ISOTROPIC RIEMANNIAN MAPS AND HELICES ALONG RIEMANNIAN MAPS

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This work has two main aims. The first aim is to study isotropic Riemannian maps as a generalization of isotropic immersions. The notion of isotropic Riemannian map is presented, an example is given and a characterization is obtained. The second aim is to study the helices along Riemannian map. By using the notion of isotropic Riemannian map and helices on the manifold, a characterization is obtained for the transportation of helices on the total manifold to the target manifold along a Riemannian map.

Keywords: isotropic Riemannian map, Riemannian map, helix, umbilical map, Riemannian manifold,

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1. **Introduction**

Since curves are basic structures of geometry, it is an important technique to arrive at geometric conclusions by looking at their behavior under certain maps. This approach was first used by Nomizu and Yano for circles and isometric immersions in [15]. They showed that when a circle on the submanifolds is carried along the immersion to the ambient manifold, such submanifolds are umbilical and their mean curvature vector field is parallel. Ikawa in [11] obtained similar characterization for helices. Later this result has been extended to the semi-Riemannian case [2] and [12], see also: [1], [3], [4], [5], [6], [13], [21]. These papers show that the behavior of a given curve under transformation will give us important information when comparing the geometry of two manifolds.

The basic properties of Riemannian submersions were firstly given by Gray [8] and O'Neill [18]. Riemannian submersions and isometric immersions are special maps between Riemannian manifolds. So, Fischer [10] defined Riemannian maps which is a generalization of Riemannian submersions and isometric immersions in 1992 as follows. Assume that $\mathcal{T}:(\mathfrak{M}_1, g_{\mathfrak{M}_1}) \to (\mathfrak{M}_2, g_{\mathfrak{M}_2})$ is a C^∞ map from the Riemannian manifold $\mathfrak{M}_1$ with $\dim \mathfrak{M}_1 = m$ to the Riemannian manifold $\mathfrak{M}_2$ with $\dim \mathfrak{M}_2 = n$, where $0 < rank\mathcal{T} < \min\{m, n\}$. Thus, we represent the kernel space of $\mathcal{T}_*$ by $\ker \mathcal{T}_*$ and $\mathcal{H} = (\ker \mathcal{T}_*)^\perp$ is orthogonal complementary space to $\ker \mathcal{T}_*$. So, we have

$$T\mathfrak{M}_1 = \ker \mathcal{T}_* \oplus (\ker \mathcal{T}_*)^\perp,$$

where $T\mathfrak{M}_1$ is the tangent bundle of $\mathfrak{M}_1$. $range\mathcal{T}_*$ denotes the range of $\mathcal{T}_*$ and $(range\mathcal{T}_*)^\perp$ denotes the orthogonal complementary space to $range\mathcal{T}_*$ in $T\mathfrak{M}_2$. The tangent bundle $T\mathfrak{M}_2$ of $\mathfrak{M}_2$ is given by

$$T\mathfrak{M}_2 = (range\mathcal{T}_*) \oplus (range\mathcal{T}_*)^\perp.$$

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Now, a smooth map $\mathcal{T} : (\mathfrak{M}_1^m, g_{\mathfrak{M}_1}) \to (\mathfrak{M}_2^n, g_{\mathfrak{M}_2})$ is called Riemannian map if $\mathcal{T}_*$ satisfies $g_{\mathfrak{M}_2}(\mathcal{T}_* X_1, \mathcal{T}_* X_2) = g_{\mathfrak{M}_1}(X_1, X_2)$, for X_1, X_2 vector fields tangent to $\mathcal{H}$ [10].

Comparing Riemannian maps with immersions and Riemannian submersions, there are few results for the characterization of such transformations with the help of curves. Indeed, there are no other studies in the literature, except for the results regarding the geodesics at reference [9] for Riemannian submersions and the results about circles at reference [19] for Riemannian maps. Since curves are the basic notion of differential geometry and their useful role in immersion theory is taken into account, obtaining new results on the characterization of Riemannian submersions and Riemannian maps by means of curves is a subject that needs to be investigated.

Also, isotropic immersions were defined by O'Neill [17] and later it has been shown that this notion is an important tool in geometric characterization [14]. The concept of isotropic submersions has been introduced by Şahin and Erdogan in [7]. As far as we know, the notion of isotropic Riemannian maps has not yet been introduced in the literature.

