EXISTENCE, STABILITY AND WELL-POSEDNESS OF FIXED POINT PROBLEM WITH APPLICATION TO INTEGRAL EQUATION

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In this paper, we consider a fixed point problem related to the notion of W_b-contraction mappings and obtained some sufficient conditions for the existence of solution for such class of mappings in a framework of orthogonal $\mathcal{F}$-metric space. As applications of the obtained results, we investigate the Ulam-Hyers stability of a fixed point problem and of a Volterra integral equation. Some illustrative examples are also provided to support the new findings.

Keywords: Fixed point; well-posedness; W_b-contraction; orthogonal set; $\mathcal{F}$-metric space; Ulam-Hyers stability.

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1. Introduction

In the theory of Ulam's stability, one can find the efficient tools to evaluate the errors, that is to study the existence of an exact solution of the perturbed functional equation which is not far from given function (see [7, 10, 16] and references cited therein). The study of the stability of functional equations has an important role in mathematics and allied sciences because of its many applications, for instance, in economics and optimization theory (see [3]).

In recent years, there are many interesting generalizations (or extensions) of the metric space concept appeared in the literature such as Czerwik [4] introduced the notion of b-metric with a coefficient 2 and this notion was further generalized by the author in [5] with a coefficient $K \geq 1$. Matthews [15] gave the concept of partial metric space, Branciari [2] introduced a notion of a v-generalized metric space, Khamsi et al. [13] reintroduced the notion of b-metric as metric type, Fagin et al. [6] gave the notion of $s - relaxed_p$ metric (see, also [14]) and thereafter many researchers gave different and wonderful concepts. Jleli et al. [11] presented a very fascinating generalization called as $\mathcal{F}$-metric space. Recently, Gordji et al. [9] introduced the concept of an orthogonal set (briefly, O-set) and presented some fixed point theorems in orthogonal metric spaces. Many researchers (see [8, 9, 12, 20]) proved the existence of fixed points using the concept of many interesting contraction mappings in various abstract spaces. In this paper, we present some results for the existence of solutions for the fixed point problem and as application to investigate the Ulam-Hyers stability problem in the setting of orthogonal $\mathcal{F}$-metric spaces. Also, some examples are provided for the usability of the results.

2. Preliminaries

In this section, we give some notations and basic definitions to be used in the sequel.

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Definition 2.1. [9] *Let $\mathscr{X} \neq \emptyset$ and $\perp \subset \mathscr{X} \times \mathscr{X}$ be a binary relation. If $\perp$ satisfies the following condition:*

there exists $\varpi_0 \in \mathscr{X}$ such that (for all $\varkappa \in \mathscr{X}$, $\varkappa \perp \varpi_0$) or (for all $\varkappa \in \mathscr{X}$, $\varpi_0 \perp \varkappa$), then it is called an orthogonal set (briefly O-set). We denote this O-set by $(\mathscr{X}, \perp)$.

Example 2.1. [9] *Let $\mathscr{X} = \mathbb{Z}$. Define $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = k\,n$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence $(\mathscr{X}, \perp)$ is an O-set.*

For more interesting examples see [9].

Definition 2.2. *Let $(\mathscr{X}, \perp)$ be an O-set. A sequence $\{\varpi_n\}_{n\in\mathbb{N}}$ is called an orthogonal sequence (briefly, O-sequence) if (for all $n, \varpi_n \perp \varpi_{n+1}$) or (for all $n, \varpi_{n+1} \perp \varpi_n$).*

Definition 2.3. *Let $(\mathscr{X}, \perp)$ be an O-set. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is said to be $\perp$-preserving if $\mathscr{T}(\varpi) \perp \mathscr{T}(\varkappa)$, then $\varpi \perp \varkappa$. Also, $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is said to be weakly $\perp$-preserving if $\mathscr{T}(\varpi) \perp \mathscr{T}(\varkappa)$ or $\mathscr{T}(\varkappa) \perp \mathscr{T}(\varpi)$, then $\varpi \perp \varkappa$.*

It is easy to see that every $\perp$-preserving mapping is weakly $\perp$-preserving. But the converse is not true (see [9]).

Let $\mathcal{F}$ be the set of function $f : (0, +\infty) \to \mathbb{R}$ satisfying the following condition:

(Θ_1) f is non-decreasing, that is $0 < \lambda < \mu \Rightarrow f(\lambda) \leq f(\mu)$

(Θ_2) for every sequence $\{\mu_n\} \subset (0, +\infty)$, we have

$$\lim_{n\to\infty} \mu_n = 0 \Leftrightarrow \lim_{n\to\infty} f(\mu_n) = -\infty.$$

Definition 2.4. [11] *Let $\mathscr{X} \neq \emptyset$, and $D : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ be a mapping. Suppose that there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that*

(D_1) $(\varpi, \varkappa) \in \mathscr{X} \times \mathscr{X}, D(\varpi, \varkappa) = 0 \Leftrightarrow \varpi = \varkappa$

(D_2) *$D(\varpi, \varkappa) = D(\varkappa, \varpi)$, for all $(\varpi, \varkappa) \in \mathscr{X} \times \mathscr{X}$.*

(D_3) *For every $(\varpi, \varkappa) \in \mathscr{X} \times \mathscr{X}$, for every $n \in \mathbb{N}, n > 1$, and for every $(\varpi_i)_{i=1}^n \subset \mathscr{X}$ with $(\varpi_i, \varpi_{i+1}) = (\varpi, \varkappa)$, we have*

$$D(\varpi, \varkappa) > 0 \Rightarrow f(D(\varpi, \varkappa)) \leq f\left(\sum_{i=1}^{n-1} D(\varpi_i, \varpi_{i+1})\right) + \alpha$$

Then D is said to be $\mathcal{F}$-metric on X, and the pair $(\mathscr{X}, D)$ is said to be a $\mathcal{F}$-metric space.

