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MAXIMAL INVARIANT SUBSPACES AND OBSERVABILITY OF MULTIDIMENSIONAL SYSTEMS

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The Geometric Approach techniques are extended to a class of multidimensional (rD, ≥ 2) linear systems. An algorithm is provided for determining the maximal subspace which is invariant with respect to r commuting drift matrices and is included in a given subspace. When this subspace is the kernel of the output matrix, this algorithm determines the subspace of the unobservable states of the system. A Matlab program is presented, which implements the algorithm and computes an orthonormal basis of this maximal invariant subspace.

Keywords: Observability, maximal invariant subspaces, multidimensional systems, discrete-time systems.

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1. Introduction

Since its birth before the middle of the 20th century, System theory was first studied as control theory by mathematicians and engineers. Now it is applied in engineering, economics, finance, political science, military science, sociology, biology, ecology etc. Different approaches have been developed during its history, which include the frequency domain techniques, the state space representations of Bellman and Kalman (around 1960), the polynomial approach (Popov and Rosenbrock, in the early 1970s). At the same time, the geometric direction enriched the field of System theory with new concepts and techniques and provided simpler and elegant solutions for many important problems such as controller synthesis, decoupling, pole-assignment, minimality, duality, etc. The history of the Geometric Approach started with the papers of Basile and Marro (see [3]) and was developed by Wonham and Morse [13], Silverman, Hautus, Willems et al. The landmark of this approach is the concept of invariance of a subspace with respect to a linear transformation.

In the past four decades a lot of published paper and books have been designed to the theory of multidimensional (rD) systems, which become a distinct and important branch of the systems theory. The reasons for the increasing interest in this domain are on one side the richness in potential application fields (signal processing,

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image processing, computer tomography, gravity and magnetic field mapping, seismology etc.) and on the other side the richness and significance of the theoretical approaches due to the fact the domain of rD systems needs a specific approach, since many aspects of the 1D systems do not generalize and there are many rD systems phenomena which have no 1D systems counterparts. Various state space 2D discrete-time models have been proposed in literature by Roesser [12], Fornasini-Marchesini [4], Attasi [1] etc.

The concept of observability, which is fundamental in control theory, originates both from classical automata theory and from basic linear systems. It was introduced for 1D systems by Kalman in [6] (see also Popov [8]), being imposed by engineering problems. The characterization of the observability of 2D discrete-time systems was provided by Attasi [1] with the use of the 2D observability matrix.

The aim of this paper is to extend the Geometric Approach techniques to multidimensional Attasi type rD linear systems by providing an algorithm to determine the maximal subspace which is invariant with respect to r commutative drift matrices and is included in the kernel of the output matrix, and to emphasize its connection with the concept of observability. The dual algorithm which determines the minimal invariant subspace that includes the image of the input matrix is described in [11].

In Section 2 the state space representation of these systems is given and the formulas of the state as well as of the input-output map are obtained on the basis of a variation-of-parameters formula.

Section 3 presents the notion of completely observable systems. Using the observability matrix, one obtains necessary and sufficient conditions of observability for LTI systems. The space X_{uo} of all unobservable states of a system is characterized as the maximal invariant subspace with respect to the drift matrices which is included in the kernel of the the output matrix. In the case of time-variable systems (which also have many applications, see [5]), observability can be studied by using the rank of a suitable observability Gramian(see [10]).

Section 4 provides an algorithm which computes the maximal invariant subspace with respect to $r \geq 2$ commuting matrices which is included in a given subspace. This algorithm is a generalization of the 1D method of G. Marro (see [7]).

Section 5 provides a Matlab program that implements the algorithm presented in Section 4 which computes an orthonormal basis of the maximal invariant subspace. An example illustrates the behavior of the provided method.

We shall use the following notations: $\overline{r} := \{1, 2, \ldots, r\}$ where $r \in \mathbf{N}^*$. A function $x(t_1, \ldots, t_r)$ is denoted by $x(t)$, where $t = (t_1, \ldots, t_r)$, and $t_i \in \mathbf{Z}^+$ are the discrete variables. By $s \leq t,\ s, t \in \mathbf{Z}^r$ we mean $s_i \leq t_i\ \forall i \in \bar{r}$ and $s < t$ means $s \leq t,\ s \neq t$. For $t^0, t^1 \in \mathbf{Z}^r,\ t^0 < t^1$, we denote by $[t^0, t^1]$ the set (the rD interval) $[t^0, t^1] = \prod_{i=1}^{r} [t_i^0, t_i^1]$.