We first define the concept of an isotropic Riemannian map and obtain a condition for isotropicity of a Riemannian map in terms of second fundamental form. Then by using characterization of an ordinary helix in a Riemannian manifold, we generalize Ikawa's theorem.

In the second section, the basic notions to be used in the paper are presented. First, the concept of helix on the manifold is given. Then, Gauss and Weingarten formulas are introduced for Riemannian maps with connections defined along a map and are presented in some detail, especially considering that the readers who study the submanifold theory may not be familiar with these notions. In the third section, the concept of isotropic Riemannian map is presented and characterization of such maps is obtained. We give the following result in the last section; if there is a helix on the base manifold of a Riemannian map, we investigate what this property tells for the Riemannian map when a helix is transformed by the Riemannian map on the target manifold.

2. **Preliminaries**

A regular curve $\alpha = \alpha(s)$ parametrized by arc lengh s is called an ordinary helix if there exist unit vector fields V_2 and V_3 along α such that

$$\nabla_{V_1} V_1 = \kappa V_2, \quad \nabla_{V_1} V_2 = -\kappa V_1 + \tau V_3, \quad \nabla_{V_1} V_1 = -\tau V_2,$$

where V_1 denotes the tangent vector field of α, κ is the curvature and τ is the torsion of α. An ordinary helix satisfies the following equation

$$\nabla^3_{V_1} V_1 + K^2 \nabla_{V_1} V_1 = 0, \tag{1}$$

where K^2 is a constant. If $\tau = 0$, helix reduces to the circle [11].

Assume that $(\mathfrak{M}_1, g_{\mathfrak{M}_1})$ and $(\mathfrak{M}_2, g_{\mathfrak{M}_2})$ are Riemannian manifolds, $\mathcal{T} : (\mathfrak{M}_1, g_{\mathfrak{M}_1}) \to (\mathfrak{M}_2, g_{\mathfrak{M}_2})$ a smooth map between them and γ a curve on $\mathfrak{M}_1$. γ is called a horizontal curve if $\dot{\gamma}(t) \in (ker\mathcal{T}_*)^{\perp}$ for any $t \in I$. If γ is a helix with $\dot{\gamma}(t) \in (ker\mathcal{T}_*)^{\perp}$ for any $t \in I$, then it is called as horizontal helix.

Let $\mathcal{T}$ be a Riemannian map between the manifolds $(\mathfrak{M}_1, g_{\mathfrak{M}_1})$ and $(\mathfrak{M}_2, g_{\mathfrak{M}_2})$, $p_2 = \mathcal{T}(p_1)$ for each $p_1 \in \mathfrak{M}_1$. Suppose that $\nabla^{\mathfrak{M}_2}$ and $\nabla^{\mathfrak{M}_1}$ represent the connections on $(\mathfrak{M}_2, g_{\mathfrak{M}_2})$ and $(\mathfrak{M}_1, g_{\mathfrak{M}_1})$, respectively. The second fundamental form of $\mathcal{T}$ can be given as follows

$$(\nabla \mathcal{T}_*)(X_1, X_2) = \overset{\mathfrak{M}_2}{\nabla^{\mathcal{T}}}_{X_1} \mathcal{T}_*(X_2) - \mathcal{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{X_1} X_2) \tag{2}$$

for $X_1,\ X_2 \in \Gamma(T\mathfrak{M}_1)$, where $\overset{\mathfrak{M}_2}{\nabla^{\mathcal{T}}}$ is the pullback connection of $\nabla^{\mathfrak{M}_2}$. For $\forall X_1, X_2 \in \Gamma((\ker \mathcal{T}_{*p_1})^{\perp})$, $(\nabla \mathcal{T}_*)$ is symmetric and has no components in $range\mathcal{T}_*$. So, we can write the following

$$g_{\mathfrak{M}_2}((\nabla \mathcal{T}_*)(X_1, X_2), \mathcal{T}_*(X_3)) = 0, \tag{3}$$

$\forall X_1, X_2, X_3 \in \Gamma((\ker \mathcal{T}_{*p_1})^{\perp})$ [20].