Definition 2.5. [11] *Let $(\mathscr{X}, D)$ be an $\mathcal{F}$-metric space. A sequence $\{\varpi_n\}$ is $\mathcal{F}$-convergent to $\varpi \in X$ if $\{\varpi_n\}$ is convergent to $\mathscr{X}$ with respect to topology $\mathcal{T}_{\mathcal{F}}$.*

Definition 2.6. [11] *Let $(\mathscr{X}, D)$ be an $\mathcal{F}$-metric space and $\{\varpi_n\}$ be a sequence in $\mathscr{X}$. A sequence $\{\varpi_n\}$ is said to be $\mathcal{F}$-Cauchy if*

$$\lim_{m,n\to\infty} D(\varpi_n, \varpi_m) = 0.$$

Definition 2.7. *(see [8, 17, 18]) Let $(\mathscr{X}, D)$ be a $\mathcal{F}$-metric space and $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a self mapping. A sequence $\{\varpi_n\}$ defined by $\varpi_{n+1} = \mathscr{T}(\varpi_n) = \mathscr{T}^n \varpi_0$ is called a Picard sequence based at point $\varpi_0 \in \mathscr{X}$. A self mapping $\mathscr{T}$ is said to be a Picard operator if it has a unique fixed point $\varkappa \in \mathscr{X}$ and $\varkappa = \lim_{n\to\infty} \mathscr{T}^n \varpi$ for all $\varpi \in \mathscr{X}$.*

Definition 2.8. [11] *Let $(\mathscr{X}, D)$ be an $\mathcal{F}$-metric space and $\{\varpi_n\}$ be a sequence in $\mathscr{X}$. A sequence $\{\varpi_n\}$ is said to be $\mathcal{F}$-complete if every $\mathcal{F}$-Cauchy sequence in $\mathscr{X}$ is $\mathcal{F}$-convergent to a certain point in $\mathscr{X}$.*

Definition 2.9. [11] *Let $(\mathscr{X}, D)$ be an $\mathcal{F}$-metric space and $\mathscr{A}$ be a non empty subset of $\mathscr{X}$. $\mathscr{A}$ is said to be $\mathcal{F}$-compact if $\mathscr{A}$ is compact with respect to the topology $\mathcal{T}_{\mathcal{F}}$ on $\mathscr{X}$.*

3. **Orthogonal Fixed Point**

To start with, we have the following notations and definitions:

Definition 3.1. *Let* $(\mathscr{X}, \perp, D)$ *be an orthogonal* $\mathcal{F}$*-metric space* ($(\mathscr{X}, \perp)$ *is an O-set and* $(\mathscr{X}, D)$ *is a* $\mathcal{F}$*-metric space).*

Example 3.1. *Let* $(\mathscr{X} = [0,1], D)$ *be a* $\mathcal{F}$*-metric space with* $\mathcal{F}$*-metric defined as*

$$D(p,q) = \begin{cases} e^{|p-q|}, p \neq q \\ 0, p = q, \end{cases}$$

for all $p, q \in \mathscr{X}$, $f(t) = \frac{-1}{t}, t > 0$ *and* $a = 1$. *Define* $p \perp q$ *as* $pq \leq p$ *or* $pq \leq q$. *Then for all* $p \in \mathscr{X}$, $0 \perp p$, *so* $(\mathscr{X}, \perp)$ *is an O-set and* $(\mathscr{X}, \perp, D)$ *is an orthogonal* $\mathcal{F}$*-metric space.*

Definition 3.2. *Let* $(\mathscr{X}, \perp, D)$ *be an orthogonal* $\mathcal{F}$*-metric space. Then* $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ *is said to be orthogonally* $\mathcal{F}$*-continuous (or* $\perp$*-*$\mathcal{F}$*-continuous) at* $a \in \mathscr{X}$ *if, for each O-sequence* $\{a_n\}$ *in* $\mathscr{X}$ *with* $a_n \to a$, *we have* $\mathscr{T}(a_n) \to \mathscr{T}(a)$. *Also,* $\mathscr{T}$ *is said to be* $\perp$*-*$\mathcal{F}$*-continuous on* $\mathscr{X}$ *if* $\mathscr{T}$ *is* $\perp$*-*$\mathcal{F}$*-continuous for each* $a \in \mathscr{X}$.

