence on the Hall effect devices performance, U. P. B. Sci. Bull., Series A, **vol. 72**, iss. 4, 2010, pp. 257-271.

[11]. *W. Versnel*, Analysis of symmetrical Hall plates with finite contacts, Appl. Phys., **vol. 52**, no. 7, 1981, pp. 4659-4666.

[12]. *W. Versnel*, Analysis of symmetrical Van der Pauw structures with finite contacts, Solid-State Electronics, 1978, **21** (10), pp. 1261–1268.
[13]. *W. von Koppenfels, F. Stallmann*, Praxis der konformen Abbildung (Springer, Berlin, 1959: part B, § 3, p. 217).