Now we give some basic formulas for Riemannian maps defined from the total manifold $(\mathfrak{M}_1, g_{\mathfrak{M}_1})$ to the target manifold $(\mathfrak{M}_2, g_{\mathfrak{M}_2})$. For $X_1, X_2 \in \Gamma((\ker \mathcal{T}_{*p_1})^{\perp})$ and $U_1 \in \Gamma((range\mathcal{T}_*)^{\perp})$, we have

$$\overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}{}_{X_1} U_1 = -S_{U_1}\mathcal{T}_* X_1 + \nabla^{\mathcal{T}^{\perp}}_{X_1} U_1, \tag{4}$$

where $S_{U_1}\mathcal{T}_* X_1$ is the tangential component of $\overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}{}_{X_1} U_1$ and $\nabla^{\mathcal{T}^{\perp}}_{X_1}$ is the orthogonal projection $\overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}{}_{X_1}$ on $\Gamma((range\mathcal{T}_*)^{\perp})$, then we have

$$g_{\mathfrak{M}_2}(S_{U_1}\mathcal{T}_* X_1, \mathcal{T}_* X_2) = g_{\mathfrak{M}_2}(U_1, (\nabla \mathcal{T}_*)(X_1, X_2)). \tag{5}$$

Since $(\nabla \mathcal{T}_*)$ is symmetric, S_{U_1} is a symmetric linear transformation of $range\mathcal{T}_*$. On the other hand, we have the following covariant derivatives

$$\begin{aligned} \left(\tilde{\nabla}_{X_1}(\nabla \mathcal{T}_*)\right)(X_2, X_3) &= \nabla^{\mathcal{T}^{\perp}}_{X_1}(\nabla \mathcal{T}_*)(X_2, X_3) \\ &- (\nabla \mathcal{T}_*)(\overset{\mathfrak{M}_1}{\nabla}_{X_1} X_2, X_3) - (\nabla \mathcal{T}_*)(X_2, \overset{\mathfrak{M}_1}{\nabla}_{X_1} X_3) \end{aligned} \tag{6}$$

and

$$\begin{aligned} \left(\tilde{\nabla}_{X_1} S\right)_{U_1} \mathcal{T}_*(X_2) &= \mathcal{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{X_1} {}^*\mathcal{T}_*(S_{U_1}\mathcal{T}_*(X_2))) \\ &- S_{(\nabla^{\mathcal{T}^{\perp}}_{X_1})U_1} \mathcal{T}_*(X_2) - S_{U_1} P \overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}_{X_1} \mathcal{T}_*(X_2), \end{aligned} \tag{7}$$

where P denotes the projection morphism on $range\mathcal{T}_*$ and ${}^*\mathcal{T}_*$ is the adjoint map of $\mathcal{T}_*$ [20]. In the following lemma we give a relation obtained from (6) and (7).

Lemma 2.1. *Let $(\mathfrak{M}_1, g_{\mathfrak{M}_1})$, $(\mathfrak{M}_2, g_{\mathfrak{M}_2})$ be Riemannian manifolds and $\mathcal{T}$ a Riemannian map between them. For $\forall X_1, X_2 \in \Gamma((\ker \mathcal{T}_{*p_1})^{\perp})$ and $U_1 \in \Gamma((range\mathcal{T}_*)^{\perp})$, we have*

$$g_{\mathfrak{M}_2}(\left(\tilde{\nabla}_{X_1}(\nabla \mathcal{T}_*)\right)(X_2, X_3), U_1) = g_{\mathfrak{M}_2}(\left(\tilde{\nabla}_{X_1} S\right)_{U_1} \mathcal{T}_*(X_2), \mathcal{T}_*(X_3)). \tag{8}$$

Proof. Taking inner product (6) with U_1, we have

$$\begin{aligned} g_{\mathfrak{M}_2}(\left(\tilde{\nabla}_{X_1}(\nabla \mathcal{T}_*)\right)(X_2, X_3), U_1) &= g_{\mathfrak{M}_2}(\nabla^{\mathcal{T}^{\perp}}_{X_1}(\nabla \mathcal{T}_*)(X_2, X_3), U_1) \\ -g_{\mathfrak{M}_2}((\nabla \mathcal{T}_*)(\overset{\mathfrak{M}_1}{\nabla}_{X_1} X_2, X_3), U_1) &- g_{\mathfrak{M}_2}((\nabla \mathcal{T}_*)(X_2, \overset{\mathfrak{M}_1}{\nabla}_{X_1} X_3), U_1). \end{aligned}$$

