CONGRUENCES INDUCED BY CERTAIN RELATIONS ON $\mathcal{AG}^{**}$-GROUPOIDS

by Faisal Yousafzai[1], Murad-ul-Islam Khan[2], Kar Ping Shum[3] and Kostaq Hila[4]

We introduce the concept of partially inverse $\mathcal{AG}^{**}$-groupoids which is almost parallel to the concepts of E-inversive semigroups and E-inversive E-semigroups. Some characterization problems are provided on partially inverse $\mathcal{AG}^{**}$-groupoids. We give necessary and sufficient conditions for a partially inverse $\mathcal{AG}^{**}$-subgroupoid E to be a rectangular band. Furthermore we determine the unitary congruence η on a partially inverse $\mathcal{AG}^{**}$-groupoid and show that each partially inverse $\mathcal{AG}^{**}$-groupoid possesses an idempotent separating congruence μ. We also study anti-separative commutative image of a locally associative $\mathcal{AG}^{**}$-groupoid. Finally, we give the concept of completely N-inverse $\mathcal{AG}^{**}$-groupoid and characterize a maximum idempotent separating congruence.

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1. Introduction

An $\mathcal{AG}$-groupoid is a useful non-associative and a noncommutative algebraic structure, midway between a groupoid and a commutative semigroup. Commutative law is given by $abc = cba$ in ternary operations. By putting brackets on the left of this equation, i.e. $(ab)c = (cb)a$, in 1972, M. A. Kazim and M. Naseeruddin introduced a new algebraic structure called a left almost semigroup abbreviated as an $\mathcal{LA}$-semigroup [14]. This identity is called the left invertive law. Several others authors that have contributed on left almost semigroups, can be found in references. Cho et al. [4] studied this structure under the name of right modular groupoid. Holgate [12] studied it as left invertive groupoid. P.V. Protić and N. Stevanović called the same structure an Abel-Grassmann's groupoid abbreviated as an $\mathcal{AG}$-groupoid [26]. $\mathcal{AG}$-groupoids have a variety of applications in flocks theory, finite mathematics, geometry and other algebras ([2, 25, 31, 33]). $\mathcal{AG}$-groups also have a geometrical interpretation that gives a rise to their application in the context of parallelogram spaces [32].

This structure is closely related to a commutative semigroup because a commutative $\mathcal{AG}$-groupoid is a semigroup [21]. It was proved in [14] that an $\mathcal{AG}$-groupoid S is medial, that is, $ab \cdot cd = ac \cdot bd$ holds for all $a,b,c,d \in S$. An $\mathcal{AG}$-groupoid may or may not contains a left identity. The left identity of an $\mathcal{AG}$-groupoid permits the inverses of elements in the structure. If an $\mathcal{AG}$-groupoid contains a left identity, then this left identity is unique [21]. In an $\mathcal{AG}$-groupoid S with left identity (unitary $\mathcal{AG}$-groupoid), the paramedial law

[1]Military College of Engineering, National University of Sciences and Technology, Islamabad, Pakistan, E-mail: yousafzai@mce.nust.edu.pk

[2]Department of Pure and Applied Mathematics, University of Haripur, Haripur, KPK, Pakistan, E-mail: muradulislam@foxmail.com

[3]Institute of Mathematics, Yunnan University, Kunming, China, E-mail: kpshum@ynu.edu.cn

[4]Department of Mathematical Egineering, Polytechnic University of Tirana, Albania, E-mail: kostaq_hila@yahoo.com

$ab \cdot cd = dc \cdot ba$ holds for all $a,b,c,d \in S$. By using medial law with left identity, we get $a \cdot bc = b \cdot ac$ for all $a,b,c \in S$. It is important to mention here that if an $\mathcal{AG}$-groupoid contains identity or even right identity, then it becomes a commutative monoid.

We should genuinely acknowledge that much of the ground work has been done by M.A. Kazim, M. Naseeruddin, Q. Mushtaq, M.S. Kamran, P.V. Protić, N. Stevanović, M. Khan, W.A. Dudek and R.S. Gigon. One can be referred to [6, 7, 15, 16, 17, 21, 22, 26, 35] in this regard.

In [22], Q. Mushtaq and S.M. Yusuf introduced the concept of a locally associative $\mathcal{AG}$-groupoid. Several fundamental theorems were proved in this paper. An exponential semigroup is a semigroup satisfying the identity $(xy)^m = x^m y^m (m \geq 2)$ and it was introduced by T. Tamura and T. Nordhal in [36]. It is noteworthy that a locally associative $\mathcal{AG}$-groupoid is exponential. In 1967, T.S. Frank [9] asked the question, whether or not there exists a subtractive group? Plainly speaking, the nature of operations of subtraction and division is such that they are non-associative and as such any structure based upon them would be the same. It wasn't till 1987 when this question was answered by Q. Mushtaq and M.S. Kamran in [13] when they successfully defined a non-associative group called an $\mathcal{AG}$-group and verified some important and known results of group theory. P.V. Protic and N. Stevanovic have done a lot of research on $\mathcal{AG}$-groupoids. The generalization of a unitary $\mathcal{AG}$-groupoid was called as an $\mathcal{AG}^{**}$-groupoid. They showed that a non-associative left simple (right simple, simple) $\mathcal{AG}^{*}$-groupoid does not exist. They introduced the notion of an $\mathcal{AG}$-band. They also introduced congruences in $\mathcal{AG}^{*}$-groupoids, $\mathcal{AG}^{**}$-groupoids and $\mathcal{AG}$-bands, and decomposed the structures using these congruences in [27] and [28]. P.V. Protić and N. Stevanović introduced a useful technique for verification of $\mathcal{AG}^{**}$-groupoids, $\mathcal{AG}^{*}$-groupoids and $\mathcal{AG}$-groupoids [35]. They defined ideals in [34], and added many interesting results to the theory of $\mathcal{AG}$-groupoids in [35]. In [15], M. Khan has studied many interesting properties of $\mathcal{AG}$-groupoids, $\mathcal{AG}^{*}$-groupoids and $\mathcal{AG}^{**}$-groupoids in his doctoral thesis. He has studied M-systems and P-systems in $\mathcal{AG}$-groupoids and characterized simple and 0-simple $\mathcal{AG}$-groupoids. In [6], W.A. Dudek and R.S. Gigon have shown that the set of all idempotents of a completely inverse $\mathcal{AG}^{**}$-groupoid A forms a semilattice and the Green's relations $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ and $\mathcal{J}$ coincide on A. The main result of this note says that any completely inverse $\mathcal{AG}^{**}$-groupoid meets the famous Lallement's Lemma for regular semigroups. They have shown that the Green's relation $\mathcal{H}$ is both the least semilattice congruence and the maximum idempotent separating congruence on any completely inverse $\mathcal{AG}^{**}$-groupoid. In [7], they have determined certain fundamental congruences on a completely inverse $\mathcal{AG}^{**}$-groupoid; namely: the maximum idempotent separating congruence, the least $\mathcal{AG}$-group congruence and the least E-unitary congruence. They have investigated the complete lattice of congruences of a completely inverse $\mathcal{AG}^{**}$-groupoid. Some other results on congruences on completely inverse $\mathcal{AG}^{**}$-groupoids have been obtained recently in [16, 17].

Many authors studied various congruences on some special classes of $\mathcal{AG}$-groupoids and described the corresponding quotient algebras as semilattices of some subgroupoids [6, 7, 15, 20, 23, 24, 27, 28]. In this paper, we introduce and study some basic results on partially inverse $\mathcal{AG}^{**}$-groupoids and completely N-inverse $\mathcal{AG}^{**}$-groupoid. These concepts allow us to study some congruences on certain classes of $\mathcal{AG}^{**}$-groupoids that have been previously explored in the structure of E-inversive semigroups, E-inversive E-semigroups and eventually regular semigroups [1, 8, 10, 18, 37, 38].

2. Congruences on partially inverse $\mathcal{AG}^{**}$-groupoids

A congruence is determined in a group if we know a single congruence class, in particular, if we know the normal subgroup. Similarly, in a ring a congruence is determined if we know the ideal. In semigroups there is no such fortunate occurrence, and we are therefore faced with the necessity of studying congruences and hence the theory of semigroups.

In this section, we give some basic results on partially inverse $\mathcal{AG}^{**}$-groupoids. Some characterization problems are provided on a partially inverse $\mathcal{AG}^{**}$-groupoid. We study some congruences on a partially inverse $\mathcal{AG}^{**}$-groupoid in terms of a semilattice E. At the beginning we determine the unitary congruence η on a partially inverse $\mathcal{AG}^{**}$-groupoid. Also, we give necessary and sufficient conditions for a partially inverse $\mathcal{AG}^{**}$-subgroupoid E to be a rectangular band. Furthermore, we provide some equivalent conditions for a partially inverse $\mathcal{AG}^{**}$-groupoid and characterized a unitary congruence η. Finally we prove that a unitary congruence τ on a partially inverse $\mathcal{AG}^{**}$-groupoid is uniquely determined by a relation $\sim$ and the set of idempotents. We show that each partially inverse $\mathcal{AG}^{**}$-groupoid possesses an idempotent separating congruence μ.

2.1. **Basic results.** A semigroup S is called E-inversive if for every $a \in S$ there exists some $x \in S$ such that ax is idempotent [18]. This concept was introduced by G. Thierrin in 1955. In [11], E-inversive semigroups are also called E-dense semigroups. Basic properties of E-inversive semigroups were given by Catino and Miccoli [3], and Mitsch and Petrich [19].

Zheng and Zhang [38] have studied some idempotent separating congruences on E-inversive semigroups whose idempotents form a rectangular band. In [10], Gigon have studied ideals and congruences in E-inversive semigroups, and also proved that E-inversive semigroup is strictly closed. In [37], Weipoltshammer have investigated a least group congruence, a semilattice congruence and an idempotent separating congruence on an E-inversive E-semigroup.

In this paper, we have generalized the concept of E-inversive semigroups and E-inversive E-semigroups. Some basic properties of E-inversive semigroups in a partially inverse $\mathcal{AG}^{**}$-groupoid have been transformed and studied different congruences on a partially inverse $\mathcal{AG}^{**}$-groupoid. It is important to note that unlike semigroup, the concepts of partially inverse $\mathcal{AG}^{**}$-groupoid and partially inverse E-$\mathcal{AG}^{**}$-groupoid coincide in a structure of an $\mathcal{AG}$-groupoid.

Thus in this paper, we have actually tried to generalize the work done for associative structures such as semigroups to non-associative structures such as $\mathcal{AG}$-groupoids.

Let us introduce the concept of soft inverses in an $\mathcal{AG}^{**}$-groupoid as follows:

For an $\mathcal{AG}^{**}$-groupoid, $U(a) = \{x \in S | x = xa \cdot x \text{ and } xa = ax\}$ is the set of all soft inverses of $a \in S$.