For $\delta = \{i_1, \ldots, i_l\}$ a subset of $\overline{r}$, we consider the notations $|\delta| := l$, $\tilde{\delta} := \overline{r} \setminus \delta$ and $|\emptyset| := 0$; for $i \in \overline{r}$, $\tilde{i} := \overline{r} \setminus \{i\}$. The notation $\delta \subset \overline{r}$ means that δ is $\emptyset$ or δ is a subset of $\overline{r}$ and $\delta \neq \overline{r}$. For $\delta = \{i_1, \ldots, i_l\}$ the operator σ_δ is defined by

$\sigma_\delta x(t) = x(t+e_\delta)$ where $e_\delta = e_{i_1} + \cdots + e_{i_l}$, $e_j = (\underbrace{0,\ldots,0}_{j-1},1,0,\ldots,0) \in \mathbf{Z}^r$; when $\delta = \overline{r}$ we denote $\sigma_\delta = \sigma$, hence $\sigma x(t_1,t_2,\ldots,t_r) = x(t_1+1,t_2+1,\ldots,t_r+1)$.

If A_i, $i \in \overline{m}$ is a family of matrices, then $\prod_{i\in\emptyset} A_i = I$.

2. The state space representation of the discrete-time rD model

Consider a field $\mathbf{K}$ and the time set $T = \mathbf{Z}^r$, $r \in \mathbf{N}$, $r \geq 2$. The $\mathbf{K}$-spaces $X = \mathbf{K}^n$, $U = \mathbf{K}^m$ and $Y = \mathbf{K}^p$ are called respectively the *state space*, the *input space* and the *output space*.

Definition 2.1. An *rD discrete-time linear system* is an ensemble $\Sigma = (A_1,\ldots,A_r; B; C; D)$ where $A_i(t_i)$, $i \in \overline{r}$ are commuting $n \times n$ matrices over $\mathbf{K}$ $\forall t_i \in \mathbf{Z}$, $\forall i \in \overline{r}$ and $B(t)$, $C(t)$, $D(t)$ are respectively $n \times m$, $p \times n$ and $p \times m$ matrices over $\mathbf{K}$; the following equations are called respectively the *state equation* and the *output equation:*

$$\sigma x(t) = \sum_{\delta\subset\overline{r}} (-1)^{r-|\delta|-1} \left(\prod_{i\in\tilde{\delta}} A_i(t_i)\right) \sigma_\delta x(t) + B(t)u(t), \tag{1}$$

$$y(t) = C(t)x(t) + D(t)u(t), \tag{2}$$

where $x(t) = x(t_1,\ldots,t_r) \in X$ is the *state*, $u(t) \in U$ is the *input* and $y(t) \in Y$ is the *output* of the system Σ.

For any set $\delta = \{i_1,\ldots,i_l\} \subset \overline{r}$ and for $t_i \in \mathbf{Z}^+$, $i \in \delta$, $t_i^0 \in \mathbf{Z}^+$, $i \in \tilde{\delta}$, we use the notation

$$x(t_\delta, t^0_{\tilde{\delta}}) := x(t_1^0,\ldots,t^0_{i_1-1},t_{i_1},t^0_{i_1+1},\ldots,t^0_{i_l-1},t_{i_l},t^0_{i_l+1},\ldots,t_r^0).$$

One defines the *discrete-time fundamental matrix* $F_i(t_i,t_i^0)$ of the matrix $A_i(t_i)$, for $t_i, t_i^0 \in \mathbf{Z}$, by

$$F_i(t_i,t_i^0) = \begin{cases} A_i(t_i-1)A_i(t_i-2)\cdots A_i(t_i^0) & \text{for} \quad t_i > t_i^0 \\ I_n & \text{for} \quad t_i = t_i^0. \end{cases}$$

$F_i(t_i,t_i^0)$ is the unique matrix solution of the following initial value problem

$$Y(t_i+1,t_i^0) = A_i(t_i)Y(t_i,t_i^0),\ Y(t_i^0,t_i^0) = I.$$

If A_i is a constant matrix then $F_i(t_i,t_i^0) = A_i^{t_i-t_i^0}$.

Since the matrices $A_i(t_i)$, $i \in \overline{r}$ commute, their discrete-time fundamental matrices $F_i(t_i,t_i^0)$ commute too.

