A 3 D SMITH CHART FOR ACTIVE AN PASSIVE MICROWAVE CIRCUITS AND VISUAL COMPLEX ANLAYSIS

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Lucrarea explică în premieră elementele de analiză complexă vizuală ale diagramei Smith 3D pentru circuite active şi pasive. Autorii îşi dezvoltă propriul concept extrem de acutal cu elemente de analiză complexă aplicate în plan şi în 3D (pe sfera lui Riemann). Proprietăţile geometrice ale diferitelor transformări din planul complex sunt exploatate şi aplicate în microunde.

The paper explains for the first time the visual complex analyisis propeties of the very recently proposed 3D Smith for active and passive circuits. The authors develop their very actual concept with elements of complex analysis applied in the complex plane and in 3D (Riemann sphere).The geometrical properties of the different transfromations of the complex plane are exploited and applied in microwave design.

Keywords: Smith chart, complex analysis

1. Introduction

In the era of Greek civilization, geometry was studied rigorously and put a firm theoretical basis for intellectual satisfaction, for intrinsic beauty of many geometrical results and the utility of the subject. Throughout the years the interest in geometry languised, it was thought old fashioned by the majority. One of the revival moments of geometry may be considered related with computer graphics and the first papers related to the fractals. The connection between nonscaling fractals, linked to one of the most difficult areas of classical mathematics [1] and visual complex analysis made a subject like " mathematical analysis" to get a new perception.

The Smith chart is one of the most widely used charts in electrical and electronic engineering [2]. Although invented in the late 30's the diagram

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survived among the time and its use has grown steadily over the years as software design tool and as a visualization instrument on the network analyzers. Its main geometrical properties are related with the beauty of the Möbius transformations and are also linked to topics from arts –Esher's art [3] and arhitecure. In the recent years several tries were done in order to increase the the capability of the original Smith chart into 3d in order to contain a wider range of impedances as in [4,5].
 These papers try to extend the previously 2d generalized approaches as [6] but lack in a pure mathematical formalism and fail to include all the loads.

In our paper we focus our attention on the most recent proposed 3D Smith chart [7,8]. This 3D Smith chart is created using the mathematical concept of the Riemann sphere. The Riemann sphere is the key tool of visualizing inversive transformations in the extended complex plane**.** It is totally difficult to visualize the infinity far apart (as it may occur in active devices) on the generalized planar Smith chart since an outer edge of the complex plane is vague definition for that. Fortunately, Riemann interpreted the numbers in the extended complex plane as points on a sphere. Using Ptolemy's theory on celestial spheres, Riemann applied a stereographic projection to perform a one-to-one correspondence between the extended complex plane and the unity radius sphere (i.e., the unit sphere). The 3D Smith chart is created using Riemann's theory on the complex plane and applying it to the Möbius transformation represented by the Smith chart mapping. Considering a single point at infinity all the circles arcs that never met on the Smith chart will get together in the South pole.

2. Möbius transformations, 2 D generalized Smith chart and the maximum modulus theorem

The main governing equation for the Smith chart is (1) and represents the reflection coefficient of the travelling waves

$$\rho = \frac{z-1}{z+1} \quad ; \quad z = r + \mathrm{j}x = Z/Z_0 . \tag{1}$$

In the case of a real characteristic impedance Z0 then all the normalized impedances (z) that represent passive circuits will have a positive real part (r>0). In this situation all the normalized passive loads will generate a reflection coefficient that has a magnitude lower than 1. An example is presented in Fig.1 where the reflection coefficient for an arbitrarily shape in the RHP (right half plane) is computed. The magnitude of the reflection coefficient in this case is always bounded by one (the white circle in Fig 1.).
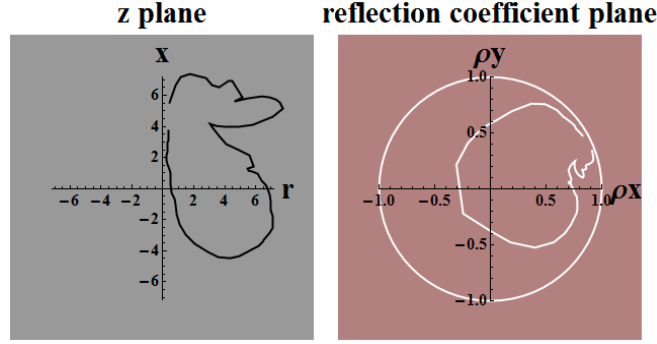
Fig. 1 Mapping of an impedance in the RHP into the reflection coefficient's plane for the travelling wave formulation