Definition 3.3. *Let* $(\mathscr{X}, \perp, D)$ *be an orthogonal* $\mathcal{F}$*-metric space. Then* $\mathscr{X}$ *is said to be orthogonally* $\mathcal{F}$*-complete (briefly, O-*$\mathcal{F}$*-complete) if every Cauchy O-sequence is* $\mathcal{F}$*-convergent.*

Definition 3.4. *Suppose that* Φ *denotes the family of functions* $\varphi : [0,\infty) \to [0,\infty)$ *satisfying following conditions:*

(i) φ *is nondecreasing*
(ii) $\varphi^n(t) \to 0$ *as* $n \to \infty$, *for* $t \in [0,\infty)$.

Now, we give W_b-contraction which will be used in our results.

Definition 3.5. *Let* $(\mathscr{X}, \perp, D)$ *be an orthogonal* $\mathcal{F}$*-metric space and* $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ *be a self mapping. A mapping* T *is called an orthogonal* W_b-*contraction if there exists a* $\varphi \in \Phi$ *such that for all* $\varpi, \varkappa \in \mathscr{X}$, *with* $\varpi \perp \varkappa$, $D(\mathscr{T}\varpi, \mathscr{T}\varkappa) > 0$ *we have*

$$D(\mathscr{T}\varpi, \mathscr{T}\varkappa) \leq \varphi(D(\varpi, \varkappa)). \tag{1}$$

Theorem 3.1. *Let* $(\mathscr{X}, \perp, D)$ *be an O-complete orthogonal* $\mathcal{F}$*-metric space and* $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ *be an orthogonal* W_b*-contraction. Suppose that* $\mathscr{T}$ *is* $\perp$*-preserving and* $\perp$*-continuous. If there exists* $\varpi_0 \in \mathscr{X}$ *such that* $\varpi_0 \perp \mathscr{T}\varpi_0$ *or* $\mathscr{T}\varpi_0 \perp \varpi_0$, *then* $\mathscr{T}$ *has a unique fixed point in* $\mathscr{X}$.

Proof. Let $\varpi_0 \in \mathscr{X}$ be such that $\varpi_0 \perp \mathscr{T}(\varpi_0)$ or $\mathscr{T}\varpi_0 \perp \varpi_0$. Take $\varpi_1 := \mathscr{T}\varpi_0$, $\varpi_2 := \mathscr{T}\varpi_1 = \mathscr{T}^2\varpi_0$. We define sequence $\{\varpi_n\}$ $\mathscr{X}$ by $\varpi_{n+1} = \mathscr{T}\varpi_n = \mathscr{T}^{n+1}\varpi_0$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\mathscr{T}$ is $\perp$-preserving, we have $\varpi_n \perp \varpi_{n+1}$ or $\varpi_{n+1} \perp \varpi_n$ for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\{\varpi_n\}$ is an orthogonal sequence.

If there exists $n_0 \in \mathbb{N}$ such that $\varpi_{n_0+1} = \varpi_{n_0}$, then ϖ_{n_0} is a fixed point of T. Therefore, we may suppose that $D(\varpi_n, \varpi_{n+1}) > 0$. Since $\mathscr{T}$ is an orthogonal W_b-contraction, we have

$$\begin{aligned} D(\varpi_n, \varpi_{n+1}) &= D(\mathscr{T}\varpi_{n-1}, \mathscr{T}\varpi_n) \\ &\leq \varphi(D(\varpi_{n-1}, \varpi_n)) = \varphi(D(\mathscr{T}\varpi_{n-2}, \mathscr{T}\varpi_{n-1})) \\ &\leq \varphi^2(D(\varpi_{n-2}, \varpi_{n-1})) \\ &\leq \dots \\ &\leq \varphi^{n-1}(D(\mathscr{T}\varpi_0, \mathscr{T}\varpi_1)) \\ &\leq \varphi^n(D(\varpi_0, \varpi_1)). \end{aligned}$$

Therefore, we have $D(\varpi_n, \varpi_{n+1}) \leq \varphi^n(D(\varpi_0, \varpi_1))$, for all $n \in \mathbb{N}$. Taking limit $n \to \infty$ and using Definition 3.4, we have

$$\lim_{n\to\infty} D(\varpi_n, \varpi_{n+1}) \leq \lim_{n\to\infty} \varphi^n(D(\varpi_0, \varpi_1) \to 0.$$

Hence

$$\lim_{n\to\infty} D(\varpi_n, \varpi_{n+1}) = 0. \tag{2}$$

Let $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ be such that (D_3) is satisfied. Suppose that $\varepsilon > 0$ is given. By (Θ_1), there exists a $\delta > 0$ such that for $0 < t < \delta$, we have

$$f(t) < f(\varepsilon) - \alpha. \tag{3}$$

We may suppose that $D(\varpi_0, \varpi_1) > 0$. Using (2), we have $\lim\limits_{n\to\infty} D(\varpi_n, \varpi_{n+1}) = 0$. Further, we have

$$\sum_{i=n}^{m-1} D(\varpi_i, \varpi_{i+1}) = D(\varpi_n, \varpi_{n+1}) + D(\varpi_{n+1}, \varpi_{n+2}) + \ldots + D(\varpi_{m-1}, \varpi_m). \tag{4}$$