Definition 2.2. *The vector $x_0 \in \mathbf{K}^n$ is called an initial state of the system Σ if*

$$x(t_\delta, t^0_{\tilde{\delta}}) = \left(\prod_{i\in\delta} F_i(t_i, t_i^0)\right) x_0 \tag{3}$$

for any $\delta \subset \overline{r}$; equalities (3) *are called initial conditions of* Σ.

By adapting the proof of Proposition 2.3 from [9] one obtains a discrete-type variation-of-parameters formula for rD systems, which can be used to obtain the following result.

Proposition 2.1. *The state of the system Σ determined by the control $u \in U$ and the initial state x_0 is*

$$x(t) = \left(\prod_{i=1}^{r} F_i(t_i, t_i^0)\right) x_0 + \sum_{l_1=t_1^0}^{t_1-1} \cdots \sum_{l_r=t_r^0}^{t_r-1} \left(\prod_{i=1}^{r} F_i(t_i, l_i+1)\right) B(l)u(l); \tag{4}$$

here $t = (t_1, \ldots, t_r) \in T$ and $l = (l_1, \ldots, l_r) \in T$.

By replacing the state $x(t)$ (4) in the output equation (2) one obtains

Proposition 2.2. *The input-output map of the system Σ is*

$$y(t) = C(t)\left(\prod_{i=1}^{r} F_i(t_i, t_i^0)\right) x_0 + \tag{5}$$

$$+\sum_{l_1=t_1^0}^{t_1-1} \cdots \sum_{l_r=t_r^0}^{t_r-1} C(t) \left(\prod_{i=1}^{r} F_i(t_i, l_i+1)\right) B(l)u(l) + D(t)u(t).$$

3. Observability of time-invariant discrete-time rD systems

The system $\Sigma = (A_1, \ldots, A_r, B, C, D)$ is said to be *time-invariant* (or *stationary*) if all its matrices are constant. In this case we can consider the time set $T = \mathbf{N}^r$ and the initial moment $t^0 = 0 \in T$.

The fundamental matrices become $F_j(t_j, 0; t_{\tilde{j}}) = A_j^{t_j}$, $j \in \overline{r}$ and the input-output map (5) can be written in the form

$$y(t) = C\left(\prod_{j=1}^{r} A_j^{t_j}\right) x^0 + \sum_{l_1=0}^{t_1-1} \cdots \sum_{l_r=0}^{t_r-1} C \left(\prod_{j=1}^{r} A_j^{t_j-l_j-1}\right) Bu(l) + Du(t). \tag{6}$$

Definition 3.1. A state $x \in \mathbf{K}^n$ is said to be *unobservable* if, for any input $u(t)$, the initial states $x^0 = x$ and $x^0 = 0$ produce the same output $y(t)$, $\forall t \in T$.

Proposition 3.1. *The state $x \in \mathbf{K}^n$ is unobservable if and only if*

$$C\left(\prod_{j=1}^{r} A_j^{t_j}\right) x = 0, \forall t_j \in \mathbf{N}, \forall j \in \overline{r}. \tag{7}$$

Proof. Let us denote by $y_x(t)$ and $y_0(t)$ the outputs produced by the initial state $x^0 = x$ and $x^0 = 0$ respectively, for an arbitrary input $u(t)$. We obtain by (6) that $y_x(t) - y_0(t) = 0 \; \forall k \in T$ if and only if (7) holds. □

In the sequel we will consider the system Σ reduced to the ensemble $\Sigma = (A_1, \ldots, A_r; C)$.

Proposition 3.2. *The state $x \in \mathbf{K}^n$ is unobservable if and only if*

$$C\left(\prod_{j=1}^{r} A_j^{t_j}\right) x = 0, \forall t_j \in \mathbf{N},\; t_j \leq n-1, \forall j \in \overline{r}. \tag{8}$$

Proof. Obviously, (7) implies (8). Conversely, assume that (8) holds. Consider the characteristic polynomial of the matrix $A_j,\; j \in \overline{r}$:

$$\det(sI - A_j) = s^n + a_{j,n-1}s^{n-1} + \cdots + a_{j,1}s + a_{j,0}.$$

By Hamilton-Cayley theorem, $p_j(A_j) = 0$, hence

$$A_j^n = -a_{j,n-1}A_j^{n-1} - \cdots + a_{j,1}A_j - a_{j,0}I. \tag{9}$$

Recurrently, one obtains from (9) that for any $N \geq n$, A_j^N can be represented as a linear combination of the matrices $A_j^{n-1}, \ldots, A_j$ and I. It follows that for any $t = (t_1, \ldots, t_r) \in T$, $C\left(\prod_{j=1}^{r} A_j^{t_j}\right) x$ can be represented as a linear combination of the matrices $C\left(\prod_{j=1}^{r} A_j^{t_j}\right) x$, where $t_j \leq n-1, \forall j \in \overline{r}$, hence (8) implies (7). □

