U.P.B. Sci. Bull., Series A, Vol. 75, Iss. 4, 2013 ISSN 1223-7027

OPERADIC STRUCTURES ON THE CONNES-KREIMER HOPF ALGEBRA

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In several papers connected with Yukawa Theory and Schwinger- Dyson equations, D. Kreimer defined two shuffle-type products on a decorated Connes-Kreimer Hopf algebra. We prove that one of these different products interacts with the coalgebra structure in an operadic manner: there are compatibilities among them and co-products which are not of classical Hopf-type. They can be described using the notion of generalized bialgebra. We identify the (co)actions of several operads already studied in Hamiltonian Physics, on the decorated version of Connes-Kreimer Hopf algebra and its graded dual, the Grossman-Larson Hopf algebra.

Keywords: renormalization, shuffle algebra, decorated Connes-Kreimer Hopf algebras, operads, generalized bialgebras.

1. Introduction

The Connes-Kreimer Hopf algebra plays a major role in the combinatorics of perturbative renormalization. It has a universal property with respect to Hochschild cohomology ([4]. Thm.2, pag 34).It is the symmetric algebra of the free co-preLie algebra of one element ([19] section 5.7). The graded dual of the Connes-Kreimer Hopf algebra is isomorphic with the Grossman-Larson Hopf algebra [9], [10],[19]. Two new associative products were defined on this Hopf algebra of trees by D.Kreimer. The first one was defined in [1] Section 2 (the definition and the proof of associativity). It has the following recursive definitition: $t_1 * t_2 = B_+^{r(t_1)}\left(u\left(B_-\left(t_1\right)\right) * t_2\right) + B_+^{r(t_2)}\left(t_1 * u\left(B_-\left(t_2\right)\right)\right)$ where the map u si also defined recursively $u\left(\prod_{i=1}^{k} t_i\right) = t_1 * u\left(\prod_{i=2}^{k} t_i\right)$, $\prod$ is the regular commutative product. The $*$product was applied to massless Yukawa theory; in this case it is associative modulo certain quantities defined by the Feynmann diagrams. It is an open question if higher coherences laws are needed to describe this lack of associativity in Yukawa theory, which is a consequence of the non-associativity of a certain product on primitive 1P1 graphs. ([2]. Section 2).

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Kreimer defined a second, different product ([3], section 2.2 ;Theorem 2),[6](section 2.3;theorem 3),[7] (section 3.1))and established a connection between Dyson–Schwinger equations, which are quantum equation of motion usually unsolvable and Euler products. He gave an affirmative answer on the existence in QFT of similar relations involved in Riemann ζ function $:\sum_n \frac{1}{n^s} = \prod_{p\,\text{prime}} \frac{1}{1-p^{-s}}, \mathrm{Re}(s) > 1$ (Euler product decomposition). A stated non-obvious theorem (from [3]) and an open question (sect. 2.2 of [3]) are connected with $\vee$, the second Kreimer product.
There is no references about an interaction between these above mentioned products $*$ and $\vee$ and the already known Connes-Kreimer Hopf algebra structure.

2. The Connes –Kreimer Hopf algebra

The Connes-Kreimer Hopf algebra, as well as its vertex-decorated version and its role in Quantum Field Theory was defined and described in the following articles: [4],[5],[6],[9],[10]. We follow these articles to define several combinatorial objects and operators.

A *rooted tree t* is a connected and simply-connected set of oriented edges and vertices such that there is one distinguished vertex which has no incoming edge, the root of *t*; every edge connects two vertices and the *fertility* $f(v)$ of a vertex *v* is the number of edges outgoing from *v*.