In (2) the general form of a Möbius transformation is given; a, b, c, d represent the coefficients of the Möbius transformation (ad-bc≠0). [M] stays for the matrix form corresponding to the Möbius transformation given by the reflection coefficient while M_1 shows the normalized Möbius transformation matrix in the Smith chart mapping case.

$$\rho_1 = \frac{Z - Z_c}{Z + Z_c}\ ; Z_c = R_c + jX_c\ ; \rho_1 = \rho_{1x} + j * \rho_{1y} \qquad (2)$$

$$\rho_1 = \frac{\dfrac{Z}{R_c} - 1 - \dfrac{jX_c}{R_c}}{\dfrac{Z}{R_1} + 1 + \dfrac{jX_c}{R_c}}\ ; \ z_1 = \frac{Z}{R_c}\ ; \ x_c = \frac{X_c}{R_c} \qquad (3)$$

$$\rho_1 = \frac{z_1 - (1 + j * x_c)}{z_1 + 1 + j * x_c}\ , z_1 = r_1 + j * x_1\ , r_1 > 0 \qquad (4)$$

In (3) one may see the general form of the reflection coefficient for complex characteristic impedances.
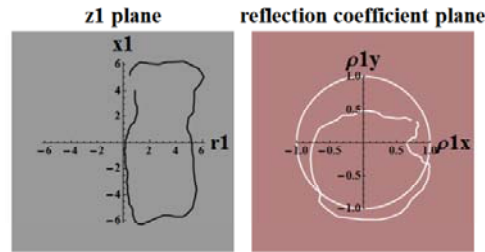
Fig. 2 Possible mapping of an impedance in the RHP into the reflection coefficient's plane for a complex characteristic impedance

In (3) the reflection coefficient is rewritten so that z1 stays for an impedance only in the RHP. However if one looks at (4) the reflection coefficient in the case of a complex characteristic impedance is just Möbius transformation of the RHP impedance plane with the coefficients: a=1,b=-1-j*xc, c=1,d=1+j*xc.
If one considers xc=0.3 in (6) this generates a mapping as in Fig 2. We may notice in Fig 2 that an arbitrarily shape in the RHP may generate a reflection coefficient that exceeds the unity. If we visualize this in a 3D manner we may get Fig 3. The interesting thing in Fig 3 is that the maximum magnitude of the reflection coefficient is obtained on the x1 axes. This value is 1.09 for xc=0.3.

However in order to give a general formula for the maximum magnitude of the reflection coefficient of a passive load one may treat (4) as a complex mapping generated by a Möbius transformation. Since z_1 is in this case just a complex number wherever in the RHP and ρ1 is analytic in the RHP one may use the maximum modulus theorem in order to establish the boundary for ρ1.
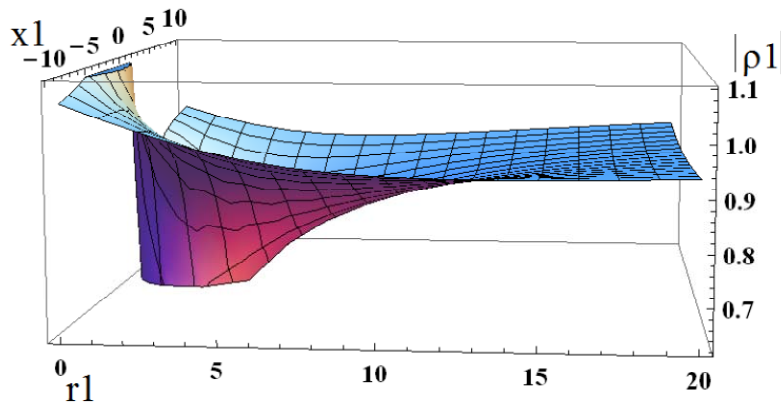

Fig. 3 Magnitude of the reflection coefficient for different values of r_1 and x_1 and a value of x_c=0.3

The maximum modulus theorem [9] states that if a function is analytic in a domain then its maximum is achieved on the frontier. Applying this theorem for z1 belonging in the RHP we find out that the maximum is always on the x_1 axis of the z1 plane. The maximum value for the magnitude of the reflection coefficient is then obtained as:

$$|\rho_1| = \sqrt{\frac{1 + x_c^2 + |x_c| * \sqrt{1 + x_c^2}}{1 + x_c^2 - |x_c| * \sqrt{1 + x_c^2}}} \qquad (5)$$