Definition 2.1. *An $\mathcal{AG}^{**}$-groupoid S is called partially inverse if for all $a \in S$, there exists $x \in S$ such that ax and xa are idempotents.*

In [3], E-inversive semigroup was characterized by Catino and Miccoli as follows: *A semigroup S is E-inversive if and only if $W(a) \neq \emptyset$ for all $a \in S$*, where $W(a) = \{x \in S \ / \ x = xax\}$ is the set of all weak inverses of $a \in S$.

By contrast, we obtain the following result in general.

Lemma 2.1. *An $\mathcal{AG}^{**}$-groupoid S is partially inverse if and only if $U(a) \neq \emptyset$ for all $a \in S$.*

Proof. Necessity. Let $a \in S$. Then there exists $x \in S$ such that ax and xa are idempotents. Thus, we have

$$\begin{aligned} ax \cdot x &= (ax \cdot ax)x = (ax \cdot ax)(ax) \cdot x = (x \cdot ax)(ax \cdot ax) = (a \cdot xx)(aa \cdot xx) \\ &= (xx \cdot aa)(xx \cdot a) = (aa \cdot xx)(x^2 a) = a^2 x^2 \cdot x^2 a = (x^2 a \cdot a)(x^2 a) \\ &= (ax \cdot x)a \cdot (ax \cdot x). \end{aligned}$$

Moreover, $(ax \cdot x)a = ax \cdot ax = a(ax \cdot x)$. It follows that $\forall a \in S$, $U(a) \neq \emptyset$.

Sufficiency. Assume that $U(a) \neq \emptyset$, for all $a \in S$. Since $x = xa \cdot x$ and $ax = xa$ for some $x \in S$, then $ax = a(xa \cdot x) = xa \cdot ax = ax \cdot ax$, which shows that S is partially inverse. □

The above fact will be used frequently without mention in the sequel.

A semigroup S is E-semigroup if the set E of idempotents of S forms a subsemigroup [1]. It is noteworthy that every $\mathcal{AG}^{**}$-groupoid is E-$\mathcal{AG}^{**}$-groupoid which can be easily followed from the following Lemma **2**.2.

Lemma 2.2. [27] *Let S be an $\mathcal{AG}^{**}$-groupoid. Then E is a semilattice.*

Example 2.1. *Let us define a set $S = \{a, b, c, d\}$ under the binary operation "$\cdot$" as follows:*

$\cdot$	a	b	c	d
a	b	b	d	d
b	b	b	b	b
c	a	b	c	d
d	a	b	a	b

By routine calculation, it is easy to see that $(S, \cdot)$ is a partially inverse $\mathcal{AG}^{**}$-groupoid as well as a partially inverse E-$\mathcal{AG}^{**}$-groupoid.

We now provide some properties of a partially inverse $\mathcal{AG}^{**}$-groupoid.

Lemma 2.3. *Let S be a partially inverse $\mathcal{AG}^{**}$-groupoid. Then the following conditions hold:*

(i) $ea' \in U(a)$ if $a' \in U(a)$ and $e \in E$;
(ii) $a'f \cdot e \in U(a)$ if $a' \in U(a)$ and $e, f \in E$;
(iii) $a'^2 f \cdot e \in U(a^2)$ if $a' \in U(a)$ and $e, f \in E$.

Proof. (i) : Let $a' \in U(a)$ and $e \in E$, then

$$\begin{aligned} ea' &= (ee)(a'a \cdot a') = (e \cdot a'a)(ea') = (ee \cdot a'a)(ea') \\ &= (aa' \cdot ee)(ea') = (aa' \cdot e)(ea') = (ea' \cdot a)(ea'), \end{aligned}$$

and $a \cdot ea' = e \cdot aa' = ee \cdot aa' = aa' \cdot e = ea' \cdot a$.

(ii) : Let $a' \in U(a)$ and $e, f \in E$, then

$$\begin{aligned} a'f \cdot e &= (a'a \cdot a')(ff) \cdot e = (a'a \cdot f)(a'f) \cdot ee = (a'a \cdot f)e \cdot (a'f)e \\ &= (ef \cdot a'a) \cdot (a'f)e = (aa' \cdot fe) \cdot (a'f)e = (fe \cdot a')a \cdot (a'f)e \\ &= (ef \cdot a')a \cdot (a'f)e = (a'f \cdot e)a \cdot (a'f)e, \end{aligned}$$

and $a(a'f \cdot e) = a'f \cdot ae = a'a \cdot fe = aa' \cdot ef = ae \cdot a'f = (a'f \cdot e)$.

(iii) : Let $a' \in U(a)$ and $e, f \in E$, then

$$\begin{aligned} a'^2 f \cdot e &= (a'a' \cdot f)e = ef \cdot a'a' = (ef) \cdot (a'a \cdot a')(a'a \cdot a') \\ &= (ef) \cdot (a'a \cdot a'a)(a'a') = (ef) \cdot (aa' \cdot aa')(a'a') \\ &= (ef)(a^2 a'^2 \cdot a'^2) = (fe)(a'^2 a'^2 \cdot a^2) = (ff \cdot a'^2 a'^2)(ea^2) \\ &= (a'^2 a'^2 \cdot ff)(ea^2) = (a'^2 f \cdot a'^2 f)(ea^2) = (a^2 e)(a'^2 f \cdot a'^2 f) \\ &= (a'^2 f \cdot a'^2 f)(ee) \cdot a^2 = (a'^2 f \cdot e)(a'^2 f \cdot e) \cdot a^2 \\ &= (aa)(a'^2 f \cdot e) \cdot (a'^2 f \cdot e) = (e \cdot a'^2 f)(aa) \cdot (a'^2 f \cdot e) \\ &= (fa'^2 \cdot e)a^2 \cdot (a'^2 f \cdot e) = (a'^2 f \cdot e)a^2 \cdot (a'^2 f \cdot e), \end{aligned}$$

and

$$a^2(a'^2 f \cdot e) = a'^2 f \cdot a^2 e = a'^2 a^2 \cdot fe = a^2 a'^2 \cdot ef = (ef \cdot a'^2)a^2 = (a'^2 f \cdot e)a^2.$$

□

Definition 2.2. *Let S be an $\mathcal{AG}^{**}$-groupoid. For any $H, B \subseteq S$, we define $H_{\varpi_B} = \{a \in S : ba \in H$ for some $b \in B\}$.*

If $B = H$, then H_{ϖ_H} will be denoted by H_{ϖ} and it is called closure of H. An $\mathcal{AG}^{**}$-subgroupoid H of an $\mathcal{AG}^{**}$-groupoid S is called closed if $H = H_{\varpi}$.

Proposition 2.1. *Let S be a partially inverse $\mathcal{AG}^{**}$-groupoid. Then E is partially inverse if E is closed.*

Proof. Let $e \in E$ and $e^{'} \in U(e)$. Since $ee^{'} \in E$, then $e^{'} \in E_{\varpi}$. Also E is closed, which follows that $e^{'} \in E$. Hence E is partially inverse. □

In general, the converse of the above proposition does not hold. The following example shows it.

Example 2.2. *Let $(S, \cdot)$ be the partially inverse $\mathcal{AG}^{**}$-groupoid of Example 2.1. Clearly $E = \{b, c\}$ is a partially inverse subset of S, but it is not closed.*

Lemma 2.4. *Let S be an $\mathcal{AG}^{**}$-groupoid. If $ab = e$ for all $a, b \in S$ and $e \in E$, then $b^2e \cdot a^2 = e$.*

Proof. Let $a, b \in S$ and $e \in E$. Then by given assumption, we get

$$b^2e \cdot a^2 = (bb \cdot e)(aa) = (eb \cdot b)(aa) = (eb \cdot a)(ba) = (ab)(a \cdot eb) = (ab)(e \cdot ab) = e.$$

□

Definition 2.3. *Let S be an $\mathcal{AG}^{**}$-groupoid. Define a relation "$\sim$" on S as follows:*

For $a \in S$ and $e \in E$, $e \sim a \Leftrightarrow e = xa$ for some $x \in S$.

For $a \in S$, a subset $E(a)$ of S is defined by $E(a) = \{e \in E \ / \ e \sim a\}$.

Lemma 2.5. *Let S be an $\mathcal{AG}^{**}$-groupoid. For all $a \in S$ and $e \in E$, $e \sim a$ if and only if $e \in Sa$.*

Proof. It is simple, hence is omitted. □

Now, we will make use of the following description of a partially inverse $\mathcal{AG}^{**}$-groupoid whose idempotents form a rectangular band.

Definition 2.4. *An $\mathcal{AG}^{**}$-groupoid S is called rectangular if S satisfies the identity $x = xy \cdot x$ for all $x, y \in S$.*

Example 2.3. *Let us consider an $\mathcal{AG}^{**}$-groupoid $S = \{a, b, c, d\}$ in the following multiplication table:*

$\cdot$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	d	c	a	b
d	c	d	b	a

It can be verified that it is a rectangular $\mathcal{AG}^{**}$-groupoid.

By an $\mathcal{AG}$-band we mean an $\mathcal{AG}$-groupoid whose elements are idempotent.

Lemma 2.6. *Let S be an $\mathcal{AG}^{**}$-groupoid and E be a rectangular band. Then E is a singleton set.*

Proof. It is simple, hence is omitted. □

Lemma 2.7. *Let S be a partially inverse $\mathcal{AG}^{**}$-groupoid. Then the following conditions are equivalent*:

(i) *E is a rectangular band*;

(ii) *For all $a, b \in S$, $U(a) \cap U(b) \neq \emptyset$ implies $U(a) = U(b)$.*

Proof. $(i) \Longrightarrow (ii)$: Take any $a, b \in S$ and $x \in U(a) \cap U(b)$. Then $x = xa \cdot x$, $xa = ax$ and $x = xb \cdot x$, $xb = bx$. Since $xa, xb \in E$, then by using Lemma **2**.6, $xa = xb$ and thus $U(a) = U(b)$.