It implies that

$$\sum_{i=n}^{m-1} D(\varpi_i, \varpi_{i+1}) \leq \varphi^n(D(\varpi_0, \varpi_1)) + \varphi^{n+1}(D(\varpi_0, \varpi_1)) + \ldots + \varphi^{m-1}(D(\varpi_0, \varpi_1)). \tag{5}$$

Hence, we have

$$\sum_{i=n}^{m-1} D(\varpi_i, \varpi_{i+1}) \leq \frac{\varphi^n(D(\varpi_0, \varpi_1))}{1 - \varphi(D(\varpi_0, \varpi_1))}.$$

Since $\lim\limits_{n\to\infty} \dfrac{\varphi^n(D(\varpi_0, \varpi_1))}{1 - \varphi(D(\varpi_0, \varpi_1))} = 0$, for a given $\delta > 0$ there exists $N \in \mathbb{N}$ such that $0 < \dfrac{\varphi^n(D(\varpi_0, \varpi_1))}{1 - \varphi(D(\varpi_0, \varpi_1))} < \delta$, for $n \geq N$. Hence by (3) and (Θ_1), we obtain

$$f\left(\sum_{i=n}^{m-1} D(\varpi_i, \varpi_{i+1})\right) \leq f\left(\frac{\varphi^n(D(\varpi_0, \varpi_1))}{1 - \varphi(D(\varpi_0, \varpi_1))}\right) < f(\varepsilon) - \alpha, \ m > n \geq N. \tag{6}$$

Using (D_3) and (6), we obtain

$$f(D(\varpi_n, \varpi_m)) \leq f\left(\sum_{i=n}^{m-1} D(\varpi_i, \varpi_{i+1})\right) + \alpha < f(\varepsilon).$$

Using Θ_1, we have

$$D(\varpi_n, \varpi_m) < \varepsilon,$$

for $m, n \geq N$. Hence $\{\varpi_n\}$ is an orthogonal $\mathcal{F}$- Cauchy.

Since $(\mathscr{X}, D)$ is an orthogonal $\mathcal{F}$-complete, there exists $\varpi^* \in \mathscr{X}$ such that $\{\varpi_n\}$ is orthogonal $\mathcal{F}$- convergent to ϖ^*, that is

$$\lim_{n\to\infty} D(\varpi_n, \varpi^*) = 0. \tag{7}$$

Using the $\perp$-continuity of $\mathscr{T}$, we have $\mathscr{T}\varpi^* = \lim\limits_{n\to+\infty} \mathscr{T}\varpi_n = \lim\limits_{n\to\infty} \varpi_{n+1} = \varpi^*$. Thus ϖ^* is a fixed point of $\mathscr{T}$.

Now, to prove the uniqueness of the fixed point, let κ^* be another fixed point of $\mathscr{T}$. Then we have $\mathscr{T}^n\kappa^* = \kappa^*$ for all $n \in \mathbb{N}$. By our choice of ϖ_0, we have $\varpi_0 \perp \kappa^*$ or $\kappa^* \perp \varpi_0$. Since $\mathscr{T}$ is $\perp$-preserving, we have $\mathscr{T}^n\varpi_0 \perp \mathscr{T}^n\kappa^*$ or $\mathscr{T}^n\kappa^* \perp \mathscr{T}^n\varpi_0$ for all $n \in \mathbb{N}$.

Suppose that $D(\mathscr{T}^n\varpi_0, \kappa^*) > 0$. By (D_3), we have

$$\begin{aligned} D(\mathscr{T}^n\varpi_0, \kappa^*) &\le \varphi(D(\mathscr{T}^{n-1}\varpi_0, \kappa^*)) \\ &\le \cdots \\ &\le \varphi^n(D(\varpi_0, \kappa^*)). \end{aligned}$$

Taking $n \to \infty$, we have $\lim_{n\to\infty} \varpi_n = \kappa^*$. Further, uniqueness of limit implies that $\varpi^* = \kappa^*$. Hence the result. □

Corollary 3.1. *Let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a self mapping on an orthogonal $\mathfrak{F}-$metric space $(\mathscr{X}, \perp, D)$. Suppose that the following conditions are satisfied:*

(i) *$\mathscr{X}$ is an orthogonal $\mathfrak{F}-$complete;*
(ii) *$\mathscr{T}$ is $\perp$-preserving and $\perp$-continuous;*
(iii) *there exists $k \in (0,1)$ such that for all $(\varpi, \varkappa) \in \mathscr{X} \times \mathscr{X}$ with $\varpi \perp \varkappa$, $D(\mathscr{T}\varpi, \mathscr{T}\varkappa) > 0$, we have*

$$D(\mathscr{T}\varpi, \mathscr{T}\varkappa) \le kD(\varpi, \varkappa).$$

If there exists $\varpi_0 \in \mathscr{X}$ such that $\varpi_0 \perp \mathscr{T}\varpi_0$ or $\mathscr{T}\varpi_0 \perp \varpi_0$, then Then $\mathscr{T}$ has a unique fixed point.