The Connes-Kreimer Hopf algebra $\mathcal{H}_R$ is the free commutative algebra of polynomials over $\mathbb{Q}$ having indeterminates rooted trees. Any rooted tree *t* with root *r* yields $f(r)$ trees $t_1,\dots,t_{f(r)}$ which are the trees attached to *r*. The unit element of this algebra is 1, corresponding, as a rooted tree, to the empty set. We denote by B_- be the operator which removes the root *r* from a tree *t*: $B_- : t \to B_-(t) = t_1 t_2 \dots t_{f(r)}$ B_+ be the operation which maps a monomial of *n* rooted trees to a new rooted tree *t* which has a root *r* with fertility $f(r) = n$ which connects to the *n* roots of $t_1,\dots,t_n$ $B_+ : t_1 \dots t_n \to B_+\left(t_1 \dots t_n\right) = t$.
An elementary cut is a cut of a rooted tree at a single chosen edge.
An admissible cut is any assignment of elementary cuts to a rooted tree t such that any path from any vertex of the tree to the root has at most one elementary cut. The coalgebra structure is given by the counit $\overline{e} : \mathcal{A} \to \mathbb{Q}$ is $\overline{e}(X) = 0$ for any $X \neq 1; \overline{e}(1) = 1$. The comultiplication(coproduct) is an algebra map. The equations $\Delta(1) = 1 \otimes 1$, $\Delta(t_1 \dots t_n) = \Delta(t_1)\dots\Delta(t_n)$, $\Delta(t) = 1 \otimes t + (B_+ \otimes id)\left[\Delta(B_-(t))\right]$
define the coproduct on trees with *n* vertices iteratively .

$$\Delta(t) = 1 \otimes t + t \otimes 1 + \sum_{admisible\ cuts\ C\ of\ t} R^C(t) \otimes P^C(t) =: 1 \otimes t + t \otimes 1 + \Delta'(t)$$

$m[(S \otimes id)\Delta(t)] = \bar{e}(t) = 0 = \sum S(t_{(1)})t_{(2)}$;we used Sweedler's notation $\Delta(t) =: \sum t_{(1)} \otimes t_{(2)}$ and *id* is the identity map $\mathcal{H}_R \to \mathcal{H}_R$.

2.1. The shuffle product on a tensor algebra.

Let A be a vector space over the complex numbers. Over the tensor algebra $T(A) = \mathbb{C} \oplus \bigoplus_{k=1}^{\infty} A^{\otimes K}$ there is a Hopf algebra structure, with comultiplication Δ

$\Delta(L_1 \otimes L_2 \otimes \ldots \otimes L_n) = \sum_{j=0}^{n} (L_1 \otimes L_2 \otimes \ldots \otimes L_j) \otimes (L_{j+1} \otimes \ldots \otimes L_n)$.The shuffle product $\blacktriangle$ between two homogenous elements is $(a_1 \otimes a_2 \otimes \ldots \otimes a_n)\blacktriangle(b_1 \otimes b_2 \otimes \ldots \otimes b_m) = \sum (c_1 \otimes c_2 \otimes \ldots \otimes c_{m+n})$, where the sum is over all $\binom{m+n}{n}$ ways to "shuffle" $a's$ among $b's$: The elements ***c***'s are equal to one of a's or b's ; the elements $a_1, a_2, \ldots, a_n$ will appear in the same order in $c's$. The same for $b\ 's$.

Example: $$(a \otimes b)\blacktriangle(x \otimes y) = x \otimes a \otimes b \otimes y + a \otimes x \otimes b \otimes y + a \otimes b \otimes x \otimes y + \\ + x \otimes y \otimes a \otimes b + x \otimes a \otimes y \otimes b + a \otimes x \otimes y \otimes b$$

2.2. The structure of the Connes-Kreimer Hopf algebra. Algebraic Operads

The importance of the decorated C-K Hopf algebra in the theory of Operads is mentioned in the articles: [18](Section 11) and [19] (Introd.,,Sections 5.7;6.6)

Theorem Let H(V) be the decorated Connes-Kreimer Hopf algebra, where the vertices of the trees are decorated by a basis of a finite-dimensional vector space V. Then H(V) is isomorphic as Hopf algebra with a shuffle Hopf algebra T(A), where A is a graded vector space **G**(V). This conclusion is based on the following fundamental results:

-the graded dual of H(V) is isomorphic with the universal enveloping algebra of a Lie algebra ([9] Prop.2.1. and [10] Prop 4.4).

-This Lie algebra is the underlying Lie algebra of the free preLie algebra over a basis of V; according to the results of [12] (Corol. 5.3) and [13](Theorem 3.3), it is the free Lie algebra generated by a basis of a vector space **G**(V), so its universal enveloping algebra is a tensor algebra. The dual of the tensor algebra is the shuffle Hopf algebra T(***A***=**G**(V)), isomorphic with the decorated Connes-Kreimer Hopf algebra, where the vertices of the trees are decorated by a basis of a finite-dimensional vector space V.