$(ii) \Longrightarrow (i)$: It is trivial to observe that if (ii) holds, then for all $e, f \in E$, $U(e) \cap U(f) \neq \emptyset$ implies $U(e) = U(f)$. Let $x \in U(ef)$, that is, $x = (x \cdot ef)x$ and $x \cdot ef = ef \cdot x$. Then

$$(xf \cdot e)e \cdot (xf \cdot e) = (e \cdot xf)(xf \cdot e) = (x \cdot ef)(ef \cdot x) = (ef \cdot x)(x \cdot ef) = ((x \cdot ef)x)(ef) = x \cdot ef = ef \cdot x = xf \cdot$$

and

$$(xf \cdot e)e = (xf \cdot e)(ee) = (ee)(e \cdot xf) = e(x \cdot ef) = e(ef \cdot x) = e(xf \cdot e).$$

Similarly we can show that $(xf \cdot e)f \cdot (xf \cdot e) = xf \cdot e$ and $(xf \cdot e)f = f(xf \cdot e)$. Thus $xf \cdot e \in U(e) \cap U(f)$, whence $U(e) = U(f)$. Since $e \in U(e)$, then it follows that $ef \cdot e = e$. Hence E is a rectangular band. □

2.2. **Unitary congruences.** The basic definitions of congruences on an $\mathcal{AG}$-groupoid are given in [20] and those definitions are analogous with those in semigroup theory. A congruence σ on an $\mathcal{AG}^{**}$-groupoid S is an equivalence relation if σ is right (left) compatible, that is, $ac\sigma bc$ $(ca\sigma cb)$ for each c in S.

Definition 2.5. *A congruence σ on an $\mathcal{AG}$-groupoid S is said to be a unitary congruence if S/σ contains a left identity.*

The group congruence σ on an E-inversive E-semigroup S was characterized by Reither [30] as follows: $a\sigma b \Longleftrightarrow ea = bf$, *for some* $e, f \in E$.

Now we will provide a theorem which will show the existence of a unitary congruence on a partially inverse $\mathcal{AG}^{**}$-groupoid as follows:

Theorem 2.1. *Let S be a partially inverse $\mathcal{AG}^{**}$-groupoid. Then the relation $\eta = \{(a, b) \in S \times S \ / \ ea^2 = b^2 f$ for some $e, f \in E\}$ is a unitary congruence on S.*

Proof. It is easy to show that η is reflexive and symmetric.

Suppose that $a\eta b$ and $b\eta c$. Then $ea^2 = b^2 f$ and $e'b^2 = c^2 f'$ for some $e, f, e', f' \in E$. Now

$$\begin{aligned} e'e \cdot a^2 &= a^2 e \cdot e' = ea^2 \cdot e' = b^2 f \cdot e'e' = e'e' \cdot fb^2 = e' \cdot fb^2 = f \cdot e'b^2 \\ &= f \cdot c^2 f' = c^2 \cdot ff', \end{aligned}$$

which follows that $a\eta c$. Thus η is transitive.

Let $a\eta b$ and $c \in S$, then $ea^2 = b^2 f$ for some $e, f \in E$. Thus

$$\begin{aligned} e(ac)^2 &= ee \cdot a^2c^2 = c^2a^2 \cdot ee = c^2a^2 \cdot e = ea^2 \cdot c^2 = b^2 f \cdot c^2 = c^2 f \cdot bb \\ &= bb \cdot fc^2 = b^2 \cdot fc^2 = f \cdot b^2c^2 = f(bc)^2. \end{aligned}$$

Thus we showed that η is right compatible. Similarly, we can show that η is left compatible. Hence η is a congruence on S.

Claim that for any $e \in E$, $e\eta$ is the left identity of S/η. Let $x \in S$ and $x' \in U(x)$. Then $(x'x)(ex)^2 = x'x \cdot e^2x^2 = x'x \cdot x^2e^2 = x^2(x'x \cdot e)$, therefore $(ex, x) \in \eta$. Thus η is a unitary congruence on S. □

Example 2.4. *Let us consider a partially inverse $\mathcal{AG}^{**}$-groupoid $S = \{1, 2, 3, 4, 5\}$ in the following multiplication table:*

$\cdot$	1	2	3	4	5
1	2	1	1	1	1
2	1	2	2	2	2
3	1	2	4	5	3
4	1	2	3	4	5
5	1	2	5	3	4

If we take $E = \{2,4\}$ and consider $\eta = \{(1,1),(1,2),(2,2),(3,1)\}$, then it is easy to show that η is a unitary congruence on S.

Corollary 2.1. *Let S be a partially inverse $\mathcal{AG}^{**}$-groupoid and let E be closed. Then $\eta = \{(a,b) \in E \times E \ / \ ea = bf$ for some $e,f \in E\}$ is the least unitary congruence on E.*

Proof. From Proposition **2**.1 and Theorem **2**.1, η is a unitary congruence on E. To show that η is least on E, let ψ be an arbitrary unitary congruence on E and $(a,b) \in \psi$. Then $ea = bf$ for some $e,f \in E$. Thus

$$a\eta = e\eta \cdot a\eta = (ea)\eta = (bf)\eta = (b^2f^2)\eta = (f^2b^2)\eta = (fb)\eta = f\eta \cdot b\eta = b\eta,$$

which implies that $(a,b) \in \eta$. Hence η is the least unitary congruence on E. □

The following characterization problem gives us an alternative description of Theorem **2**.1.

Theorem 2.2. *Let S be a partially inverse $\mathcal{AG}^{**}$-groupoid. Then the unitary congruence η on S is given by $a\eta b \Leftrightarrow U(a^2) \cap U(b^2) \neq \emptyset$.*

Proof. Necessity. Let $a,b \in S$ such that $a\eta b$. Then by Theorem **2**.1, there exist $e,f \in E$ with $ea^2 = b^2f$. Let $a' \in U(a)$. Then by Lemma **2**.3 *(iii)*, $a'^2f \cdot e \in U(a^2) \cap U(b^2)$, whence $U(a^2) \cap U(b^2) \neq \emptyset$.

Sufficiency. Let $x \in U(a^2) \cap U(b^2)$. Then $xa^2 = a^2x$ and $xb^2 = b^2x$. It is easy to see that $b^2x, xa^2 \in E$. Thus $b^2x \cdot a^2 = a^2x \cdot b^2 = bb \cdot xa^2 = b \cdot xa^2$. Hence by Theorem **2**.1, $a\eta b$. □

Corollary 2.2. *Let S be a partially inverse $\mathcal{AG}^{**}$-groupoid and let E be closed. Then the unitary congruence η on E is given by $\eta = \{(a,b) \in E \times E \ / \ U(a) = U(b)\}$ if and only if E is a rectangular band.*

Proof. Necessity. Let $e,f \in E$. From Corollary **2**.1, η is a unitary congruence on E, $e\eta f$. By hypothesis, therefore we have $U(a) = U(b)$. Since $e \in U(e) = U(f)$, then it follows that $ef \cdot e = e$. Thus E is a rectangular band.

Sufficiency. Let $a\eta b$ and E be a rectangular band. By Theorem **2**.2, $U(a^2) \cap U(b^2) \neq \emptyset$, whence by Lemma **2**.7, $U(a) = U(b)$. Conversely, if $U(a) = U(b)$, then $U(a) \cap U(b) = U(a) \neq \emptyset$ (see Lemma **2**.1) which implies that $U(a^2) \cap U(b^2) \neq \emptyset$. Hence by Theorem **2**.2, $a\eta b$. □

Definition 2.6. *A subset H of an $\mathcal{AG}^{**}$-groupoid S is called full if $E \subseteq H$. An $\mathcal{AG}^{**}$-subgroupoid H of an $\mathcal{AG}^{**}$-groupoid S is called softly self conjugate if for all $a \in S$, $x \in H$, $a' \in U(a)$; $a'x \cdot a$, $a' \cdot xa \in H$, $ax = xa$, $a'x = xa'$.*

Example 2.5. *Let S be the partially inverse $\mathcal{AG}^{**}$-groupoid of Example 2.4. Consider $E = \{2,4\}$ and $H = \{2\}$. Then it is easy to verify that H is a softly self conjugate subset of S.*

For an $\mathcal{AG}^{**}$-groupoid S, the following notations will be used: P is the class of all full and softly self conjugate $\mathcal{AG}^{**}$-subgroupoids of S and P^* is the set of all closed $\mathcal{AG}^{**}$-subgroupoids of S.

If we consider $H \in P$ instead of E in Theorem **2**.1, then we can get some important characterization problems connected to a unitary congruences of a partially inverse $\mathcal{AG}^{**}$-groupoid.

Theorem 2.3. *Let S be a partially inverse $\mathcal{AG}^{**}$-groupoid and $H \in P$. Then the relation $\Omega_H = \{(a,b) \in S \times S \ / \ xa^2 = b^2y$ for some $x, y \in H\}$ is a unitary congruence on S.*

Proof. To show that Ω_H is a congruence on S, let $a, b, c \in S$ such that $a' \in U(a)$. It is easy to see that $a'a, aa' \in E$ and since H is full, then $a'a, aa' \in H$. Thus $aa' \cdot a^2 = a^2 \cdot a'a$, which shows that $a\Omega_H a$.

Suppose that $a\Omega_H b$. Then $xa^2 = b^2y$ for some $x, y \in H$. Let $b' \in U(b)$. Thus

$$\begin{aligned}
(b'y \cdot b)(aa') \cdot b^2 &= (b^2 \cdot aa')(by \cdot b') = (b^2 \cdot by)(aa' \cdot b') = (b \cdot b^2y)(aa' \cdot b') \\
&= (b \cdot xa^2)(aa' \cdot b') = (b' \cdot aa')(xa^2 \cdot b) = (xa^2) \cdot (b' \cdot aa')b \\
&= b(b' \cdot aa') \cdot (a^2x) = a^2 \cdot (b(b' \cdot aa'))x = x(b(b' \cdot aa')) \cdot a^2 \\
&= a^2(b(b' \cdot aa')) \cdot x = b(a^2(b' \cdot aa')) \cdot x = b(b'(a^2 \cdot aa')) \cdot x \\
&= b(b'(a'a \cdot a^2)) \cdot x = b(a'a \cdot b'a^2) \cdot x = (a'a)(b \cdot b'a^2) \cdot x \\
&= (b'a^2 \cdot b)(aa') \cdot x = (x \cdot aa')(b'a^2 \cdot b) = (x \cdot a'a)(b'a^2 \cdot b) \\
&= (a' \cdot xa)(b'a^2 \cdot b) = (b'a^2) \cdot (a' \cdot xa)b = b'(a' \cdot xa) \cdot (a^2b) \\
&= a^2 \cdot (b'(a' \cdot xa))b.
\end{aligned}$$

Since $(b'y \cdot b)(aa')$, $(b'(a' \cdot xa))b \in H$, then $b\Omega_H a$.

Again suppose that $a\Omega_H b$ and $b\Omega_H c$. Then $xa^2 = b^2y$ and $zb^2 = c^2w$ for some $w, x, y, z \in H$. Since

$$\begin{aligned}
(xw \cdot zy)a^2 &= (a^2 \cdot zy)(xw) = (a^2x)(zy \cdot w) = (w \cdot zy)(xa^2) = (w \cdot zy)(b^2y) \\
&= (z \cdot wy)(b^2y) = (zb^2)(wy \cdot y) = (c^2w)(y^2w) = (y^2w \cdot w)c^2 \\
&= w^2y^2 \cdot c^2 = c^2 \cdot y^2w^2,
\end{aligned}$$

then it follows that $a\Omega_H c$.