Example 3.2. *Let $\mathscr{X} = [0, 3]$ and $D : \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ be the mapping defined by*

$$D(\varpi, \varkappa) = (\varpi - \varkappa)^2$$

for all $(\varpi, \varkappa) \in \mathscr{X} \times \mathscr{X}$. Define a relation $\perp$ on $\mathscr{X}$ by $\varpi \perp \varkappa$ if and only if $\varpi\varkappa \in \{\varpi, \varkappa\} \subseteq \mathscr{X}$. Then $(\mathscr{X}, \perp, D)$ is an O-complete orthogonal $\mathfrak{F}$-metric space with $f(t) = ln(t)$ and $\alpha = 1$.

Let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a mapping defined by

$$\mathscr{T}\varpi = \frac{\varpi}{2} + 1$$

for all $x \in \mathscr{X}$. Define $\varphi : [0, \infty) \to [0, \infty)$ as $\varphi(t) = t$, $t \ge 0$. So it satisfies the following conditions:

(i) *φ is non decreasing*
(ii) *$\varphi^n(t) \to 0$ as $n \to \infty$, for $t \in [0, \infty)$.*

Then $\mathscr{T}$ is an orthogonal W_b-contraction and has a unique fixed point.

Proof. Let $\mathscr{T}\varpi = \frac{\varpi}{2} + 1$, for each $\varpi \in \mathscr{X}$. Consider

$$\begin{aligned} D(\mathscr{T}\varpi, \mathscr{T}\varkappa) &= \left(\frac{\varpi}{2} + 1 - \frac{\varkappa}{2} - 1\right)^2 \\ &= \left(\frac{\varpi}{2} - \frac{\varkappa}{2}\right)^2 \\ &= \frac{(\varpi - \varkappa)^2}{4} \\ &= \frac{D(\varpi, \varkappa)}{4}. \end{aligned}$$

Therefore, $D(\mathscr{T}\varpi, \mathscr{T}\varkappa) \le \varphi(D(\varpi, \varkappa))$. Therefore $\mathscr{T}$ has a unique fixed point and fixed point of $\varpi = \frac{\varpi}{2} + 1$ is 2 (see Figure 1). □

Example 3.3. *Let $\mathscr{X} = \{\varpi_n = \ln\left(\frac{n(n+1)}{2}\right)\} : n \in \mathbb{N}$ endowed with $\mathfrak{F}$-metric given by*

$$D(\varpi, \varkappa) = \begin{cases} e^{|p-q|}, p \ne q \\ 0, p = q, \end{cases}$$

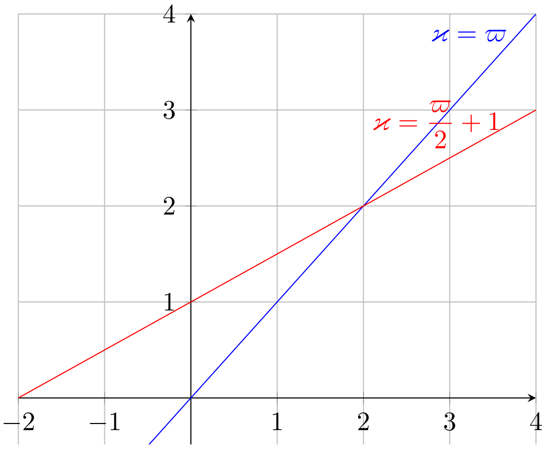

FIGURE 1. Graph of $\varpi = \dfrac{\varpi}{2} + 1$, showing the intersecting point (fixed point) of curve is 2.

with $f(\mu) = \frac{-1}{\mu}$ *and* $\alpha = 1$*. For all* $\varpi_n, \varpi_w \in \mathscr{X}$*, define* $\varpi_n \perp \varpi_w$ *if and only if* $(w \geq 2 \wedge n = 1)$*. Then* $(\mathscr{X}, \perp, D)$ *is an orthogonal* $\mathfrak{F}$*-metric space.*

Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ *as*

$$\mathscr{T}(\varpi_n) = \begin{cases} \varpi_1, n = 1 \\ \varpi_{n-1}, n > 1. \end{cases}$$

Take $\phi(t) = t, t \geq 0$*. Here* $D(\mathscr{T}\varpi_n, \mathscr{T}\varpi_w) > 0$*, so for every* $w \geq 2$*, we have*

$$\begin{aligned} \frac{D(\mathscr{T}\varpi_1, \mathscr{T}\varpi_w)}{D(\varpi_1, \varpi_w)} e^{D(\mathscr{T}\varpi_1, \mathscr{T}\varpi_w) - D(\varpi_1, \varpi_w)} &= \frac{e^{\varpi_{w-1} - \varpi_1}}{e^{\varpi_w - \varpi_1}} e^{\varpi_{w-1} - \varpi_w} \\ &= \frac{w-1}{w+1} e^{-w} < e^{-1}. \end{aligned}$$

Hence all the hypothesis of Theorem 3.1 are satisfied and $\mathscr{T}$ *has a unique fixed point.*

4. **Ulam-Hyers stability**

The Ulam-Hyers stability of various functional equations has been investigated by many authors in various abstract spaces. In this section, we investigate the Ulam-Hyers stability result using the fixed point techniques by generalizing the results of [1, 19] in the setting of orthogonal $\mathfrak{F}-$ metric space.