About *a basis of* the vector space **G**(V) not many things are known; **G**(V) is the free **G-algebra** generated by V over an operad **G** defined in [11], [12]. G is a sub-operad of the preLie operad and it was conjectured in [11] to be a free operad.

Definition 1: a preLie algebra is a vector space V together with a binary operation such that $(x*y)*z - x*(y*z) = (x*z)*y - x*(z*y)$ for any x,y,z.

In this case x*y –y*x defines a Lie bracket.

2.3 Definition ([15],[17],[20],[25]) A non-Σ operad *O* is a collection of sets *O*(*n*), $n \geq 1$ such that: there is a composition law $f: O(m) \otimes O(n_1) \otimes \ldots \otimes O(n_m) \to O(n_1 + \ldots + n_m)$ There is a unit $e \in O(1)$. $f(g; e, e, e, \ldots e) = g$ for any $g \in O(k)$. The composition law *f* is associative:

$$f\left[f\left(g; g_1, g_2, \ldots, g_n\right); r_1^1, r_2^1, \ldots, r_{x_1}^1, r_1^2, r_2^2, \ldots, r_{x_2}^2, \ldots, r_1^n, r_2^n, \ldots, r_{x_n}^n\right] =$$
$$= f\left(g; f\left(g_1; r_1^1, r_2^1, \ldots, r_{x_1}^1\right), \left(g_2; r_1^2, r_2^2, \ldots, r_{x_2}^2\right) \ldots \left(g_n; r_1^n, r_2^n, \ldots, r_{x_n}^n\right)\right).$$

A vector space V is an *O-algebra* if there is a morphism of operads between *O and End(V)*-the endomorphism operad of V. So each element of *O*(*n*) defines an algebraic operation $V^{\otimes n} \longrightarrow$ V, subject to the composition law above. Associative, Lie, Poisson algebras are all *O-algebras for* various operads.

2.4 The dual notion is that of a co-operad and a co-algebra over a co-operad. In the finite-dimensional case, taking the duals will “change the arrows”. The graded vector space C is an co-operad if and only if C^* is an operad. If D is a C-coalgebra, every $\delta \in C(n)$ define a cooperation from *D* to $D^{\otimes n}$.

Prim(D,r) $:= \left\{x \in D \mid \delta(x) = 0 \text{ for every } \delta \in C(n),\ n \geq r\right\}$. The primitive part of D, Prim(D)=Prim(D,1). A coalgebra D is connected if $D = \bigcup_{r \geq 1} \operatorname{Pr} im(D, r)$.

2.5 Definition ([17],[19],[20],[27]) A generalized bialgebra is a vector space $\mathcal{H}$ which is a *C*-coalgebra, an A-algebra and there are compatibility relations (also called distributivity) between operations and cooperations denoted by (Ω).$\mathcal{H}$ is called an $(C^c,(\Omega),\mathcal{A})$-algebra. These compatibility relations are formalized by the following axiom : (Ω) For every co-operation $\delta \in C(m)$ and every operation $\mu \in \mathcal{A}(n)$ there is the following equality of maps $\delta \circ \mu = \sum_i \left(\mu_1^i \otimes \ldots \otimes \mu_m^i\right) \circ \omega \circ \left(\delta_1^i \otimes \ldots \otimes \delta_m^i\right) : \mathrm{H}^{\otimes n} \longrightarrow \mathrm{H}^{\otimes m}$, where :

$$\begin{cases} \mu \in \mathcal{A}(n), \mu_1^i \in \mathcal{A}(k_1), \ldots, \mu_m^i \in \mathcal{A}(k_m), \\ \delta \in \mathcal{C}(n), \delta_1^i \in \mathcal{C}(k_1), \ldots, \delta_n^i \in \mathcal{C}(k_m), \\ k_1 + \ldots k_m = l_1 + \ldots + l_n = r, \\ \omega \in K[S_r]. \end{cases}$$

The axiom above is the generalization of the classical bialgebra case, where we have the Hopf compatibility condition : a product of two elements followed by co-multiplication is equal to the product, in a tensor algebra, of separate co-products (we can switch the order of operations and co-operations).