Let $a\Omega_H b$ and $c \in S$. Then $xa^2 = b^2y$ for some $x, y \in H$. Let $b' \in U(b)$ and $c' \in U(c)$. Then

$$\begin{aligned}
&x(bb' \cdot cc') \cdot (ac)^2 = x(bb' \cdot cc') \cdot (a^2c^2) = (xa^2) \cdot (bb' \cdot cc')c^2 \\
&= (b^2y) \cdot (bb' \cdot cc')c^2 = (b^2y) \cdot c^2(cc' \cdot bb') \\
&= (b^2c^2) \cdot y(cc' \cdot bb') = (bc)^2 \cdot y(cc' \cdot bb'),
\end{aligned}$$

which shows that Ω_H is right compatible. Similarly, we can show that Ω_H is left compatible. Hence Ω_H is a congruence on S.

Fix $x \in H$ and claim that $x\Omega_H$ is the left identity of S/Ω_H. Let $a \in S$ and $a' \in U(a)$. Then $(x^2 \cdot aa')a^2 = (a'a \cdot x^2)a^2 = a^2x^2 \cdot a'a = (xa)^2(a'a)$, which implies that $(a, xa) \in \Omega_H$. Hence Ω_H is a unitary congruence on S. □

Theorem 2.4. *Let S be a partially inverse $\mathcal{AG}^{**}$-groupoid with $H \in P^*$. If $a, b \in S$, then the following statements are equivalent*:

(i) *For all $b' \in U(b)$, $a^2b'^2 \in H$;*
(ii) *For all $a' \in U(a)$, $a'^2b^2 \in H$;*
(iii) *For all $a' \in U(a)$, $b^2a'^2 \in H$;*
(iv) *For all $b' \in U(b)$, $b'^2a^2 \in H$;*
(v) *For all $b' \in U(b)$, there exists $x \in H$ such that $a^2x \cdot b'^2 \in H$;*
(vi) *For all $a' \in U(a)$, there exists $x \in H$ such that $a'^2x \cdot b^2 \in H$;*
(vii) *For all $a' \in U(a)$, there exists $x \in H$ such that $b^2x \cdot a'^2 \in H$;*
$(viii)$ *For all $b' \in U(b)$, there exists $x \in H$ such that $b'^2x \cdot a^2 \in H$;*
(ix) *There exist $x, y \in H$ such that $xa^2 = b^2y$;*
(x) *There exist $x, y \in H$ such that $a^2x = yb^2$;*

(xi) $Ha^2 \cap Hb^2 \neq \emptyset$.

Proof. $(i) \Longrightarrow (ii)$: Let $a^{'} \in U(a)$ and $b^{'} \in U(b)$. Then $a^2b^{'2} = ab^{'} \cdot ab^{'} \in H$ implies $ab^{'} \in H_{\varpi} = H$ and

$$\begin{aligned}(a^{'2} \cdot a^2b^{'2})a^2 &= ((a^{'}a^{'})(ab^{'} \cdot ab^{'}))(aa) = ((a^{'} \cdot ab^{'})(a^{'} \cdot ab^{'}))(aa) \\ &= (a^{'} \cdot ab^{'})a \cdot (a^{'} \cdot ab^{'})a \in H.\end{aligned}$$

Now

$$\begin{aligned}&(a^{'2} \cdot a^2b^{'2})a^2 \cdot (a^{'2}b^2) = ((a^{'} \cdot ab^{'})a \cdot (a^{'} \cdot ab^{'})a)(a^{'}b \cdot a^{'}b) \\ &= ((a^{'} \cdot ab^{'})a \cdot (a^{'}b))((a^{'} \cdot ab^{'})a \cdot (a^{'}b)) = ((ab^{'} \cdot a^{'})a \cdot (a^{'}b))((ab^{'} \cdot a^{'})a \cdot (a^{'}b)) \\ &= ((aa^{'} \cdot ab^{'})(a^{'}b))((aa^{'} \cdot ab^{'})(a^{'}b)) = ((a^{'}b \cdot ab^{'})(aa^{'}))((a^{'}b \cdot ab^{'})(aa^{'})) \\ &= ((a^{'}a)(ab^{'} \cdot a^{'}b))((a^{'}a)(ab^{'} \cdot a^{'}b)) = ((aa^{'})(aa^{'} \cdot bb^{'}))((aa^{'})(aa^{'} \cdot bb^{'})) \in HH \subseteq H.\end{aligned}$$

It follows that $a^{'2}b^2 \in H_{\varpi} = H$.

$(ii) \Longrightarrow (i)$: It is similar to the proof of $(i) \Longrightarrow (ii)$.

The proof of $(ii) \Longleftrightarrow (iii) \Longleftrightarrow (iv)$ are similar to the proof of $(i) \Longleftrightarrow (ii)$.

$(iv) \Longrightarrow (v)$: Let $b^{'} \in U(b)$ and $a^{'} \in U(a)$. Then $b^{'2}a^2 = b^{'}a \cdot b^{'}a \in H$, which implies that $b^{'}a \in H_{\varpi} = H$ and $(b^{'2} \cdot a^2a^{'2})b^2 \in H$. Thus

$$\begin{aligned}(a^2 \cdot (b^{'2} \cdot a^2a^{'2})b^2)b^{'2} &= (b^{'2} \cdot a^2a^{'2})(a^2b^2) \cdot b^{'2} = ((b^{'} \cdot aa^{'})(b^{'} \cdot aa^{'}) \cdot (a^2b^2))b^{'2} \\ &= ((b^{'} \cdot aa^{'})(ab) \cdot (b^{'} \cdot aa^{'})(ab))b^{'2} \\ &= ((b^{'} \cdot aa^{'})(ab) \cdot b^{'})((b^{'} \cdot aa^{'})(ab) \cdot b^{'}) \\ &= ((b^{'} \cdot ab)(b^{'} \cdot aa^{'}))((b^{'} \cdot ab)(b^{'} \cdot aa^{'})) \\ &= ((a \cdot b^{'}b)(b^{'} \cdot aa^{'}))((a \cdot b^{'}b)(b^{'} \cdot aa^{'})) \\ &= ((b^{'} \cdot aa^{'})(bb^{'}) \cdot a)((b^{'} \cdot aa^{'})(bb^{'}) \cdot a) \\ &= ((b^{'}b)(aa^{'} \cdot b^{'}) \cdot a)((b^{'}b)(aa^{'} \cdot b^{'}) \cdot a) \\ &= (a(aa^{'} \cdot b^{'}) \cdot (b^{'}b))(a(aa^{'} \cdot b^{'}) \cdot (b^{'}b)) \\ &= ((aa^{'} \cdot ab^{'})(b^{'}b))((aa^{'} \cdot ab^{'}) \cdot (b^{'}b)) \\ &= ((b^{'}a \cdot a^{'}a)(b^{'}b))((b^{'}a \cdot a^{'}a) \cdot (b^{'}b)) \in HH \subseteq H.\end{aligned}$$

$(v) \Longrightarrow (iv)$: Let $a^{'} \in U(a)$ and $b^{'} \in U(b)$. Then there exists $x \in H$ such that $x^2 \in H$, which follows that $a^2x^2 \cdot b^{'2} \in H$. Since $a^{'}(a^2x^2 \cdot b^{'2}) \cdot a \in H$, then

$$\begin{aligned}a^{'}(a^2x^2 \cdot b^{'2}) \cdot a &= (a^2x^2 \cdot a^{'}b^{'2})a = (x^2a^2 \cdot a^{'}b^{'2})a \\ &= (b^{'2}a^2 \cdot a^{'}x^2)a = (a \cdot a^{'}x^2)(b^{'2}a^2) \in H.\end{aligned}$$

Thus we showed that $b^{'2}a^2 \in H_{\varpi} = H$.

$(v) \Longrightarrow (vi)$: Let $a^{'} \in U(a)$ and $b^{'} \in U(b)$. Then there exists $x \in H$ such that $a^2x \cdot b^{'2} \in H$. Therefore

$$\begin{aligned}a^{'2}(a^2x \cdot b^{'2}) \cdot b^2 &= b^2(a^2x \cdot b^{'2}) \cdot a^{'2} = (a^2x \cdot b^2b^{'2})a^{'2} = (b^{'2}b^2 \cdot xa^2)a^{'2} \\ &= (a^{'2} \cdot xa^2)(b^{'2}b^2) = (x \cdot a^{'2}a^2)(b^{'2}b^2) \in H.\end{aligned}$$

$(vi) \Longrightarrow (v)$: It is similar to the proof of $(v) \Longrightarrow (iv)$.

Again, we can show that $(vi) \Longleftrightarrow (vii) \Longleftrightarrow (viii)$.

$(viii) \Longrightarrow (ix)$: Let $a^{'} \in U(a)$ and $b^{'} \in U(b)$. Then there exists $x \in H$ such that $b^{'2}x \cdot a^2 \in H$. Since $(a^{'}(bb^{'} \cdot x) \cdot a)(b^2 b^{'2}), ((b^{'2}x)a^2)(b^{'}b \cdot aa^{'}) \in H$, then we have

$$\begin{aligned}(a^{'}(bb^{'} \cdot x) \cdot a)(b^2 b^{'2}) \cdot a^2 &= (b^2 \cdot (a^{'}(bb^{'} \cdot x) \cdot a)b^{'2})a^2 = (a^2 \cdot (a^{'}(bb^{'} \cdot x) \cdot a)b^{'2})b^2 \\ &= b^2 \cdot ((a^{'}(bb^{'} \cdot x) \cdot a)b^{'2})a^2 = b^2 \cdot (a^2 b^{'2})(a^{'}(bb^{'} \cdot x) \cdot a) \\ &= b^2 \cdot (a^2 b^{'2})((bb^{'} \cdot x)a^{'} \cdot a) = b^2 \cdot (a^2 b^{'2})((aa^{'})(bb^{'} \cdot x)) \\ &= b^2 \cdot (a^2 b^{'2})((x \cdot bb^{'})(a^{'}a)) = b^2 \cdot (a^2 b^{'2})((a^{'}a \cdot bb^{'})x) \\ &= b^2 \cdot (a^2(a^{'}a \cdot bb^{'}))(b^{'2}x) = b^2 \cdot ((b^{'}b \cdot aa^{'})a^2)(b^{'2}x) \\ &= b^2 \cdot ((b^{'2}x)a^2)(b^{'}b \cdot aa^{'}).\end{aligned}$$

$(ix) \Longrightarrow (x)$: Let $a^{'} \in U(a)$ and $b^{'} \in U(b)$. By assumption there exist $x, y \in H$ such that $xa^2 = b^2 y$. We know that $(bb^{'})(a^{'}x \cdot a), (aa^{'} \cdot b^{'}b)y \in H$. Thus

$$\begin{aligned}a^2 \cdot (bb^{'})(a^{'}x \cdot a) &= a^2 \cdot (bb^{'})(xa^{'} \cdot a) = a^2 \cdot (bb^{'})(aa^{'} \cdot x) = a^2 \cdot (aa^{'})(bb^{'} \cdot x) \\ &= (aa^{'}) \cdot a^2(bb^{'} \cdot x) = (aa^{'})(bb^{'} \cdot a^2 x) = (aa^{'})(xa^2 \cdot b^{'}b) \\ &= (aa^{'})(b^2 y \cdot b^{'}b) = (b^2 y)(aa^{'} \cdot b^{'}b) = (aa^{'} \cdot b^{'}b)y \cdot b^2.\end{aligned}$$

$(x) \Longrightarrow (xi)$: Suppose that $a^2 x = yb^2$ for some $x, y \in H$. Then

$$\begin{aligned}y^2 x^2 \cdot a^2 &= a^2 x^2 \cdot y^2 = x^2 a^2 \cdot y^2 = (a^2 x \cdot x)y^2 = (yb^2 \cdot x)y^2 \\ &= y^2 x \cdot yb^2 = b^2 y \cdot xy^2 = (xy^2 \cdot y)b^2,\end{aligned}$$

which implies that $Ha^2 \cap Hb^2 \neq \emptyset$.