If we take inner product (7) with $\mathcal{T}_*(X_3)$, we obtain

$$\begin{aligned} g_{\mathfrak{M}_2}(\left(\tilde{\nabla}_{X_1} S\right)_{U_1} \mathcal{T}_*(X_2), \mathcal{T}_*(X_3)) &= g_{\mathfrak{M}_2}(\mathcal{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{X_1} {}^*\mathcal{T}_*(S_{U_1}\mathcal{T}_*(X_2))), \mathcal{T}_*(X_3)) \\ -g_{\mathfrak{M}_2}(S_{(\nabla^{\mathcal{T}^{\perp}}_{X_1})U_1} \mathcal{T}_*(X_2), \mathcal{T}_*(X_3)) &- g_{\mathfrak{M}_2}(S_{U_1} P \overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}_{X_1} \mathcal{T}_*(X_2), \mathcal{T}_*(X_3)). \end{aligned}$$

Using (5), we can write the following equalites

$$g_{\mathfrak{M}_2}((\nabla \mathcal{T}_*)(\overset{\mathfrak{M}_1}{\nabla}_{X_1} X_2, X_3), U_1) = g_{\mathfrak{M}_2}(S_{U_1} P \overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}_{X_1} \mathcal{T}_*(X_2), X_3) \tag{9}$$

and

$$g_{\mathfrak{M}_2}(S_{U_1}\mathcal{T}_*(X_2), \mathcal{T}_*(X_3)) = g_{\mathfrak{M}_2}((\nabla \mathcal{T}_*)(X_2, X_3), U_1).$$

If we take derivative of the last equation and use (3), we get

$$\begin{aligned} &g_{\mathfrak{M}_2}(\mathcal{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{X_1}{}^*\mathcal{T}_*(S_{U_1}\mathcal{T}_*(X_2))),\mathcal{T}_*(X_3)) + g_{\mathfrak{M}_2}((\nabla\mathcal{T}_*)(X_2,\overset{\mathfrak{M}_1}{\nabla}_{X_1}X_3),U_1) \\ &= g_{\mathfrak{M}_2}(\nabla^{\mathcal{T}^\perp}_{X_1}(\nabla\mathcal{T}_*)(X_2,X_3),U_1) + g_{\mathfrak{M}_2}(S_{(\nabla^{\mathcal{T}^\perp}_{X_1})U_1}\mathcal{T}_*(X_2),\mathcal{T}_*(X_3)). \end{aligned} \tag{10}$$

If we take into consideration (9) and (10), we have (8). □

We consider that $\mathcal{T}$ is a Riemannian map from a connected Riemannian manifold $(\mathfrak{M}_1, g_{\mathfrak{M}_1})$, $dim\mathfrak{M}_1 \geq 2$ to a Riemannian manifold $(\mathfrak{M}_2, g_{\mathfrak{M}_2})$. We know that $\mathcal{T}$ is an umbilical Riemannian map at $p_1 \in \mathfrak{M}_1$, if the following is satisfied

$$S_{U_1}\mathcal{T}_{*p_1}((X_1)_{p_1}) = g_{\mathfrak{M}_2}(H_2, U_1)\mathcal{T}_{*p_1}((X_1)_{p_1})$$

for $X_1 \in \Gamma(range\mathcal{T}_*)$, $U_1 \in \Gamma((range\mathcal{T}_*)^\perp)$ and $H_2 \in (range\mathcal{T}_*)^\perp$. If $\mathcal{T}$ is umbilical for $\forall p_1 \in \mathfrak{M}_1$, we know that $\mathcal{T}$ is umbilical [20].

3. **Isotropic Riemannian maps**

Now, we present the notion of isotropic Riemannian map and get a characterization for such maps.

Definition 3.1. ***(h-isotropic Riemannian map).*** *A Riemannian map $\mathcal{T} : \mathfrak{M}_1 \to \mathfrak{M}_2$ is said to be h-isotropic at $p \in \mathfrak{M}_1$ if*

$$\lambda(X_1) = \|(\nabla\mathcal{T}_*)(X_1, X_1)\| \,/\, \|\mathcal{T}_*X_1\|^2$$

doesn't depend upon the selection of $X_1 \in \Gamma((\ker\mathcal{T}_*)^\perp$. If the map is h-isotropic at all points, the map is called as h-isotropic. Also, if $\lambda = \lambda(p)$ is constant along $\mathcal{T}$, $\mathcal{T}$ is called a constant $(\lambda h-)$isotropic map.