Our result is : the Kreimer * product is a part of a generalized bialgebra structure on H(V) for which there is a compatibility relation as above with respect to the regular Hopf-algebra co-product; we have to define several other operations and co-operations to be in the framework of the definition **2.5.** We conjecture that the second product is also part of a generalized bialgebra structure which satisfies the conditions of the following theorem. The interest in these structures came from the structure theorem to be given below

Theorem ([17]Theorem 2.5.1.). Let $\mathcal{H}$ be a generalized $(C^{\circ},(\Omega),\mathcal{A})$-bialgebra. Under additional hypotheses, the following statements are equivalent :

$\mathcal{H}$ is connected $\Leftrightarrow$ There is a bialgebra isomorphism $\mathcal{H} \cong U(\mathrm{Prim}\mathcal{H})$ $\Leftrightarrow$ $\mathcal{H}$ is cofree, i.e. $\mathcal{H} \cong C^c(\mathrm{Prim}\mathcal{H})$, isomorphism of connected coalgebras.

3. Shuffle products on the decorated Connes –Kreimer Hopf algebra H(V)

3.1 We define the following co-product **d:** $\mathrm{H(V)} \to \mathrm{H(V)} \otimes \mathrm{H(V)}$

$$d(t) = 1 \otimes t + \sum_{\text{super-elementary cuts } C \text{ of } t} R^C(t) \otimes T^C(t)$$

A super-elementary cut is an elementary cut such that the falling tree which does not contain the root is a single vertex or has the fertility greater than 1.

d has the following recursive definition: $\mathrm{d}\left(B_+(t)\right) = 1 \otimes B_+(t) + \left(B_+(t) \otimes id\right)\mathrm{d}(t)$

On the entire H(V), d is defined as a co-derivation: $\mathrm{d}(xy) = xy_1 \otimes y_2 + yx_1 \otimes x_2$

Contrary to the regular coproduct, d is not co-associative; d is a *co-preLie* operation: its dual operation in the graded dual of H(V) satisfies Definition 1.

Also, there are compatibility relations between **d** and Δ :
Lemma 3.2 Any linear combinations of **d** and Δ is a preLie co-product.
According to [9] and [10], the graded dual of H(V) is isomorphic to the Grossman-Larson Hopf algebra. $f: H_{GL} \to H_k^{gr}$ $f(t)(u) = (B_-(t), u) = (t, B_+(u))$ $(u_1, u_2) = (B_+(u_1), B_+(u_2))$ the inner product $(t_1, t_2) = |SG(t_1)| \delta_{t_1 t_2}$, where $|SG(t_1)|$ is the cardinality of the symmetry group of a tree. Based on this isomorphism, the relations between **d** and Δ are exactly the relations between the duals of these co-operations in H_{GL}, which are binary operations. Δ^* will be the associative product $\circ$ and d^* will be a pre-Lie operation denoted by $*$. Then for every x,y,z there is the following relation: $(x \circ y) * z - x \circ (y * z) = (x * z) \circ y - x * (z \circ y)$

Remark. Let $[\circ]$ and $[\cdot]$ be Lie brackets on a common vector space .One can then define $[a,b] := \alpha[a \circ b] + \beta[a \cdot b]$, for any $\alpha, \beta \in C$. The Jacobi identity for [,] $[[a,b],c] + [[b,c],a] + [[c,a],a] = 0$ is equivalent to:

$$[[a \circ b] \cdot c] + [[b \circ c] \cdot a] + [[c \circ a] \circ b] + [[a \circ b] \circ c] + [[b \cdot c] \circ a] + [[c \cdot a] \circ b] = 0.$$

Two pre-Lie products $\circ$ and $\cdot$ on V are compatible if any linear combination of the two products, $\alpha a \circ b + \beta a \cdot b$, is a pre-Lie product for any $\alpha, \beta \in K$. This property is equivalent to the condition that
$(a \circ b) \cdot c - a \circ (b \cdot c) + (a \cdot b) \circ c - a \cdot (b \circ c) = (a \circ c) \cdot b - a \circ (c \cdot b) + (a \cdot c) \circ b - a \cdot (c \circ b)$
for any $a, b, c \in V$. Both compatibility conditions, for preLie and for the associated Lie products are implied by the relation: $(x \circ y) * z - x \circ (y * z) = (x * z) \circ y - x * (z \circ y)$. This relation appears in [25] (page 52).
The algebraic operads which encode these compatible operations were recently studied in: [22].[23],[24].