$(xi) \Longrightarrow (v)$: Suppose that $Ha^2 \cap Hb^2 \neq \emptyset$. Let $xa^2 = yb^2$ for some $x, y \in H$ and $a^{'} \in U(a)$, $b^{'} \in U(b)$. Then

$$(aa^{'})(xa^2 \cdot a^{'2}) = (aa^{'})(a^{'2}a^2 \cdot x) = (aa^{'}) \cdot (a^{'}a \cdot a^{'}a)x \in H.$$

It follows that

$$\begin{aligned}a^2((aa^{'})(xa^2 \cdot a^{'2})) \cdot b^{'2} &= a^2((aa^{'})(yb^2 \cdot a^{'2})) \cdot b^{'2} = (aa^{'})(a^2(yb^2 \cdot a^{'2})) \cdot b^{'2} \\ &= (aa^{'})(yb^2 \cdot a^2 a^{'2}) \cdot b^{'2} = (aa^{'})((yb^2)(aa^{'} \cdot aa^{'})) \cdot b^{'2} \\ &= (aa^{'})((aa^{'} \cdot aa^{'})(b^2 y)) \cdot b^{'2} = b^{'2}(aa^{'} \cdot b^2 y) \cdot (aa^{'}) \\ &= (aa^{'})(b^{'2} \cdot b^2 y) \cdot (aa^{'}) = (aa^{'})(yb^2 \cdot b^{'2}) \cdot (aa^{'}) \\ &= (aa^{'})(b^{'2}b^2 \cdot y) \cdot (aa^{'}) \in H.\end{aligned}$$

□

Corollary 2.3. *Let S be a partially inverse $\mathcal{AG}^{**}$-groupoid with $H \in P^*$ and $a, b \in S$. Then $a\Omega_H b$ if and only if one of the equivalent conditions in Theorem **2**.4 holds.*

Theorem 2.5. *Let S be a partially inverse $\mathcal{AG}^{**}$-groupoid. Then the relation $\tau = \{(a, b) \in S \times S \ / \ E(a) = E(b)\}$ is a unitary congruence on S.*

Proof. Clearly, τ is an equivalence relation. We shall show that τ is compatible. Let $a, b, c \in S$ be such that $a\tau b$. Suppose that $e \in E$ with $e \sim ac$. Then there exists $x \in S$ such that $x \cdot ac = e$. By Lemma **2**.4, we have

$$e = (ac)^2 e \cdot x^2 = (a^2 c^2 \cdot e)x^2 = x^2 e \cdot a^2 c^2 = (aa)(x^2 e \cdot c^2) = (x^2 e \cdot c^2)a \cdot a.$$

Hence $e \sim a$. Since $E(a) = E(b)$, $e \sim b$ and therefore there exists $y \in S$ such that $yb = e$. Thus $(yb)(x \cdot ac) = e$. Now by using Lemma **2**.4, we have

$$\begin{aligned} e &= (x \cdot ac)^2 e \cdot (yb)^2 = (x^2 \cdot a^2c^2)e \cdot y^2b^2 = (x^2 \cdot c^2a^2)e \cdot y^2b^2 \\ &= (c^2 \cdot x^2a^2)e \cdot y^2b^2 = (e \cdot x^2a^2)c^2 \cdot y^2b^2 = b^2y^2 \cdot c^2(e \cdot x^2a^2) \\ &= b^2c^2 \cdot y^2(e \cdot x^2a^2) = (bc)(bc) \cdot y^2(e \cdot x^2a^2) = (y^2(e \cdot x^2a^2) \cdot bc)(bc), \end{aligned}$$

which implies that $e \sim bc$. Thus $E(ac) \subseteq E(bc)$. Similarly we can show that $E(bc) \subseteq E(ac)$. Hence $E(ac) = E(bc)$, which follows that $ac\tau bc$. The similar argument will holds for $ca\tau cb$. Therefore τ is a congruence on S.

To show that $x\tau$ is a left identity of S/τ for some $x \in E$, let $a \in S$. If $e \sim a$, then there exists $x \in S$ such that $e = xa$. By Lemma **2**.4, we have

$$\begin{aligned} e &= a^2e \cdot x^2 = x^2 \cdot ea^2 = e \cdot x^2a^2 = e(xa \cdot xa) = e(ax \cdot ax) \\ &= (ax)(e \cdot ax) = (ax \cdot e)(xa). \end{aligned}$$

Thus we showed that $e \sim xa$ and so $E(a) \subseteq E(xa)$. Similarly we can show that $E(xa) \subseteq E(a)$. Hence $(a, xa) \in \tau$, and thus τ is a unitary congruence on S. □

2.3. **Idempotent separating congruences.** We investigate an idempotent separating congruence on a partially inverse $\mathcal{AG}^{**}$-groupoid S concerning centralizer H_ϑ of H in S, where $H_\vartheta = \{x \in S \ / \ xh = hx \text{ for all } h \in H\}$.

Definition 2.7. *A congruence σ on an $\mathcal{AG}^{**}$-groupoid is called idempotent separating if each σ-class contains at most one idempotent.*

Theorem 2.6. *Let S be a partially inverse $\mathcal{AG}^{**}$-groupoid and $H \in P$. Then the relation $\lambda = \{(a,b) \in S \times S \ / $ for each $a' \in U(a)$, there exists $b' \in U(b)$ (for all $b' \in U(b)$, there exists $a' \in U(a)$) such that $a'x \cdot a = b'x \cdot b$ and $a' \cdot xa = b' \cdot xb$ for all $x \in H\}$ is an idempotent separating congruence on S.*

Proof. Obviously λ is reflexive and symmetric. Take any $a, b, c \in S$ with $a\lambda b$ and $b\lambda c$. Let $a' \in U(a)$. Then there exists $b' \in U(b)$ such that $a'x \cdot a = b'x \cdot b$ and $a' \cdot xa = b' \cdot xb$ for all $x \in H$. Since $b\lambda c$ and $b' \in U(b)$, then there exists $c' \in U(c)$ such that $b'x \cdot b = c'x \cdot c$ and $b' \cdot xb = c' \cdot xc$ for all $x \in H$. Thus $a'x \cdot a = c'x \cdot c$ and $a' \cdot xa = c' \cdot xc$ for all $x \in H$. Similarly we can show that for all $c' \in U(c)$, there exists $a' \in U(a)$ such that $a'x \cdot a = c'x \cdot c$ and $a' \cdot xa = c' \cdot xc$ for all $x \in H$. Thus we showed that λ is transitive.

Let $a, b, c \in S$ with $a\lambda b$ and let $a' \in U(a)$. Then there exists $b' \in U(b)$ such that $a'x \cdot a = b'x \cdot b$ and $a' \cdot xa = b' \cdot xb$ for all $x \in H$. Let $c' \in U(c)$. Then by $(ac)' \in U(ac)$, $(bc)' \in U(bc)$ and for all $x \in H$, we get

$$\begin{aligned} &((ac)' \cdot x)(ac) = (ac \cdot x)(a'c') = (xc \cdot a)(a'c') = (c'a')(a \cdot xc) = (c'a')(a \cdot cx) \\ &= (c'a')(c \cdot ax) = (c'c)(a' \cdot ax) = (cc')(a' \cdot ax) = (ca')(c' \cdot ax) \\ &= (ax \cdot c')(a'c) = (ax \cdot a')(c'c) = c' \cdot (ax \cdot a')c = c' \cdot (a'x \cdot a)c \\ &= c' \cdot (b'x \cdot b)c = c' \cdot (bx \cdot b')c = (bx \cdot b')(c'c) = (bx \cdot c')(b'c) \\ &= (cb')(c' \cdot bx) = (cc')(b' \cdot bx) = (c'c)(b' \cdot bx) = (c'b')(c \cdot bx) \\ &= (c'b')(b \cdot cx) = (c'b')(b \cdot xc) = (xc \cdot b)(b'c') = (bc \cdot x)(b'c') = ((bc)' \cdot x)(bc), \end{aligned}$$

and

$$\begin{aligned}(ac)^{'}(x\cdot ac) &= (a^{'}c^{'})(x\cdot ac) = (a^{'}x)(c^{'}\cdot ac) = (xa^{'})(c^{'}\cdot ac) = (xc^{'})(a^{'}\cdot ac)\\ &= a^{'}(xc^{'}\cdot ac) = a^{'}(xa\cdot c^{'}c) = a^{'}(ax\cdot c^{'}c) = a^{'}\cdot(c^{'}c\cdot x)a\\ &= b^{'}\cdot(c^{'}c\cdot x)b = b^{'}(bx\cdot c^{'}c) = b^{'}(xb\cdot c^{'}c) = b^{'}(xc^{'}\cdot bc)\\ &= (xc^{'})(b^{'}\cdot bc) = (xb^{'})(c^{'}\cdot bc) = (b^{'}x)(c^{'}\cdot bc) = (b^{'}c^{'})(x\cdot bc) = (bc)^{'}(x\cdot bc),\end{aligned}$$

which shows that $ac\lambda bc$ and therefore λ is right compatible. Similarly we can show that λ is left compatible as well. Hence λ is a congruence relation on S.