The value stated in (5) is the maximum value that the reflection coefficient may reach in a case of a complex impedance. One may see that considering xc =0 we get the well known $\rho 1$ =1 while in the case of xc=0.3 from Fig 3 we get the value 1.09. The results obtained for ρ1 in [10], [11], are a consequence of (5). The value is in perfect concordance with the results presented in [12] but the maximum modulus theorem applied to form (4) extend the point of view presented in [12] and prove that (5) is the global maximum for the magnitude of the reflection coefficient for passive loads when normalized to a complex impedance.
In Fig 4 is presented the mapping of an impedance from the RHP to the reflection coefficient's plane. The extended 2 D Smith chart is also drawn.

In Figs 4-5 one may see the effect that an active device may generate. (the same for a complex normalization impedance). The reflection coefficient's magnitude may exceed the unity and this leads to a reflection coefficient outside of the classical Smith chart (red-dashed).
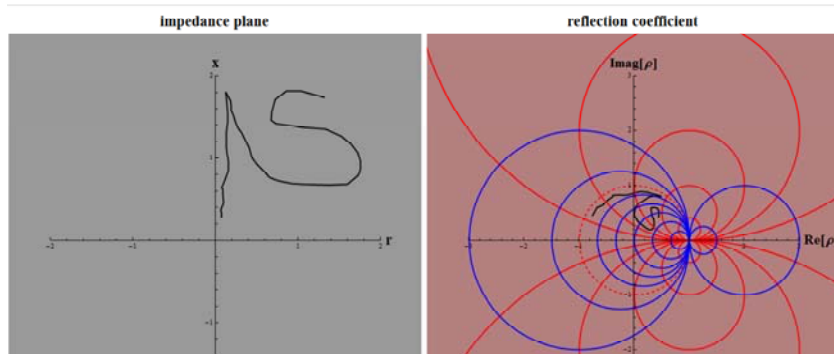

Fig. 4 Mapping of an arbitrary shape from the impedance plane into the reflection coefficients plane from the RHP using the extended Smith chart as a visual tool
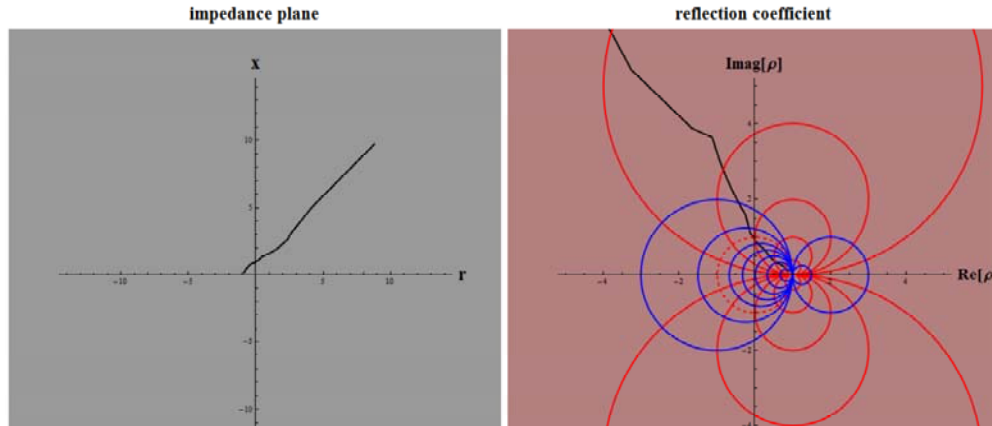
Fig.5.Mapping of an arbitrary shape in the reflection coefficients plane from the RHP and LHP using the extended Smith chart as a visual tool. The presence in the LHP makes the reflection coefficient jump outside the Smith chart

3. A 3D Smith chart and its latitude-reflection coefficient correspondence

Since one needs different zoom scales in order to work with the phenomena presented in Fig 5, the letter [7] proposed the Riemann sphere as the new 3D Smith chart that may include any types of loads. However the concept is just mathematically described.