Definition 4.1. *Let* $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ *be an operator on an orthogonal* $\mathfrak{F}-$ *metric space* $(\mathscr{X}, D)$*. The fixed point equation*

$$\varpi = \mathscr{T}(\varpi), \ \varpi \in \mathscr{X} \tag{8}$$

is Ulam-Hyers stable if there exists a strictly increasing and surjective function $\beta : [0, \infty) \to [0, \infty)$ *with* $\beta(r) = r - \varphi(r)$*,* $r \in [0, \infty)$*, where* φ *is a non-decreasing function* $\varphi : [0, \infty) \to [0, \infty)$ *and* $\lim_{n \to \infty} \varphi^n(t) = 0$ *and such that for each* $\varepsilon > 0$ *and each solution* $\varkappa^*$ *of the inequality* $D(\varkappa, \mathscr{T}(\varkappa)) < \varepsilon$*, for each* $\varkappa \in \mathscr{X}$*, there exists a solution* ϖ^* *of equation* (8) *such that*

$$D(\varkappa^*, \varpi^*) < \beta^{-1}(\varepsilon).$$

Definition 4.2. *The fixed point problem (8) for* $\mathscr{T}$ *is said to be well-posed if it satisfies the following conditions:*

(i) *$\mathscr{T}$ has a unique fixed point $\varpi^* \in \mathscr{X}$*
(ii) *if for any O-sequence $\{\varpi_n\}$ in $\mathscr{X}$ such that*

$$\lim_{n\to\infty} D(\mathscr{T}\varpi_n, \varpi_n) = 0,$$

then

$$\lim_{n\to\infty} D(\varpi_n, \varpi^*) = 0.$$

Theorem 4.1. *Suppose that all the hypotheses of Theorem 3.1 are satisfied. Then the following conditions hold:*

(a) *The fixed point problem (8) is Ulam-Hyers stable, that is, if for each $\varepsilon > 0$ and each solution $\varkappa^*$ of the inequality $D(\varkappa, T(\varkappa)) < \varepsilon$, for each $\varkappa \in \mathscr{X}$, there exists a solution ϖ^* of equation (8) such that*

$$D(\varkappa^*, \varpi^*) < \beta^{-1}(\varepsilon).$$

(b) *If $\{\varpi_n\}$ is an O-sequence in $\mathscr{X}$ such that $\lim_{n\to\infty} D(\mathscr{T}\varpi_n, \varpi_n) = 0$ and ϖ^* is a fixed point of $\mathscr{T}$ then the fixed point problem (8) is well posed.*

Proof. (a) Using Theorem 3.1, there is a unique $\varpi^* \in \mathscr{X}$ such that $\varpi^* = \mathscr{T}\varpi^*$ that is $\varpi^* \in \mathscr{X}$ is solution of the fixed point equation ($\varpi = \mathscr{T}\varpi$). Assume that $\varepsilon > 0$ and $\varkappa^* \in \mathscr{X}$. Using ($D_3$), we have

$$\begin{aligned} f(D(\varkappa^*, \varpi^*)) &\le f[D(\varkappa^*, \mathscr{T}\varkappa^*) + D(\mathscr{T}\varkappa^*, \varpi^*)] + \alpha \\ &\le f[\varepsilon + D(\mathscr{T}\varkappa^*, \mathscr{T}\varpi^*)] + \alpha \\ &\le f[\varepsilon + \varphi(D(\varkappa^*, \varpi^*))] + \alpha. \end{aligned}$$

It implies that $f(D(\varkappa^*, \varpi^*)) - \alpha \le f[\varepsilon + \varphi(D(\varkappa^*, \varpi^*))]$ or $f(D(\varkappa^*, \varpi^*)) - \alpha \le f(D(\varkappa^*, \varpi^*)) \le f[\varepsilon+\varphi(D(\varkappa^*, \varpi^*))]$. Therefore, $f(D(\varkappa^*, \varpi^*)) \le f[\varepsilon+\varphi(D(\varkappa^*, \varpi^*))]$. Hence using property of (θ_1), we have $D(\varkappa^*, \varpi^*) \le \varepsilon+\varphi(D(\varkappa^*, \varpi^*))$, or $D(\varkappa^*, \varpi^*) - \varphi D(\varkappa^*, \varpi^*) \le \varepsilon$. Further, we have $\beta(D(\varkappa^*, \varpi^*) \le \varepsilon$. Hence

$$D(\varkappa^*, \varpi^*) \le \beta^{-1}(\varepsilon),$$

which completes the proof.