Definition 3.2. A subspace $\mathcal{V}$ of $\mathbf{K}^n$ is said to be $(A_1, \ldots, A_r)$-*invariant* if $A_j v \in \mathcal{V},\; \forall v \in \mathcal{V},\; \forall j \in \overline{r}$.

Let $\mathcal{C}$ be a proper subspace of $\mathbf{K}^n$. A subspace $\mathcal{V}$ of $\mathbf{K}^n$ is said to be $(A_1, \ldots, A_r; \mathcal{C})$-*invariant* if $\mathcal{V}$ is $(A_1, \ldots, A_r)$-invariant and it is included in $\mathcal{C}$. $\mathcal{V}$ is called *maximal* if, for any subspace $\widetilde{\mathcal{V}}$ which is $(A_1, \ldots, A_r)$-invariant and included in $\mathcal{C}$, $\widetilde{\mathcal{V}} \subset \mathcal{V}$.

Let us denote by X_{uo} the set of the unobservable states of Σ. Now we give a geometric characterization of the set of unobservable states of Σ.

Theorem 3.1. *X_{uo} is the maximal $(A_1,\dots,A_r;\mathcal{C})$-invariant subspace of $\mathbf{K}^n$, where $\mathcal{C}$=KerC.*

Proof. Let x be an unobservable state of Σ. Obviously, one obtains from (7), for $t_j=0,\ \forall j\in\overline{r}$, that $Cx=0$, hence $X_{uo}\subset \mathrm{Ker}C$.

For arbitrary $i\in\overline{r}$ and $t_j\in\mathbf{N},\ j\in\overline{r}$ one obtains from (7): $0=C(\prod_{\substack{j=1\\ j\neq i}}^{r}A_j^{t_j})A_i^{t_i+1}x=C(\prod_{j=1}^{r}A_j^{t_j})A_ix$, hence $A_ix\in X_{uo},\forall i\in\overline{r}$, i.e. X_{uo} is $(A_1,\dots,A_r;\mathcal{C})$-invariant.

Now, consider an arbitrary $(A_1,\dots,A_r;\mathcal{C})$-invariant subspace $\mathcal{V}$ of $\mathbf{K}^n$ and let v be an element of $\mathcal{V}$. Since $\mathcal{V}$ is $(A_1,\dots,A_r)$-invariant and it is included in Ker$C$ one obtains $(\prod_{j=1}^{r}A_j^{t_j})v\in\mathcal{V}$ and $C(\prod_{j=1}^{r}A_j^{t_j})v=0,\ \forall t_j\geq 0,\ \forall j\in\overline{r}$. By (7), $v\in X_{uo}$, hence $\mathcal{V}\subset X_{uo}$. □

Definition 3.3. The matrix

$$O_\Sigma=[C^T\ A_1^TC^T\dots(A_1^T)^{n-1}C^T\ A_2^TC^TA_1^TA_2^TC^T\dots(A_1^T)^{n-1}A_2^TC^T\dots \tag{10}$$
$$\dots(\prod_{i=1}^{r-1}(A_i^T)^{n-1})C^T\ A_r^T(\prod_{i=1}^{r-1}(A_i^T)^{n-1})C^T\dots(\prod_{i=1}^{r}(A_i^T)^{n-1})C^T]^T.$$

is called the *observability matrix* of the system Σ.

Definition 3.4. The system Σ is said to be *completely observable* if there is no unobservable state $x\neq 0$.

Theorem 3.2. *The system $\Sigma=(A_1,\dots,A_r,C)$ is completely observable if and only if $\{0\}$ is the greatest subspace of $\mathbf{K}^n$ which is $(A_1,\dots,A_r;C)$-invariant.*

Proof. This criterion is a consequence of theorem 3.1 and of the fact that the system $\Sigma=(A_1,\dots,A_r,C)$ is completely observable if and only if $X_{uo}=\{0\}$. □

One obtains by Proposition 3.2 the following result.