Let $e, f \in E$ such that $e\lambda f$. Then for any $e \in U(e)$, there exists some $f \in U(f)$ such that $ex\cdot e = f^{'}x\cdot f$ for all $x \in H$. As H is full, $e \in H$ and $e = ee\cdot e = f^{'}e\cdot f$, we have

$$ef = (f^{'}e\cdot f)f = ff\cdot f^{'}e = f\cdot f^{'}e = f^{'}\cdot fe = f^{'}\cdot ef = fe\cdot f^{'} = f^{'}e\cdot f = e.$$

Now if $f \in U(f)$, then there exists $e^{'} \in U(e)$ such that $fx\cdot f = e^{'}x\cdot e$ for all $x \in H$. Since $f \in H$,then we have $f = ff\cdot f = e^{'}f\cdot e$. It follows that

$$ef = e(e^{'}f\cdot e) = e(ef\cdot e^{'}) = ef\cdot ee^{'} = e^{'}e\cdot fe = e^{'}f\cdot ee = e^{'}f\cdot e = f.$$

Thus $e = f$, and so λ is an idempotent separating congruence on S. □

Theorem 2.7. *Let S be a partially inverse $\mathcal{AG}^{**}$-groupoid with $H \in P$and let μ be the relation given by $\mu = \{(a,b) \in S\times S$ / for each $a^{'} \in U(a)$, there exists $b^{'} \in U(b)$ (for all $b^{'} \in U(b)$, there exists $a^{'} \in U(a)$) such that $a^{'}a = b^{'}b$, $a^{'}b \in H_\vartheta$ for all $x \in H\}$. If H is commutative, then μ is an idempotent separating congruence on S.*

Proof. Let $a, b \in S$ with $a\lambda b$ and let $a^{'} \in U(a)$. Then there exists $b^{'} \in U(b)$ such that $a^{'}x\cdot a = b^{'}x\cdot b$ and $a^{'}\cdot xa = b^{'}\cdot xb$ for all $x \in H$. Also note that $a^{'}a \in E \subseteq H$. Thus it follows that

$$\begin{aligned}a^{'}a &= (a^{'}a\cdot a^{'})a = aa^{'}\cdot a^{'}a = a^{'}(aa^{'}\cdot a) = b^{'}(aa^{'}\cdot b) = aa^{'}\cdot b^{'}b = bb^{'}\cdot a^{'}a\\ &= (a^{'}a\cdot b^{'})b = b^{''}b,\end{aligned}$$

where by Lemma **2**.3 (i), $a^{'}a\cdot b^{'} \in U(b)$. For any $h \in H$, we have

$$\begin{aligned}a^{'}b\cdot h &= (a^{'}a\cdot a^{'})b\cdot h = (hb)(a^{'}a\cdot a^{'}) = (h\cdot a^{'}a)(ba^{'}) = (a^{'}a\cdot h)(ba^{'})\\ &= (a^{'}a\cdot b)(ha^{'}) = (a^{'}a\cdot b)(a^{'}h) = (a^{'}a\cdot a^{'})(bh) = (a^{'}a\cdot a^{'})(hb)\\ &= h\cdot(a^{'}a\cdot a^{'})b = h\cdot a^{'}b.\end{aligned}$$

Therefore $a^{'}b \in H_\vartheta$. Similarly we can prove it for the second case as well. Hence $\lambda \subseteq \mu$.

Now we further suppose that $a, b \in S$ such that $a\mu b$ and $a^{'} \in U(a)$. Then there exists $b^{'} \in U(b)$ such that $a^{'}a = b^{'}b$, $a^{'}b \in H_\vartheta$ for all $x \in H$. By Lemma **2**.3 (ii), $b^{''} = (b^{'}\cdot a^{'}a)(a^{'}a) \in U(b)$. Indeed, if $x \in H$, then

$$\begin{aligned}a^{'}x\cdot a &= (a^{'}a\cdot a^{'})x\cdot a = (b^{'}b\cdot a^{'})x\cdot a = (a^{'}b\cdot b^{'})x\cdot a = (xb^{'}\cdot a^{'}b)a\\ &= (b^{'}x\cdot a^{'}b)a = (a^{'}b\cdot x)b^{'}\cdot a = (x\cdot a^{'}b)b^{'}\cdot a = (ab^{'})(x\cdot a^{'}b)\\ &= (ab^{'})(a^{'}\cdot xb) = (xb\cdot a^{'})(b^{'}a) = (b^{'}a\cdot a^{'})(xb) = (bx)(a^{'}\cdot b^{'}a)\\ &= (a^{'}\cdot b^{'}a)x\cdot b = (x\cdot b^{'}a)a^{'}\cdot b = (b^{'}\cdot xa)a^{'}\cdot b = (a^{'}\cdot xa)b^{'}\cdot b\\ &= (x\cdot a^{'}a)b^{'}\cdot b = (a^{'}a\cdot x)b^{'}\cdot b = (b^{'}x\cdot a^{'}a)\cdot b = (xb^{'}\cdot a^{'}a)\cdot b\\ &= (a^{'}a\cdot b^{'})x\cdot b = ((a^{'}a\cdot a^{'})a\cdot b^{'})x\cdot b = ((b^{'}a)(a^{'}a\cdot a^{'})\cdot x)b\\ &= ((b^{'}\cdot a^{'}a)(a^{'}a)\cdot x)b = b^{''}x\cdot b,\end{aligned}$$

and since $aa' \in E \subseteq H$, then

$$\begin{aligned} a' \cdot xa &= (a'a \cdot a')(xa) = (xa \cdot a')(a'a) = (a'a \cdot x)(a'a) = (aa' \cdot x)(a'a) \\ &= (xa' \cdot a)(a'a) = (a'x \cdot a)(a'a) = (a'a)(a'x \cdot a) = (a \cdot a'x)(aa') \\ &= (a' \cdot ax)(aa') = (aa')(a' \cdot ax) = (bb')(a' \cdot xa) = (bb')(x \cdot a'a) \\ &= (bx)(b' \cdot a'a) = (a'a \cdot b')(xb) = ((a'a \cdot a')a \cdot b')(xb) \\ &= (b'a)(a'a \cdot a') \cdot xb = (a'a)(b'a \cdot a') \cdot xb = (a'a)(a'a \cdot b') \cdot xb \\ &= (b' \cdot a'a)(a'a) \cdot xb = b'' \cdot xb. \end{aligned}$$

Similarly we can easily show that for every $b' \in U(b)$, there exists $a' \in U(a)$ such that $a'x \cdot a = b''x \cdot b$ and $a' \cdot xa = b'' \cdot xb$, for all $x \in H$. It follows that $a\lambda b$ and hence $\mu \subseteq \lambda$. Thus it was shown that μ is an idempotent separating congruence on S. □

3. Maximal anti-separative decomposition of locally associative $\mathcal{AG}^{**}$-groupoids

Recall that an $\mathcal{AG}$-groupoid S is called a locally associative $\mathcal{AG}$-groupoid if $a \cdot aa = aa \cdot a$, for all $a \in S$ [23].

Definition 3.1. *A locally associative $\mathcal{AG}^{**}$-groupoid is an $\mathcal{AG}^{**}$-groupoid S satisfying an identity $a \cdot aa = aa \cdot a$, for all $a \in S$.*

It is easy to note that every locally associative $\mathcal{AG}^{**}$-groupoid is a partially inverse $\mathcal{AG}^{**}$-groupoid, but the converse implication is not true in general which can be followed from Example **2**.1.

The following basic facts will be used frequently without mention in the sequel and can be found in [23, 22, 24].

Lemma 3.1. *Every locally associative $\mathcal{AG}^{**}$-groupoid S has associative powers, that is, $aa^n = a^n a$, $\forall a \in S$ and $n \in N$.*

Lemma 3.2. *In an $\mathcal{AG}^{**}$-groupoid S, $a^m a^n = a^{m+n}$, $\forall a \in S$ and $m, n \in N$.*

Lemma 3.3. *In a locally associative $\mathcal{AG}^{**}$-groupoid S, $(a^m)^n = a^{mn}$, $\forall a \in S$ and $m, n \in N$.*

Lemma 3.4. *If S is a locally associative $\mathcal{AG}^{**}$-groupoid and $a, b \in S$, then $(ab)^n = a^n b^n$ for any $n \geq 1$ and $(ab)^n = b^n a^n$ for any $n \geq 2$.*

Note that $a^{n-1}a = ((((aa)a)a)...a)a$ and $aa^{n-1} = a((((aa)a)a)...a)$.

Lemma 3.5. *Let S be a locally associative $\mathcal{AG}^{**}$-groupoid. Then $a^n = a^{n-1}a = aa^{n-1}$, $\forall a \in S$ and $\forall n > 1$.*

Lemma 3.6. *If S is a locally associative $\mathcal{AG}^{**}$-groupoid and $a, b \in S$, then $a^n b^m = b^m a^n$ for $m, n > 1$.*

Let define a relation τ as $a\tau b \iff ab^n = b^{n+1}$ and $ba^n = a^{n+1}$, $\forall a, b \in S$ and $n \in N$.

Lemma 3.7. *The relation τ on a locally associative $\mathcal{AG}^{**}$-groupoid S is a congruence relation.*

Proof. Clearly τ is reflexive and symmetric. For transitivity, let $a\tau b$ and $b\tau c$. Then there exist positive integers m, n such that $ab^n = b^{n+1}$, $ba^n = a^{n+1}$ and $bc^m = c^{m+1}$, $cb^m = b^{m+1}$. Let $k = (n+1)(m+1) - 1$, that is, $k = n(m+1) + m$. Then

$$\begin{aligned} ac^k &= ac^{n(m+1)+m} = a \cdot c^{n(m+1)}c^m = a \cdot (c^{m+1})^n c^m = a \cdot (bc^m)^n c^m \\ &= a \cdot (b^n c^{mn})c^m = b^n c^{mn} \cdot ac^m = b^n a \cdot c^{mn}c^m = c^m c^{mn} \cdot ab^n \\ &= c^m c^{mn} \cdot b^{n+1} = c^m c^{mn} \cdot b^n b = bb^n \cdot c^{mn}c^m = b^{n+1}c^{m(n+1)} \\ &= (bc^m)^{n+1} = (c^{m+1})^{n+1} = c^{(m+1)(n+1)} = c^{k+1}. \end{aligned}$$

Similarly, we can show that $ca^k = a^{k+1}$. Thus τ is an equivalence relation. To show that τ is compatible, assume that $a\tau b$ and let $c \in S$. Then $(ac)(bc)^n = ac \cdot b^n c^n = ab^n \cdot cc^n = b^{n+1}c^{n+1} = (bc)^{n+1}$.