We extend the ideas and present for the first time the connection between latitude and the reflection coefficient's plane and the steeographical map of the world and the Smith reflection plane (Fig 6 and Fig 7)
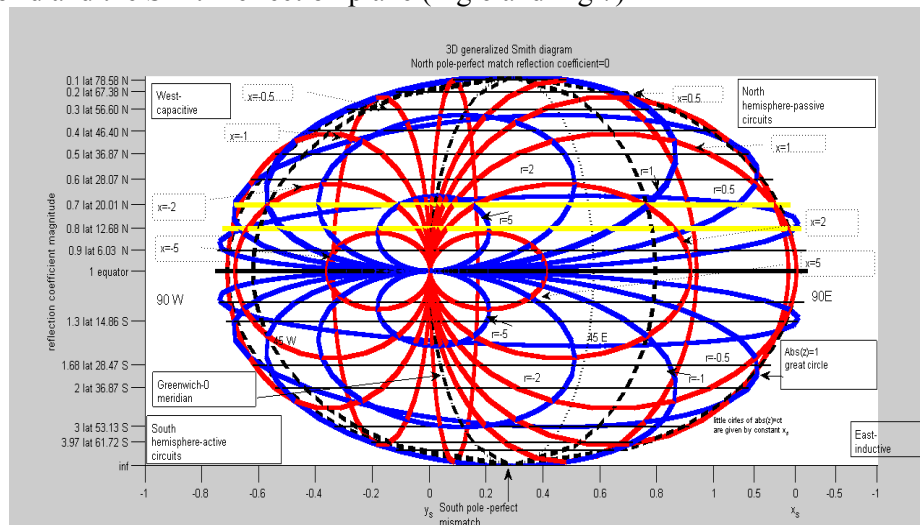

Fig. 6 A 3D Smith chart and the relationship between latitude and refelction coefficient
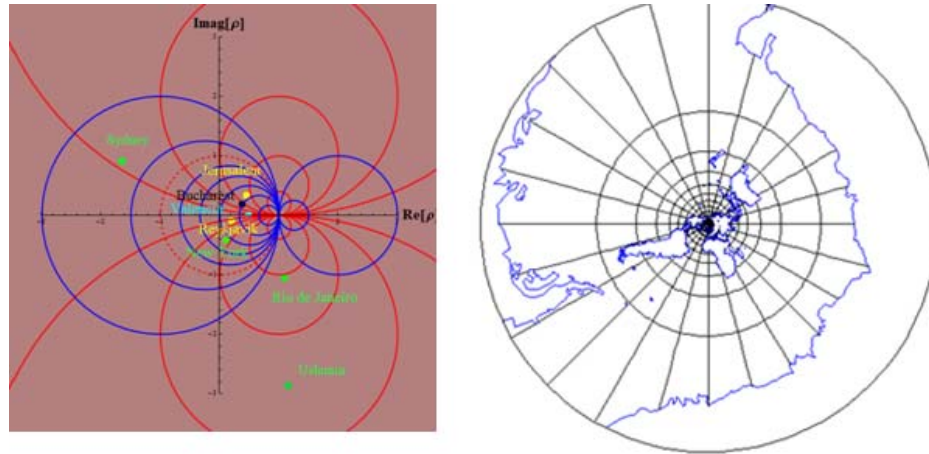
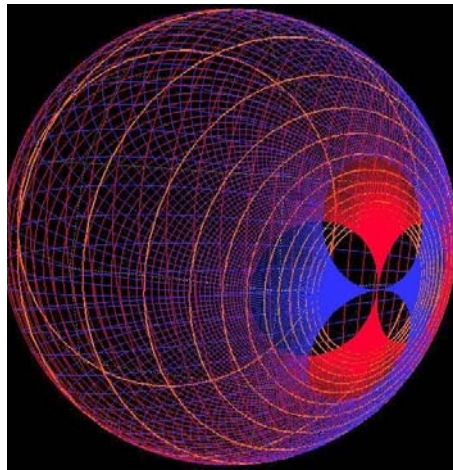Fig. 7 Stereographical map of the world and the Smith chart

Fig. 8 3D Smith chart [www.3dsmithchart.com]

4. Conclusions

The paper used elements of visual complex analysis like the maximum modulus theorem or geometry of the Möbius transformations in order to establish a boundary for the reflection coefficient's magnitude in the case of microwave circuits with complex characteristic impedance. It presented a new vision on the recently proposed 3D Smith chart and proposed a latitude –reflection coefficient scale for the 3D Smith chart. The results can be a useful starting point for a microwave engineer dealing with active circuits and complex characteristic impedances. Nevertheless the paper showed the connection between Möbius transformations and microwaves. Plotting the Möbius transformations that occur

in microwave theory on the 3D Smith chart one may get a handy tool to solve any matching problems.

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