(b) If $\{\xi_n\}$ is an O-sequence in $\mathscr{X}$ such that $\lim_{n\to\infty} D(\mathscr{T}\xi_n, \xi_n) = 0$ and ϖ^* is a unique fixed point of $\mathscr{T}$ (using Theorem 3.1). From the contractive condition and triangle inequality, we have

$$\begin{aligned} f(D(\xi_n, \varpi^*)) &\le f[D(\xi_n, \mathscr{T}\xi_n) + D(\mathscr{T}\xi_n, \varpi^*)] + \alpha \\ &\le f[D(\xi_n, \mathscr{T}\xi_n) + D(\mathscr{T}\xi_n, \mathscr{T}\varpi^*)] + \alpha \\ &\le f[D(\xi_n, \mathscr{T}\xi_n) + \varphi(D(\xi_n, \varpi^*))] + \alpha, \end{aligned}$$

or we have

$$f(D(\xi_n\varpi^*)) - \alpha \le f[D(\xi_n, \mathscr{T}\xi_n) + \varphi D(\xi_n, \varpi^*)].$$

On the same lines of above cases, we have $\beta(D(\xi_n, \varpi^*)) \le D(\xi_n, \mathscr{T}\xi_n)$. Taking limit $n \to \infty$, we get

$$\lim_{n\to\infty} \beta(D(\xi_n, \varpi^*)) \le \lim_{n\to\infty} D(\xi_n, \mathscr{T}\xi_n).$$

Therefore, $\lim_{n\to\infty} \beta(D(\xi_n, \varpi^*)) = 0$. Hence $D(\xi_n, x^*) = 0$. This shows that the fixed point problem (8) is well-posed.

□

Theorem 4.2. *Assume that all the hypotheses of Theorem (3.1) are satisfied. If $\mathscr{G} : \mathscr{X} \to \mathscr{X}$ be a mapping such that there exists $\eta > 0$ with*

$$D(\mathscr{T}\xi, \mathscr{G}\xi) < \eta,$$

for all $\xi \in \mathscr{X}$, then for any fixed point $\varkappa^*$ of $\mathscr{G}$, we have

$$D(\varpi^*, \varkappa^*) \leq \beta^{-1}(\eta).$$

Proof. Assume that $\mathscr{G} : \mathscr{X} \to \mathscr{X}$ is a mapping such that there exists $\eta > 0$, with $D(\mathscr{T}\xi, \mathscr{G}\xi) < \eta$, for all $\xi \in \mathscr{X}$. Choose $\varkappa^*$ be the fixed point of $\mathscr{G}$ then by triangle inequality, we have

$$\begin{aligned} f(D(\varpi^*, \varkappa^*) &\leq f(D(\varpi^*, \varkappa^*)) + \alpha \\ &\leq f(D(\mathscr{T}\varpi^*, \mathscr{G}\varkappa^*)) + \alpha \\ &\leq f[D(\mathscr{T}\varpi^*, \mathscr{T}\varkappa^*) + D(\mathscr{T}\varkappa^*, \mathscr{G}\varkappa^*)] + \alpha \\ &\leq f[\varphi(D(\varpi^*, \varkappa^*)) + D(\mathscr{T}\varkappa^*, \mathscr{G}\varkappa^*)] + \alpha \\ &\leq f[\varphi(D(\varpi^*, \varkappa^*)) + \eta] + \alpha. \end{aligned}$$

Therefore, we have

$$f(D(\varpi^*, \varkappa^*) - \alpha \leq f[\varphi(D(\varpi^*, \varkappa^*)) + \eta],$$

or

$$f(D(\varpi^*, \varkappa^*) - \alpha \leq f(D(\varpi^*, \varkappa^*) \leq f[\varphi(D(\varpi^*, \varkappa^*)) + \eta].$$

Hence $f(D(\varpi^*, \varkappa^*) \leq f[\varphi(D(\varpi^*, \varkappa^*)) + \eta]$, $D(\varpi^*, \varkappa^*) \leq \varphi(D(\varpi^*, \varkappa^*)) + \eta$, $D(\varpi^*, \varkappa^*) - \varphi(D(\varpi^*, \varkappa^*)) \leq \eta$. Therefore, we get $\beta(D(\varpi^*, \varkappa^*)) \leq \eta$, or $D(\varpi^*, \varkappa^*) \leq \beta^{-1}(\eta)$. Hence the result. □

Remark 4.1. *Let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a self mapping on an orthogonal $\mathfrak{F}-$ metric space $(\mathscr{X}, \perp, D)$. Suppose that the following condition are satisfied:*

(i) *$\mathscr{X}$ is an orthogonal $\mathfrak{F}-$complete;*
(ii) *$\mathscr{T}$ is $\perp$-preserving and $\perp$-continuous;*
(iii) *there exists $k \in (0, 1)$ such that for all $(\varpi, \varkappa) \in \mathscr{X} \times \mathscr{X}$ with $\varpi \perp \varkappa$, $D(\mathscr{T}\varpi, \mathscr{T}\varkappa) > 0$ we have*

$$D(\mathscr{T}\varpi, \mathscr{T}\varkappa) \leq kD(\varpi^*, \varkappa^*),\ (\varpi^*, \varkappa^*) \in \mathscr{X} \times \mathscr{X}.$$

Then the fixed point problem for $\mathscr{T}$ is well posed.
Indeed, $\mathscr{T}\varpi^* = \varpi^*$ and let $\varpi_n \in \mathscr{X}, n \in \mathbb{N}$, be such that
$D(\varpi_n, \mathscr{T}\varpi_n) \to 0$ as $n \to \infty$. we have

$$\begin{aligned} f(D(\varpi_n, \varpi^*)) &\leq f[D(\varpi_n, \mathscr{T}\varpi_n) + D(\mathscr{T}\varpi_n, \varpi^*)] + \alpha \\ &\leq f[D(\varpi_n, \mathscr{T}\varpi_n) + kD(\varpi_n, \varpi^*)] + \alpha. \end{aligned}$$