3.2.1 On H(V) it was defined in [8] an associative flat-shuffle product $a \varpi b$ .The shuffle algebra T(V) is seen as subspace of H(V), monomials being the linear trees. $\rho: H(V) \to H(T) \subset H(V)$. ρ is defined inductively, being identity on linear (unbranched)trees and $\rho(B_+(u)) = B_+(\rho(u))$. $\rho(ab)$=shuffle product between $\rho(a)$ and $\rho(b) = a \varpi b$

For example: $\rho\left({}^{b}\mathrm{V}_{a}^{c}\right) = \rho(abc) + \rho(acb)$ = abc + acb.

3.2.2 We define an associative quasi-tensor product **x** on H(V): $(a \times b) \times c = a \times (b \times c)$

$a \times b = \rho(a) \to b$, where each monomial from $\rho(a)$ is glued (notation $\to$)on top of the tree b (we apply the linear combination of iterated Hochschild operators with the decorations provided by $\rho(a)$; $\Delta(a \times b) = a_1 \times 1 \otimes a_2 \times b + a \times b_1 \otimes b_2$

Theorem 3.3 $a * b = \rho(a_1 b) \to a_2 + \rho(b_1 a) \to b_2 = (a_1 b) \times a_2 + (b_1 a) \times b_2$,

where * is the Kreimer shuffle product and d(a)= $a_1 \otimes a_2$ is the Sweedler's notation for the d-coproduct.

Corollary: Δ and * have compatibility relations of the type described by Definition 2.5. H(V) is a generalized bialgebra, the relevant co-operations are Δ and **d.** The operations are the regular product, *, $\times$, ϖ. We recall several relations satisfied by the binary operations: $(ab) * c = a * b * c$, $\left(ab\right) \times c = \left(a \varpi b\right) \times c = \left(a * b\right) \times c$

Proof. We prove by induction over the number of vertices of involved trees that the two products $t_1 * t_2 = B_+^{r(t_1)}\left(u\left(B_-\left(t_1\right)\right) * t_2\right) + B_+^{r(t_2)}\left(t_1 * u\left(B_-\left(t_2\right)\right)\right)$ and

$\rho(a_1 b) \to a_2 + \rho(b_1 a) \to b_2 = (a_1 b) \times a_2 + (b_1 a) \times b_2$ are equal.

Let $a = B_+(xyz..)$ and $b = B_+(\alpha\beta S..)$ be two trees.

$\mathrm{d}(B_+(xyz)) = 1 \otimes a + \left(B_+ \otimes id\right)\mathrm{d}(xyz) = 1 \otimes a + B_+^{r(a)}\left(\left(xyz\right)_1\right) \otimes (xyz)_2$

(Thm.3.3) $\Leftrightarrow B_+^{r(a)}(x * y * z * b) + B_+^{r(b)}(\alpha * \beta * S * a) =$

$B_+^{r(a)}\left((xyz)_1\right) b \to (xuz)_2 + B_+^{r(b)}\left((\alpha\beta S)_1\right) a \to (\alpha\beta S)_2$

$= B_+^{r(a)}\left[(xyz)_1 b \to (xyz)_2 + B_+^{r(b)}\left((\alpha\beta S)_1\right) xyz \to (\alpha\beta S)_2\right] +$

$B_+^{r(b)}\left[(\alpha\beta S)_1 a \to (\alpha\beta S)_2 + B_+^{r(a)}\left((xyz)_1\right) \alpha\beta S \to (xyz)_2\right]$

$\Leftrightarrow x * y * z * b = (xyz)_1 b \to (xyz)_2 + B_+^{r(b)}\left((\alpha\beta S)_1\right) xyz \to (\alpha\beta S)_2$

(which is true according to the induction hypothesis applied to x,y,z,b)

$= \sum x_1 yzb \to x_2 + B_+^{r(b)}\left((\alpha\beta S)_1\right) x * y * z \to (\alpha\beta S)_2$

$d(x * y * z) = (\rho \otimes id)\sum x_1 yz \otimes x_2 = (\rho \otimes id) d(xyz)$