Proposition 3.3. *$X_{uo}=KerO_\Sigma$.*

Theorem 3.3. *The system $\Sigma=(A_1,\dots,A_r,C)$ is completely observable if and only if*

$$rank\, O_\Sigma=n. \tag{11}$$

Proof. This follows from Proposition 3.3 and the fact that $X_{uo}=\mathrm{Ker}O_\Sigma=\{0\}$ if and only if (11) holds. □

4. **Algorithm of minimal invariant subspaces**

Let $\mathcal{C}$ be a proper subspace of $\mathbf{K}^n$ and $A_1, \ldots, A_r \in \mathbf{K}^{n\times n}$ commuting matrices. Let us denote by $\max I(A_1, \ldots, A_r; \mathcal{C})$ the maximal $(A_1, \ldots, A_r)$-invariant subspace included in $\mathcal{C}$.

For a subspace $\mathcal{V}$ of $\mathbf{K}^n$, we consider the subspaces $(\prod_{i=1}^r A_i^{-k_i})\mathcal{V} = \{v \in \mathbf{K}^n | (\prod_{i=1}^r A_i^{k_i})v \in \mathcal{V}\}$, $k_i \in \mathbf{N}$, where $(\prod_{i=1}^r A_i^{-0})\mathcal{V} = \mathcal{V}$. If $v \in A_i^{-j}\mathcal{V}$, then $A_i v \in A_i^{-(j-1)}\mathcal{V}, \forall i \in \overline{r}, \forall j \geq 1$.

Proposition 4.1. *The maximal $(A_1, \ldots, A_r)$-invariant subspace included in $\mathcal{C}$ is*

$$maxI(A_1, \ldots, A_r; \mathcal{C}) = \bigcap_{k_1=0}^{\infty} \cdots \bigcap_{k_r=0}^{\infty} (\prod_{i=1}^{r} A_i^{-k_i})\mathcal{C}. \tag{12}$$

Proof. Let us denote by $\mathcal{V}_1$ the subspace $\bigcap_{k_1=0}^{\infty} \cdots \bigcap_{k_r=0}^{\infty} (\prod_{i=1}^r A_i^{-k_i})\mathcal{C}$. If $v \in \mathcal{V}_1$ then $v \in (\prod_{\substack{i=1\\ i\neq j}}^r A_i^{-k_i}) A_j^{-(k_j+1)}\mathcal{C}$, hence $A_j v \in (\prod_{i=1}^r A_i^{-k_i})\mathcal{C}$, $\forall k_i \in \mathbf{N}$, $j \in \overline{r}$. It follows that $A_j v \in \mathcal{V}_1$, $\forall j \in \overline{r}$ i.e. $\mathcal{V}_1$ is $(A_1, \ldots, A_r)$-invariant. We can write by (12) $\mathcal{V}_1 = \mathcal{C} \cap \bigcap_{k_1=0}^{\infty} \cdots \bigcap_{k_r=0}^{\infty} (\prod_{i=1}^r A_i^{-k_i})\mathcal{C}$ where $(k_1, \ldots, k_r) \neq (0, \ldots, 0)$, hence $\mathcal{V}_1$ is included in $\mathcal{C}$.

Now, let $\mathcal{V}$ be an $(A_1, \ldots, A_r)$-invariant subspace included in $\mathcal{C}$. Then, for any $v \in \mathcal{V}$, $(\prod_{i=1}^r A_i^{k_i})v \in \mathcal{V} \subset \mathcal{C}$, hence $v \in (\prod_{i=1}^r A_i^{-k_i})\mathcal{C}, \forall k_i \geq 0$, $\forall i \in \overline{r}$, which implies $v \in \mathcal{V}_1$. Therefore $\mathcal{V} \subset \mathcal{V}_1$, i.e. $\mathcal{V}_1$ is the maximal such subspace. □

Proposition 4.2. *The maximal $(A_1, \ldots, A_r)$-invariant subspace included in $\mathcal{C}$ is*

$$maxI(A_1, \ldots, A_r; \mathcal{C}) = \bigcap_{k_1=0}^{n-1} \cdots \bigcap_{k_r=0}^{n-1} (\prod_{i=1}^{r} A_i^{-k_i})\mathcal{C}. \tag{13}$$

Proof. Let us denote by $\mathcal{V}_2$ the subspace given in the right-hand member of (13). Obviously, by Proposition 4.1, $\mathcal{V}_1 \subset \mathcal{V}_2$, where $\mathcal{V}_1 = \max I(A_1, \ldots, A_r; \mathcal{C})$.