Since f is increasing and taking α is 0, we have

$$D(\varpi_n, \varpi^*) \leq D(\varpi_n, \mathscr{T}\varpi_n) + kD(\varpi_n, \varpi^*).$$

So

$$D(\varpi_n, \varpi^*) \leq \frac{1}{1-k} D(\varpi_n, \mathscr{T}\varpi_n) \to 0 \text{ as } n \to \infty.$$

5. **Stability of integral equation**

Now using the theorems proved in previous section, we have the Ulam-Hyers stability of a Volterra integral equation on $[0, \infty)$.

Consider the equation

$$\varpi(t) = \int_0^t \varpi(s)ds, \ t \in \mathbb{R}_+ \tag{9}$$

in $\mathscr{X} = C(\mathbb{R}_+)$ endowed with the $\mathcal{F}-$metric

$$D(\varpi, \varkappa) = \sup_{t \in \mathbb{R}_+} \mid \varpi(t) - \varkappa(t) \mid .$$

Define the orthogonality relation $\perp$ on $\mathscr{X}$ by $\varpi \perp \varkappa \Leftrightarrow \varpi(t)\varkappa(t) \geq \varpi(t)$ or $\varpi(t)\varkappa(t) \geq \varkappa(t)$ for all $t \in \mathbb{R}_+$. Then $(\mathscr{X}, \perp, D)$ is an orthogonal $\mathcal{F}$-metric space.

The equation (9) has in $\mathscr{X} = C(\mathbb{R}_+)$ a unique solution $\varpi^* = 0$. On the other hand, we remark that $\varkappa^*(t) = \varepsilon e^t$ is a solution of the inequation

$$\left| \varkappa(t) - \int_0^1 \varkappa(s)ds \right| \ \leq \ \varepsilon \text{ for all } t \in \mathbb{R}_+.$$

Now using (D_3), for every $(\varpi^*, \varkappa^*) \in \mathscr{X} \times \mathscr{X}$, for every $n \in \mathbb{N}$, $n > 1$, and for every $\{\varpi_i\}_{i=1}^n \subset X$ with $(\varpi_1, \varpi_n) = (\varpi^*, \varkappa^*)$, we have

$$D(\varpi^*, \varkappa^*) > 0 \Rightarrow f(D(\varpi^*, \varkappa^*)) \leq f\left(\sum_{i=1}^{n-1} D(\varpi_i, \varpi_{i+1})\right) + \alpha \tag{10}$$

$$\begin{aligned}
\sum_{i=1}^{n-1} D(\varpi_i, \varpi_{i+1}) &= \sum_{i=1}^{n-1} \sup \mid \varpi_i - \varpi_{i+1} \mid \\
&= \sup \mid \varpi_1 - \varpi_2 \mid + \sup \mid \varpi_2 - \varpi_3 \mid + \ldots\ldots. + \sup \mid \varpi_{n-1} - \varpi_n \mid \\
&= \sup (\mid \varpi_1 - \varpi_2 \mid + \mid \varpi_2 - \varpi_3 \mid + \ldots\ldots. + \mid \varpi_{n-1} - \varpi_n \mid) \\
&\geq \sup (\mid \varpi_1 - \varpi_2 + \varpi_2 - \varpi_3 + \ldots\ldots. + \varpi_{n-1} - \varpi_n \mid) \\
&= \sup \mid \varpi_1 - \varpi_n \mid \\
&= \sup \mid \varpi^* - \varpi^* \mid .
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\sum_{i=1}^{n-1} \sup \mid \varpi_i - \varpi_{i+1} \mid &\geq \sup \ \mid \varpi^* - \varpi^* \mid \\
&= \sup \mid 0 - \varepsilon e^t \mid .
\end{aligned}$$

Further, it implies that

$$\frac{1}{\sum_{i=1}^{n-1} \sup \mid \varpi_i - \varpi_{i+1} \mid} \leq \frac{1}{\sup \ \mid \varepsilon e^t \mid}.$$

Taking limit $t \to \infty$, we get

$$\frac{1}{\sum_{i=1}^{n-1} \sup \mid \varpi_i - \varpi_{i+1} \mid} \ \leq \ 0 \tag{11}$$

Now from (10) and (11) with taking $f(t) = -\dfrac{1}{t}$, $t > 0$ and $\alpha = 0$, we get

$$f(D(\varpi^*, \varkappa^*)) \leq \varepsilon,$$

Further, we get

$$D(\varpi^*, \varkappa^*) \leq f^{-1}(\varepsilon).$$

Hence the result.

6. Conclusions

In this paper, we study some sufficient conditions for the existence of solution for a fixed point problem in the setting of orthogonal $\mathcal{F}-$metric space by generalizing many known results of the literature. Also, we obtain some results on the Ulam-Hyers stability problem and well-posedness of the fixed point problem.

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