Remark 3.1. *It is easy to see that the above notion is generalization of isotropic immersion, however there is no inclusion relation between isotropic Riemannian submersions and $h-$isotropic Riemannian maps.*

Proposition 3.1. *Let $\varphi : \mathfrak{M}_1^m \to \mathfrak{M}_2^n$ be a Riemannian submersion and $\psi : \mathfrak{M}_2^n \to \mathfrak{M}_3^k$ a $h-$isotropic immersion. Then the Riemannian map $\mathcal{T} = \psi \circ \varphi$ is a $h-$isotropic map.*

Proof. We consider that $\mathcal{T} = \psi \circ \varphi$ is a Riemannian map, where $\varphi : \mathfrak{M}_1^m \to \mathfrak{M}_2^n$ is a Riemannian submersion and $\psi : \mathfrak{M}_2^n \to \mathfrak{M}_3^k$ is a $h-$isotropic immersion. Then we have for $X_1, X_2 \in (ker\mathcal{T}_*)^\perp$

$$(\nabla(\psi\circ\varphi)_*)(X_1, X_2) = \psi_*((\nabla\varphi_*)(X_1, X_2)) + (\nabla\psi_*)(\varphi_*(X_1), \varphi_*(X_2)). \tag{11}$$

On the other hand, we get

$$g_{\mathfrak{M}_1}((\psi\circ\varphi)_*(X_1), (\psi\circ\varphi)_*(X_2)) = g_{\mathfrak{M}_2}(\psi_*(X_1), \psi_*(X_2)) = g_{\mathfrak{M}_3}(X_1, X_2). \tag{12}$$

From (11), we obtain

$$\begin{aligned} &\|(\nabla(\psi\circ\varphi)_*)(X_1, X_1)\|^2 = \|\psi_*((\nabla\varphi_*)(X_1, X_1))\|^2 \\ &+2g_{\mathfrak{M}_3}(\psi_*((\nabla\varphi_*)(X_1, X_1)), (\nabla\psi_*)(\varphi_*(X_1), \varphi_*(X_1)) \\ &+\|(\nabla\psi_*)(\varphi_*(X_1), \varphi_*(X_1)\|^2. \end{aligned} \tag{13}$$

Because ψ is a $h-$isotropic immersion and due to Riemannian submersion φ, $(\nabla\varphi_*)(X_1, X_1) = 0$ for $X_1 \in (ker\mathcal{T}_*)^\perp$, (13) and (12) show that $\mathcal{T}$ is $h-$isotropic map. □

The following example can be given as an application of Proposition 3.1.

Example 3.1. *Let $\mathcal{T}$ be a Riemannian map given by $\mathcal{T} : \mathbb{R}^4 \to \mathbb{R}^3$*

$$(x_1, x_2, x_3, x_4) \to (\frac{(x_1 - x_2)^2}{2} - x_3^2, \sqrt{2}(x_1 - x_2)x_3, 0).$$

Since $\varphi : \mathbb{R}^4 \to \mathbb{R}^2$, $(x_1, x_2, x_3, x_4) \to (\frac{x_1 - x_2}{\sqrt{2}}, x_3)$ is a Riemannian submersion and $\psi : \mathbb{R}^2 \to \mathbb{R}^3$, $(x_1, x_2) \to (x_1^2 - x_2^2, 2x_1x_2, 0)$, where $(x_1^2 + x_2^2)^2 = 1$ (see, [16]), a $h-$isotropic immersion, the Riemannian map $\mathcal{T} = \psi \circ \varphi$ is a $h-$isotropic Riemannian map.

The following lemma gives a criteria for a h-isotropic Riemannian map.

Lemma 3.1. *Let $\mathcal{T} : \mathfrak{M}_1 \to \mathfrak{M}_2$ be a Riemannian map. $\mathcal{T}$ is h-isotropic at $p \in \mathfrak{M}_1$ iff the second fundamental form $\nabla\mathcal{T}_*$ satisfies*

$$g_{\mathfrak{M}_2}((\nabla\mathcal{T}_*)(X_1, X_1), (\nabla\mathcal{T}_*)(X_1, X_2)) = 0 \tag{14}$$

for an arbitrary orthogonal couple $X_1, X_2 \in \Gamma((\ker \mathcal{T}_{*p_1})^\perp)$.