We esentially used that d is a co-derivation $\mathrm{d}(xy) = xy_1 \otimes y_2 + yx_1 \otimes x_2$ and the following property of Kreimer's *: $a \varpi b = \rho(a * b) \Rightarrow$ (ab) $\times$T=$(a \varpi b) \times$T=$(a * b) \times$T which can be easily proved by induction: $a * b$ and $a \varpi b$ have the same image under ρ. $\Delta(a * b) = \Delta(a_1 b \to a_2) + \Delta(b_1 a \to b_2)$

$= \Delta a_1 b \to (| \otimes a_2) + \Delta a b_1 \to (| \otimes b_2) + (a_1 b \otimes |) \to \Delta a_2 + (a b_2 \otimes |) \to \Delta b_2$

So, the Kreimer * product , Δ and d have a distributivity relation in the sense of Defn. 2.5. $\Delta(a * b)$ can be computed by first applying co-operations and after that $\times$ and ϖ.

Definition 1. A dendriform algebra is a vector space V, together with two binary operations $\wedge$ and * such that: * is associative, *= $\vee$ + $\wedge$, and :
$(x \wedge y) \wedge z = x \wedge (y * z)$ $(x \vee y) \vee z = x \vee (y \wedge z)$ $(x * y) \vee z = x \vee (y \vee z)$
Definition 2. A Zinbiel algebra is a vector space V and one binary operation such that $(x > y) > z = x > (y > z) + x > (z > y) = x > (y * z)$, where $x * y = x > y + y > x$
We can decompose Kreimer * product into 4 different binary operations: $\nearrow, \swarrow, \searrow, \nwarrow$ $a \searrow b = b \nwarrow a$ $a \nearrow b = b \swarrow a$
$a * b = a_1 b \to a_2 + b_1 a \to b_2 = B_+^{r(a)}\left(u\left(B_-(a)\right) * b\right) + B_+^{r(b)}\left(u\left(B_-(b)\right) * a\right)$
$a > b = a_1 b \to a_2 = a \nearrow b + a \searrow b$ =sum of trees with root r(a) + sum of trees with root r(b)
$a < b = b_1 a \to b_2 = a \nwarrow b + a \swarrow b$ =sum of trees with root r(a) + sum of trees with root r(b)
$a \wedge b = B_+^{r(a)}\left(u\left(B_-(a)\right) * b\right) = \; = a \nearrow b + a \nwarrow b$
$a \vee b = B_+^{r(b)}\left(u\left(B_-(b)\right) * a\right) = \; = a \searrow b + a \swarrow b$

One can verify that ($\wedge$, $\vee$ and *) forms a dendriform algebra on H(V).
Also, x>(y$*$z) + y> (x$*$z) = (x $*$ y) > z. This is the relation which quantify the fact H(V) is *not* a Zinbiel algebra. The relations above were checked following the program initiated on [16], on quadri-algebras: associative algebras where the associative product is splitted into four different products satisfying several axioms.

6. Conclusions

a)Remark 4.3, pag. 11 [26](„Any quasi-shuffle algebra associated with a commutative algebra V is isomorphic as a Hopf algebra with the shuffle Hopf algebra **T(V)** “) and Theorem 3.3 has the following corollary: the shuffle-type algebra structures defined in [1] section 2 equations (5) and (6) are isomorphic for a commutative algebra V. It is not a Hopf algebra isomorphism because of the intricate operadic structures involved.
b) We conjecture a good triple of operads in the sense of [17], Section 2.5.6., which would imply a structure or rigidity theorem for the V-decorated Connes-Kreimer Hopf algebras(see also [18] sect. 11), the two operads being given by (one associative,the other one preLie operation satisfying $(x \circ y) * z - x \circ (y * z) = (x * z) \circ y - x * (z \circ y)$) and an operad of four operations (*, X, ϖ and a commutative associative product, satisfying the axioms of Section 3.2.)

c) Exotic algebraic structures encoded by specific operads already appeared in Theoretical Physics: [14] preLie algebras, vertex algebras, [23] Poisson algebras, quantization [22] Bi-Hamiltonial operad and Bi-Hamiltonian structures [28], [25] and [15]. This is also the case of the first Kreimer * product, where various operadic structures met in: [13],[16],[17],[19],[21]-[27] appeared.

Aknowledgements
This paper is suported by the Sectorial Operational Programme Human Resources Development (SOP HRD), financed from the European Social Fund and by the Romanian Government under the contract number SOP HRD/89/1.5/S/62988

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