Now, for any $v \in \mathcal{V}_2$, $(\prod_{i=1}^r A_i^{k_i})v \in \mathcal{C}$, $\forall k_i \in \mathbf{N}$, $0 \leq k_i \leq n-1$, $\forall i \in \overline{r}$. Let $p_j(s) = \det(sI - A_j) = s^n + a_{n-1,j}s^{n-1} + \cdots + a_{1,j}s + a_{0,j}$ be the characteristic polynomial of the matrix A_j, $j \in \overline{r}$. By Hamilton-Cayley Theorem, $p_j(A_j) = 0_n$, hence

$$A_j^n = -a_{n-1,j}A_j^{n-1} - \cdots - a_{1,j}A_j - a_{0,j}I_n. \tag{14}$$

Then, for any vector $v \in \mathcal{V}_2$, $A_j^n v = -\sum_{l=0}^{n-1} a_{l,j}A_j^l v$. Since A_j are commutative matrices, we can premultiply this equality by $(\prod_{\substack{i=1\\ i\neq j}}^r A_i^{k_i})$ and we obtain $(\prod_{\substack{i=1\\ i\neq j}}^r A_i^{k_i})A_j^n v = -\sum_{i=0}^{n-1} a_{l,j}(\prod_{\substack{i=1\\ i\neq j}}^{r} A_i^{k_i})A_j^l v$, hence $(\prod_{\substack{i=1\\ i\neq j}}^r A_i^{k_i})A_j^n v \in \mathcal{C}$ since $(\prod_{\substack{i=1\\ i\neq j}}^r A_i^{k_i})A_j^l v \in \mathcal{C}$ for $0 \leq l \leq n-1$ and $\mathcal{C}$ is a subspace. Similarly, by postmultiplying (14) by $(\prod_{\substack{i=1\\ i\neq j}}^r A_i^{k_i})A_j^t v$, $t = 1, 2, \ldots$, one obtains recurrently that $(\prod_{\substack{i=1\\ i\neq j}}^r A_i^{k_i})A_j^{n+t} v \in \mathcal{C}$ and finally that $(\prod_{i=1}^r A_i^{k_i})v \in$

$\mathcal{C}$, $\forall k_i \geq 0$, hence $v \in \mathcal{V}_1$. It follows that $\mathcal{V}_2 \subset \mathcal{V}_1$, hence $\mathcal{V}_2 = \mathcal{V}_1 = \max I(A_1, \ldots, A_r; \mathcal{C})$. □

Algorithm 4.1.

Stage 1. Determine the sequence of subspaces $(S_{i_1,0,\ldots,0,0})_{0 \leq i_1 \leq n}$ of the space $X = \mathbf{K}^n$:

$$S_{0,0,\ldots,0,0} = \mathcal{C}; \tag{15}$$

$$S_{i_1,0,\ldots,0,0} = \mathcal{C} \cap A_1^{-1} S_{i_1-1,0,\ldots,0,0},\ i_1 = 1, \ldots, n; \tag{16}$$

Stage 2. Determine i_1^0, the first index in $\{0, 1, \ldots, n-1\}$ which verifies

$$S_{i_1^0+1,0,\ldots,0,0} = S_{i_1^0,0,\ldots,0,0}. \tag{17}$$

If $i_1^0 = n - 1$, then $\max I(A_1, \ldots, A_r; \mathcal{C}) = \{0\} \subset \mathbf{K}^n$. STOP

If $i_1^0 < n - 1$, put $j := 1$ and GO TO Stage 3.

Stage 3. Determine the sequence of subspaces $(S_{i_1^0,i_2^0,\ldots,i_j^0,i_{j+1},0,\ldots,0})_{0 \leq i_{j+1} \leq n}$ of the space $X = \mathbf{K}^n$: for $i_{j+1} = 1, 2, \ldots, n$,

$$S_{i_1^0,i_2^0,\ldots,i_j^0,i_{j+1},0,\ldots,0} = S_{i_1^0,i_2^0,\ldots,i_j^0,i_{j+1}-1,0,\ldots,0} \cap A_{j+1}^{-1} S_{i_1^0,i_2^0,\ldots,i_j^0,i_{j+1}-1,0,\ldots,0}. \tag{18}$$

Stage 4. Determine i_{j+1}^0, the first index i_{j+1} in $\{0, 1, \ldots, n-1\}$ which verifies

$$S_{i_1^0,i_2^0,\ldots,i_j^0,i_{j+1}^0+1,0,\ldots,0} = S_{i_1^0,i_2^0,\ldots,i_j^0,i_{j+1}^0,0,\ldots,0}. \tag{19}$$

If $i_{j+1}^0 = n - 1$ then $\max I(A_1, \ldots, A_r; \mathcal{C}) = \{0\} \subset \mathbf{K}^n$. STOP

If $i_{j+1}^0 < n - 1$ then GO TO Stage 5.