Proof. Let $f : \Gamma((\ker \mathcal{T}_{*p_1})^\perp) \to \mathbb{R}$ be a quadrilinear function such that for $\forall x_1$, x_2, u_1, $u_2 \in \Gamma((\ker \mathcal{T}_{*p_1})^\perp)$

$$\begin{aligned} f(x_1, x_2, u_1, u_2) = {} & g_{\mathfrak{M}_2}((\nabla\mathcal{T}_*)(x_1, x_2), (\nabla\mathcal{T}_*)(u_1, u_2)) \\ & -\lambda^2 g_{\mathfrak{M}_2}(\mathcal{T}_*(x_1), \mathcal{T}_*(x_2)) g_N(\mathcal{T}_*(u_1), \mathcal{T}_*(u_2)). \end{aligned}$$

If a Riemannian map $\mathcal{T}$ is $h-$isotropic, we have for $\forall u_3 \in \Gamma((\ker \mathcal{T}_{*p_1})^\perp)$

$$\begin{aligned} B(u_3) &= f(u_3, u_3, u_3, u_3), \\ &= g_N((\nabla\mathcal{T}_*)(u_3, u_3), (\nabla\mathcal{T}_*)(u_3, u_3)) - \lambda^2 g_M(u_3, u_3) g_M(u_3, u_{31}), \\ &= \lambda^2 \|u_3\|^4 - \lambda^2 \|u_3\|^4 = 0. \end{aligned}$$

If we use $B(x_1 + x_2) + B(x_1 - x_2) = 0$, we get

$$f(x_1, x_1, x_2, x_2) + 2f(x_1, x_2, x_1, x_2) = 0.$$

Changing x_2 into $x_1 + x_2$, we have

$$f(x_1, x_2, x_2, x_2) = 0.$$

In the last equation, if we change x_1 into u_2 and x_2 into u_1 where $u_1 \perp u_2$, we obtain

$$\begin{aligned} f(u_2, u_1, u_1, u_1) = {} & g_{\mathfrak{M}_2}((\nabla\mathcal{T}_*)(u_2, u_1), (\nabla\mathcal{T}_*)(u_1, u_1)) \\ & - \lambda^2 g_{\mathfrak{M}_1}(u_2, u_1) g_{\mathfrak{M}_1}(u_1, u_1) = 0. \end{aligned}$$

So, we have

$$g_{\mathfrak{M}_2}((\nabla\mathcal{T}_*)(u_2, u_1), (\nabla\mathcal{T}_*)(u_1, u_1)) = 0.$$

Conversely, we suppose that (11) is satisfied along the Riemannian map $\mathcal{T}$. For an arbitrary orthogonal pair $x_1, x_2 \in \Gamma((\ker \mathcal{T}_{*p_1})^\perp)$, we have

$$\begin{aligned} f(x_1, x_1, x_1, x_2) = {} & g_{\mathfrak{M}_2}((\nabla\mathcal{T}_*)(x_1, x_1), (\nabla\mathcal{T}_*)(x_1, x_2)) \\ & -\lambda^2 g_{\mathfrak{M}_2}(\mathcal{T}_*(x_1), \mathcal{T}_*(x_1)) g_{\mathfrak{M}_2}(\mathcal{T}_*(x_1), \mathcal{T}_*(x_2)) = 0. \end{aligned}$$

If we write $x_1 + x_2$ instead of x_2, we have

$$f(x_1, x_1, x_1, x_1) = 0.$$

So we have

$$g_{\mathfrak{M}_2}((\nabla\mathcal{T}_*)(x_1, x_1), (\nabla\mathcal{T}_*)(x_1, x_1)) = \lambda^2 g_{\mathfrak{M}_2}(\mathcal{T}_*(x_1), \mathcal{T}_*(x_1)) g_N(\mathcal{T}_*(x_1), \mathcal{T}_*(x_1)),$$

that is, $\mathcal{T}$ is $h-$isotropic. □

Let $(\mathfrak{M}_1, g_{\mathfrak{M}_1})$ and $(\mathfrak{M}_2, g_{\mathfrak{M}_2})$ be Riemannian manifolds and suppose that $\mathcal{T} : (\mathfrak{M}_1, g_{\mathfrak{M}_1}) \to (\mathfrak{M}_2, g_{\mathfrak{M}_2})$ is a Riemannian map between them. Let α be a horizontal curve with curvature κ in $\mathfrak{M}_1$ and $\gamma = \mathcal{T} \circ \alpha$ a curve with curvature $\tilde{\kappa}$ in $\mathfrak{M}_2$ along $\mathcal{T}$. Using (2) and (3), we obtain for $\forall t \in \mathbb{R}$