Similarly, we can show that $(bc)(ac)^n = (ac)^{n+1}$. Hence τ is a congruence relation on S. □

Definition 3.2. *A congruence σ is said to be anti-separative congruence in S, if $ab\sigma a^2$ and $ba\sigma b^2$ implies that $a\sigma b$.*

Theorem 3.1. *Let S be a locally associative $\mathcal{AG}^{**}$-groupoid. Then S/τ is a maximal anti-separative commutative image of S.*

Proof. It is easy to see that if $ab^m = b^{m+1}$ and $ba^n = a^{n+1}$ for $a, b \in S$ and positive integers m, n such that $n > m$, then $a\tau b$. Let $a, b \in S$ such that $ab\tau a^2$ and $ba\tau b^2$. Then by definition of τ, there exist positive integers m and n such that $(ab)(a^2)^m = (a^2)^{m+1}$, $a^2(ab)^m = (ab)^{m+1}$, and $(ba)(b^2)^n = (b^2)^{n+1}$, $b^2(ba)^n = (ba)^{n+1}$. It follows that

$$\begin{aligned} ba^{2m+1} &= b \cdot a^{2m}a = a^{2m} \cdot ba = a^m a^m \cdot ba = ab \cdot a^m a^m = ab \cdot a^{2m} \\ &= (ab)(a^2)^m = (a^2)^{m+1} = a^{2m+2}, \end{aligned}$$

and

$$\begin{aligned} ab^{2n+1} &= a \cdot b^{2n}b = b^{2n} \cdot ab = b^n b^n \cdot ab = ba \cdot b^n b^n = ba \cdot b^{2n} \\ &= (ba)(b^2)^n = (b^2)^{n+1} = b^{2n+2}. \end{aligned}$$

which implies that $a\tau b$. Thus τ is anti-separative, and hence S/τ is anti-separative. We now show that τ is contained in every anti-separative congruence relation ξ on S. Let $a\tau b$ so that there exists a positive integer n such that, $ab^n = b^{n+1}$ and $ba^n = a^{n+1}$.

We need to show that $a\xi b$, where ξ is any anti-separative congruence on S. Let k be a positive integer such that, $ab^k \xi b^{k+1}$ and $ba^k \xi a^{k+1}$.

Suppose that $k > 2$, then $(ab^{k-1})^2 = ab^{k-1} \cdot ab^{k-1} = aa \cdot b^{k-1}b^{k-1} = a^2 b^{2k-2}$,

$$\begin{aligned} a^2 b^{2k-2} &= aa \cdot b^{k-2}b^k = ab^{k-2} \cdot ab^k \xi ab^{k-2} \cdot b^{k+1} \\ &= ab^{k-2} \cdot b^k b = ab^k \cdot b^{k-2}b = ab^k \cdot b^{k-1}, \end{aligned}$$

and from above, it follows that

$$a^2 b^{2k-2} \xi ab^k \cdot b^{k-1} = b^{k-1}b^k \cdot a = b^k b^{k-1} \cdot a = ab^{k-1} \cdot b^k.$$

Thus $(ab^{k-1})^2 \xi ab^k \cdot b^{k-1}$. Since $ab^k \xi b^{k+1}$ implies that $ab^k \cdot b^{k-1} \xi b^{k+1} \cdot b^{k-1}$. Hence $(ab^{k-1})^2 \xi (b^k)^2$. It further implies that $(ab^{k-1})^2 \xi a^2 b^{2k-2} = b^{2k-2}a^2 \xi (b^k)^2$.

Thus it was shown that $ab^{k-1} \xi b^k$. Similarly, we can show that $ba^{k-1} \xi a^k$.

By induction down from k, it follows that for $k = 1$, $ab\xi b^2$ and $ba\xi a^2$. Also it is easy to see that τ is commutative. Hence by using anti-separativity and commutativity, it follows that S/τ is a maximal anti-separative commutative image of S. □

3.1. Maximum idempotent separating congruences on a completely N-inverse $\mathcal{AG}^{**}$-groupoid.

In this section, we introduce the concept of a completely N-inverse $\mathcal{AG}^{**}$-groupoid. We study some properties of a completely N-inverse $\mathcal{AG}^{**}$-groupoid. Also, a maximum idempotent separating congruence on a completely N-inverse $\mathcal{AG}^{**}$-groupoid is studied.

A completely N-inverse $\mathcal{AG}^{**}$-groupoid S is a locally associative $\mathcal{AG}^{**}$-groupoid satisfying the identity $xa^n = a^n x$, where x is a unique inverse of a^n, that is, $a^n = a^n x \cdot a^n, x = xa^n \cdot x$, for all $a \in S$.

Example 3.1. *Let $S = \{x, y, z\}$ be an $\mathcal{AG}$-groupoid defined in the following multiplication table:*

$\cdot$	x	y	z
x	z	y	z
y	x	z	z
z	z	z	z

Then S is a locally associative $\mathcal{AG}^{**}$-groupoid. It can be verified that S is a completely N-inverse $\mathcal{AG}^{**}$-groupoid.

Note that $a^n = a^n x \cdot a^n$ implies that a^n is a regular element of a completely N-inverse $\mathcal{AG}^{**}$-groupoid. It is easy to see that xa^n is idempotent in S.

Recall that if S is a locally associative $\mathcal{AG}^{**}$-groupoid, then $a^n = a^{n-1}a = aa^{n-1}$ for all $a \in S$ and $n > 1$, and $(ab)^n = a^n b^n$ for all $a, b \in S$ and $n \geq 1$. These basic facts are obviously valid in a completely N-inverse $\mathcal{AG}^{**}$-groupoid and will be used without mention in the sequel.

By convention, if we write xa^0 or a^0x, for some $x \in S$, then we mean x.

Lemma 3.8. *Let S be a completely N-inverse $\mathcal{AG}^{**}$-groupoid and $a \in S$. Then for any $n \in \mathbb{N}$ such that a^n is regular:*

(i) $|V(a^n)| = 1$, that is, a^n has a unique inverse that we denote by a^{-n};

(ii) $aa^{-n} \cdot a^{n-1}$, $a \cdot a^{-n}a^{n-1} \in E$;

(iii) $af \cdot a^{-n}a^{n-1} \in E$, for all $f \in E$;

(iv) If $a, b \in S$ such that a^n and b^m are regular for some m, $n \in \mathbb{N}$, then $a^n b^m$ is regular with inverse $a^{-n}b^{-m}$;

(v) $(ab)^{-n} = a^{-n}b^{-n}$, for all $a, b \in S$.

Proof. Let a, $b \in S$ and m, $n \in \mathbb{N}$ such that a^n and b^m are regular.

(i) : Let $y, z \in V(a^n)$, then

$$\begin{aligned} y &= ya^n \cdot y = y(a^n z \cdot a^n) \cdot y = (a^n z \cdot ya^n)y = (za^n \cdot ya^n)y = (ya^n \cdot a^n)z \cdot y \\ &= (yz)(a^n y \cdot a^n) = yz \cdot a^n = (yz)(a^n z \cdot a^n) = (y \cdot a^n z)(za^n) = (a^n \cdot yz)(a^n z) \\ &= (za^n)(yz \cdot a^n) = (yz \cdot a^n)a^n \cdot z = (a^n a^n \cdot yz)z = (a^n y \cdot a^n z)z = (a^n y \cdot za^n)z \\ &= z(a^n y \cdot a^n) \cdot z = za^n \cdot z = z. \end{aligned}$$

(ii) : Let $n > 2$ and suppose that a^{-n} is the inverse of a^n, then

$$\begin{aligned} &(aa^{-n} \cdot a^{n-1})(aa^{-n} \cdot a^{n-1}) = (a^{n-1} \cdot aa^{-n})(a^{n-1} \cdot aa^{-n}) \\ &= (a^{n-2}a \cdot aa^{-n})(a^{n-1} \cdot aa^{-n}) = (a^{-n}a \cdot aa^{n-2})(a^{n-1} \cdot aa^{-n}) \\ &= (a^{-n}a \cdot a^{n-1})(a^{n-2}a \cdot aa^{-n}) = (a^{n-1}a \cdot a^{-n})(a^{-n}a \cdot aa^{n-2}) \\ &= (a^n a^{-n})(a^{-n}a \cdot a^{n-1}) = (a^{n-1} \cdot a^{-n}a)(a^{-n}a^n) \\ &= (a^{-n}a^n \cdot a^{-n}a)a^{n-1} = (a^{-n}a^{-n} \cdot a^n a)a^{n-1}. \end{aligned}$$

Since $a^n a = a^{n-1}a \cdot a = aa \cdot a^{n-2}a = aa^{n-2} \cdot aa = a(aa^{n-2} \cdot a) = aa^n$, then it follows from above that,

$$\begin{aligned} &(aa^{-n} \cdot a^{n-1})(aa^{-n} \cdot a^{n-1}) = (a^{-n}a^{-n} \cdot aa^n)a^{n-1} = a(a^{-n}a^{-n} \cdot a^n) \cdot a^{n-1} \\ &= a(a^n a^{-n} \cdot a^{-n}) \cdot a^{n-1} = a(a^{-n}a^n \cdot a^{-n}) \cdot a^{n-1} = aa^{-n} \cdot a^{n-1}. \end{aligned}$$

Thus it was shown that $aa^{-n} \cdot a^{n-1} \in E$. Similarly, we can show that $a \cdot a^{-n}a^{n-1} \in E$.

(iii) : Let $f \in E$ and $n > 2$. Then

$$\begin{aligned} &(af \cdot a^{-n}a^{n-1})(af \cdot a^{-n}a^{n-1}) = (aa^{-n} \cdot fa^{n-1})(af \cdot a^{-n}a^{n-1}) \\ &= (aa^{-n})(ff \cdot a^{n-2}a) \cdot (af \cdot a^{-n}a^{n-1}) = (aa^{-n})(aa^{n-2} \cdot ff) \cdot (af \cdot a^{-n}a^{n-1}) \\ &= (aa^{-n} \cdot a^{n-1}f)(af \cdot a^{-n}a^{n-1}) = (aa^{n-1} \cdot a^{-n}f)(a^{n-1}a^{-n} \cdot fa) \\ &= (a^n \cdot a^{-n}f)(a^{n-1}a^{-n} \cdot fa) = (a^n \cdot a^{n-1}a^{-n})(a^{-n}f \cdot fa). \end{aligned}$$

Since

$$\begin{aligned} a^{-n}f &= (a^{-n}a^n \cdot a^{-n})f = f(a^{-n} \cdot a^{-n}a^n) = a^{-n}(f \cdot a^{-n}a^n) = a^{-n}(a^n a^{-n} \cdot f) \\ &= a^{-n}(a^{-n}a^n \cdot f) = a^{-n}(fa^n \cdot a^{-n}) = a^{-n}(a^n f \cdot a^{-n}) = a^n f \cdot a^{-n}a^{-n} \\ &= a^{-n}a^{-n} \cdot fa^n = f(a^{-n}a^{-n} \cdot a^n) = f(a^n a^{-n} \cdot a^{-n}) = f(a^{-n}a^n \cdot a^{-n}) \\ &= fa^{-n}, \end{aligned}$$

then it follows that,

$$\begin{aligned} &(af \cdot a^{-n}a^{n-1})(af \cdot a^{-n}a^{n-1}) = (a^n \cdot a^{n-1}a^{-n})(a^{-n}f \cdot fa) \\ &= (a^n \cdot a^{n-1}a^{-n})(fa^{-n} \cdot fa) = (a^n \cdot a^{n-1}a^{-n})(f \cdot a^{-n}a) \\ &= (a^{-n}a \cdot f)(a^{n-1}a^{-n} \cdot a^n) = (fa \cdot a^{-n})(a^n a^{-n} \cdot a^{n-2}a) \\ &= (fa \cdot a^{-n})(aa^{n-2} \cdot a^{-n}a^n) = (fa \cdot a^{-n})(a^{n-1} \cdot a^n a^{-n}) \\ &= (fa \cdot a^{n-1})(a^{-n} \cdot a^{-n}a^n) = (a^{-n}a^n \cdot a^{-n})(a^{n-1} \cdot fa) \\ &= a^{-n}(a^{n-1} \cdot fa) = (a^{n-2}a)(a^{-n} \cdot fa) = (fa \cdot a^{-n})a^{n-1} = (a^{n-1}a^{-n})(fa) \\ &= (af)(a^{-n}a^{n-1}). \end{aligned}$$

Thus it is shown that $af \cdot a^{-n}a^{n-1} \in E$.