Stage 5. If $j < r - 1$ then put $j := j + 1$ and GO TO Stage 3.

If $j = r - 1$, then $\max I(A_1, \ldots, A_r; \mathcal{C}) = S_{i_1^0,i_2^0,\ldots,i_j^0,i_{j+1}^0,\ldots,i_r^0}$. STOP

The proof is based upon Propositions 4.1 and 4.2, but it is too long and it is omitted for lack of space.

5. **Matlab program**

The *Matlab* program presented below and based upon the algorithm above calculates the dimension and an orthonormal basis of the unobservable states space. The instructions make use of the m-functions *ints*, *invt* and *ker* included in the Geometric Approach toolbox published by G. Marro and G. Basile at
http://www3.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm;
this GA toolbox works with Matlab and Control System Toolbox.

More precisely, given the matrices A_1, A_2 .... A_r that commute and the matrix C, the next commands will compute and display the dimension of a basis and an orthonormal one in the space $S = \max\mathrm{I}(A_1, A_2, \ldots A_r; \mathcal{C})$, where $\mathcal{C} = \mathrm{Ker}C$. The matrices are loaded from the m-File *MatricesAC*, where A_1, A_2 .... A_r are stored in an $n \times n \times r-$ dimensional array A.

```
% begin m-file
% ints(A,B)=an orthonormal basis for Im(A) intersected with Im(B)
% invt(A,X)=an orthonormal basis for the inverse image of X through A
% ker(C) = an othhonormal basis in the kernel of C
MatricesAC;                 % for loading A and C
[n, ~, r] = size(A);        % A is nxnxr, its "pages" being A1...Ar
Osigma= [];                 % The observability matrix
Ctemp = C; for j = 1: r
    A_current = A(:, :, j); % successively A1, A2,..., Ar
    Osigma = obsv(A_current, Ctemp);
    Ctemp = Osigma;
end rk = rank(Osigma); if (rk == n) disp(strcat('The
(',int2str(r),'D)-system is completely observable.')) else S =
ker(C);
[n, dimUno] = size(S);      % will be the dim of the unobs. sp.
index = zeros(1, r);        % the index of the calculated subspace
for j = 1:r                 % loop for index position
    A_current = A(:, :, r-j+1);  % successively Ar, A(r-1),..., A1
for i= 1:n-1                % loop for index value
  S = ints(S, invt(A_current,S)); [n, m1] = size(S); index(j) = i;
  if (m1 == dimUno)  break;    else  dimUno = m1;   end
end if (dimUno == n) break; end end disp(['The dimension of the
unobservable space is ']) disp([ num2str(dimUno)]) disp('and an
orthonormal basis is:') disp(S) end
% end m-file
```

For example, given the matrices

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ -1 & -4 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, C = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix},$$

the m-File *MatricesAC* will be

```
A = [1 0 0; 0 1 0; 1 0 2]; A(:,:,2) =[2 4 0; 3 1 0; -1 -4 1];
A(:,:,3) = [1 0 0; 0 1 0; 0 0 3 ]; C= [1 -1 0];
```

and the above Matlab program will give the answers:

```
The dimension of the unobservable space is 1 and an orthonormal
basis is:
     0
     0
     1
```

while, with the same A_1, A_2, A_3 but with $C = [1 \ -1 \quad 1]$ the answer will be

```
The (3D)-system is completely observable.
```

6. Conclusions

In the lines of the Geometric Approach, the observability of a class of multidimensional LTI systems was studied in this paper. The list of the complete observability criteria can be extended by using the duality of the concepts of controllability and observability or by using the rank of a suitable observability Gramian, in the case of time-variable systems (see [8]). The algorithm which determines the maximal invariant subspace can be used to find the canonical form of the unobservable systems and the Kalman canonical decomposition of a system.

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