(iv) : Suppose that a^n and b^m are regular with inverses a^{-n} and b^{-m}, then

$$(a^n b^m \cdot a^{-n}b^{-m})(a^n b^m) = (a^n a^{-n} \cdot b^m b^{-m})(a^n b^m) = (a^n a^{-n} \cdot a^n)(b^m b^{-m} \cdot b^m) = a^n b^m.$$

Similarly, we can show that $(a^{-n}b^{-m} \cdot a^n b^m)(a^{-n}b^{-m}) = a^{-n}b^{-m}$. Moreover $a^n b^m \cdot a^{-n}b^{-m} = a^{-n}b^{-m} \cdot a^n b^m$, which is what we set out to prove.

(v) : It is simple. □

Note that if σ is a congruence relation on a completely N-inverse $\mathcal{AG}^{**}$-groupoid S, then S/σ is a completely N-inverse $\mathcal{AG}^{**}$-groupoid and if $(a, b) \in \sigma$, then $(a^{-n}, b^{-n}) \in \sigma$ and conversely.

Theorem 3.2. *Let S be a completely N-inverse $\mathcal{AG}^{**}$-groupoid. Then the relation ω defined as $a\omega b$ if and only if $a^n a^{-n} \cdot e = b^n b^{-n} \cdot e$ is a maximum idempotent separating congruence on S, where $n \geq 1$ and $e \in E$.*

Proof. It is easy to see that ω is an equivalence relation on S. Now let $a\omega b$, then $a^n a^{-n} \cdot e = b^n b^{-n} \cdot e$, for every $e \in E$. Thus

$$\begin{aligned} &(ac)^n(ac)^{-n} \cdot e = (ac)^{-n}(ac)^n \cdot e = e(ac)^n \cdot (ac)^{-n} = (ee \cdot (ac)^{n-1}(ac))(ac)^{-n} \\ &= ((ac)(ac)^{n-1} \cdot ee)(ac)^{-n} = (ac)^n e \cdot (ac)^{-n} = (ac)^{-n}e \cdot (ac) \\ &= (a^{-n}c^{-n} \cdot ee)(ac)^n = (a^{-n}e \cdot c^{-n}e)(a^n c^n) = (a^{-n}e \cdot a^n)(c^{-n}e \cdot c^n) \\ &= (a^n e \cdot a^{-n})(c^n e \cdot c^{-n}) = (ea^n \cdot a^{-n})(ec^n \cdot c^{-n}) \\ &= (a^{-n}a^n \cdot e)(c^{-n}c^n \cdot e) = (a^n a^{-n} \cdot e)(c^n c^{-n} \cdot e) = (b^n b^{-n} \cdot e)(c^n c^{-n} \cdot e) \\ &= (b^n b^{-n} \cdot c^n c^{-n})(ee) = (b^n c^n \cdot b^{-n}c^{-n})e = (bc)^n(bc)^{-n} \cdot e. \end{aligned}$$

Thus $ac\omega bc$. Similarly, we can show that $ca\omega cb$. Hence ω is a congruence relation on S.

Furthermore, let $e\omega f$ for $e, f \in E$. Then for every $g \in E$, $e^n e^{-n} \cdot g = f^n f^{-n} \cdot g$, and therefore $e^n = e^n e^{-n} \cdot e^n = f^n f^{-n} \cdot e^n$. Thus

$$\begin{aligned} ef &= e^n f = (f^n f^{-n} \cdot e^n)f = fe^n \cdot f^n f^{-n} = e^n f \cdot f^n f^{-n} = (f^n f^{-n} \cdot f)e^n \\ &= (f^{-n}f^n \cdot f)e^n = (ff^n \cdot f^{-n})e^n = f^n f^{-n} \cdot e^n = e^n = e, \end{aligned}$$

and similarly, we can show that $ef = f$, which shows that $e = f$. Hence ω is an idempotent separating congruence on S.

Now let ϖ be any other idempotent separating congruence on S. We shall show that $\varpi \subseteq \omega$. Let $x\varpi y$. Then clearly $x^n \varpi y^n$. Also $x^{-n}\varpi y^{-n}$, and therefore $x^{-n} \cdot ex^n = e \cdot x^{-n}x^n \varpi y^{-n} \cdot ey^n = e \cdot y^{-n}y^n$, but both $e \cdot x^{-n}x^n$ and $e \cdot y^{-n}y^n$ are idempotents, and so it follows that $e \cdot x^{-n}x^n = e \cdot y^{-n}y^n$, which implies that $x^n x^{-n} \cdot e = y^n y^{-n} \cdot e$. Thus $x\omega y$. Hence ω is maximum. □

Theorem 3.3. *Let S be a completely N-inverse $\mathcal{AG}^{**}$-groupoid and let ω be the idempotent separating congruence on S. Then for $n \geq 1$, $(a, b) \in \omega$ if and only if $a^{-n}a^n = b^{-n}b^n$ and $a^n b^{-n} \in H_\vartheta$.*

Proof. Let $(a, b) \in \omega$. Then $a^n a^{-n} \cdot e = b^n b^{-n} \cdot e$ for all $e \in E$. From

$$(a^n a^{-n} \cdot e)(a^n a^{-n} \cdot e) = (b^n b^{-n} \cdot e)(b^n b^{-n} \cdot e),$$

it follows that $(a^n a^{-n} \cdot a^n a^{-n})e = (b^n b^{-n} \cdot b^n b^{-n})e$. Thus

$$\begin{aligned} a^{-n}a^n &= (a^{-n}a^n \cdot a^{-n})(a^n a^{-n} \cdot a^n) = (a^n a^{-n} \cdot a^n a^{-n})(a^{-n}a^n) \\ &= (b^n b^{-n} \cdot b^n b^{-n})(a^{-n}a^n) = b^n b^{-n} \cdot a^{-n}a^n. \end{aligned}$$

Similarly, we can show that $b^{-n}b^n = b^n b^{-n} \cdot a^{-n}a^n$. Thus we have shown that $a^{-n}a^n = b^{-n}b^n$.

Since $a^n a^{-n} \cdot e = b^n b^{-n} \cdot e$, for all $e \in E$, then $a^n(a^n a^{-n} \cdot e) \cdot b^{-n} = a^n(b^n b^{-n} \cdot e) \cdot b^{-n}$, whence

$$\begin{aligned} a^n(a^n a^{-n} \cdot e) \cdot b^{-n} &= (a^n a^{-n} \cdot a^n e) \cdot b^{-n} = (b^{-n} \cdot a^n e)(a^n a^{-n}) = (b^{-n} \cdot ea^n)(a^n a^{-n}) \\ &= (e \cdot b^{-n}a^n)(a^n a^{-n}) = (a^n b^{-n} \cdot e)(a^n a^{-n}) = (a^n a^{-n} \cdot e)(a^n b^{-n}). \end{aligned}$$

Similarly, we can show that $a^n(b^n b^{-n} \cdot e) \cdot b^{-n} = (a^n b^{-n})(a^n a^{-n} \cdot e)$. Hence $a^n b^{-n} \in H_\vartheta$.

Conversely, let $a^{-n}a^n = b^{-n}b^n$ and $a^n b^{-n} \in H_\vartheta$. Then $e \cdot a^n b^{-n} = a^n b^{-n} \cdot e$, $\forall e \in E$, which further implies that $a^{-n}(e \cdot a^n b^{-n}) \cdot b^n = a^{-n}(a^n b^{-n} \cdot e) \cdot b^n$. Thus

$$\begin{aligned} a^{-n}(e \cdot a^n b^{-n}) \cdot b^n &= b^n(e \cdot a^n b^{-n}) \cdot a^{-n} = e(b^n \cdot a^n b^{-n}) \cdot a^{-n} = e(a^n \cdot b^n b^{-n}) \cdot a^{-n} \\ &= (ee)(a^n \cdot a^n a^{-n}) \cdot a^{-n} = (a^n a^{-n} \cdot a^n)(ee) \cdot a^{-n} = a^n e \cdot a^{-n} \\ &= ea^n \cdot a^{-n} = a^{-n}a^n \cdot e = a^n a^{-n} \cdot e, \end{aligned}$$

$$\begin{aligned} a^{-n}(a^n b^{-n} \cdot e) \cdot b^n &= a^{-n}(a^n b^{-n} \cdot ee) \cdot b^n = a^{-n}(a^n e \cdot b^{-n}e) \cdot b^n \\ &= (a^n e)(a^{-n} \cdot b^{-n}e) \cdot b^n = (b^{-n}e \cdot a^{-n})(ea^n) \cdot b^n \\ &= (b^{-n}e \cdot e)(a^{-n}a^n) \cdot b^n = (eb^{-n})(a^{-n}a^n) \cdot b^n \\ &= (eb^{-n})(b^{-n}b^n) \cdot b^n = (b^n \cdot b^{-n}b^n)(eb^{-n}) \\ &= (b^{-n}e)(b^{-n}b^n \cdot b^n) = (b^{-n}e)(b^n b^{-n} \cdot b^n) \\ &= b^{-n}e \cdot b^n = b^n e \cdot b^{-n} = eb^n \cdot b^{-n} \\ &= b^{-n}b^n \cdot e = b^n b^{-n} \cdot e. \end{aligned}$$

It follows that $a^n a^{-n} \cdot e = b^n b^{-n} \cdot e$. Hence $a\omega b$. □

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