$$\begin{aligned}
\tilde{\kappa}^2 &= g_{\mathfrak{M}_2}(\overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}_{\dot{\alpha}} \mathcal{T}_*(\dot{\alpha}), \overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}_{\dot{\alpha}} \mathcal{T}_*(\dot{\alpha})), \\
&= g_{\mathfrak{M}_2}(\mathcal{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{\dot{\alpha}} \dot{\alpha}) + (\nabla \mathcal{T}_*)(\dot{\alpha}, \dot{\alpha}), \mathcal{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{\dot{\alpha}} \dot{\alpha}) + (\nabla \mathcal{T}_*)(\dot{\alpha}, \dot{\alpha})), \\
&= g_{\mathfrak{M}_1}(\overset{\mathfrak{M}_1}{\nabla}_{\dot{\alpha}} \dot{\alpha}, \overset{\mathfrak{M}_1}{\nabla}_{\dot{\alpha}} \dot{\alpha}) + \|(\nabla \mathcal{T}_*)(\dot{\alpha}, \dot{\alpha})\|^2, \\
&= \kappa^2 + \|(\nabla \mathcal{T}_*)(\dot{\alpha}, \dot{\alpha})\|^2,
\end{aligned}$$

where $\overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}$ denotes the pullback connection of $\overset{\mathfrak{M}_2}{\nabla}$. So, we can write

$$\tilde{\kappa} = \tilde{\kappa}(t) = \sqrt{\kappa(t)^2 + \|(\nabla \mathcal{T}_*)(\dot{\alpha}(t), \dot{\alpha}(t))\|^2}. \tag{15}$$

We suppose that $\mathcal{T} : \mathfrak{M}_1 \to \mathfrak{M}_2$ is a Riemannian map and $\alpha = \alpha(s)$ is a horizontal circle parametrized by arc length s with

$$\overset{\mathfrak{M}_1}{\nabla}_{\dot{\alpha}} \dot{\alpha} = \kappa Y, \quad \overset{\mathfrak{M}_1}{\nabla}_{\dot{\alpha}} Y = -\kappa \dot{\alpha}, \tag{16}$$

where $Y = Y_s$ of unit vector along α and κ is the curvature of α on $\mathfrak{M}_1$. For each point $p \in \mathfrak{M}_1$, each orthonormal couple $u_1, u_2 \in \Gamma(ker\mathcal{T}_*)^\perp$ at p and each constant $\kappa > 0$, there is locally a unique horizontal circle $\alpha = \alpha(s)$ on $\mathfrak{M}_1$ with initial condition that $\alpha(0) = p$, $\dot{\alpha}(0) = u_1$ and $\overset{\mathfrak{M}_1}{\nabla}_{\dot{\alpha}} \dot{\alpha}(0) = \kappa u_2$ (see, [14, 19]).

Theorem 3.1. *Let $\mathcal{T} : \mathfrak{M}_1 \to \mathfrak{M}_2$ be a smooth map between Riemannian manifolds $(\mathfrak{M}_1, g_{\mathfrak{M}_1})$ and $(\mathfrak{M}_2, g_{\mathfrak{M}_2})$. The following are equivalent:*

(i) *$\mathcal{T}$ is $h-$isotropic Riemannian map,*
(ii) *There is $\kappa > 0$ satisfying that for each horizontal circle α with curvature κ on $\mathfrak{M}_1$, $\gamma = \mathcal{T} \circ \alpha$ on $\mathfrak{M}_2$ has constant curvature $\tilde{\kappa}$ along γ.*

Proof. Assume that $\mathcal{T} : \mathfrak{M}_1 \to \mathfrak{M}_2$ is $h-$isotropic Riemannin map. From (16) and (15), curvature of γ

$$\tilde{\kappa}(t) = \sqrt{\kappa^2 + \lambda^2} \tag{17}$$

is a constant. Conversely, we suppose that there is $\kappa > 0$ satisfying that for each circle α with curvature κ on $\mathfrak{M}_1$, the curve $\gamma = \mathcal{T} \circ \alpha$ on $\mathfrak{M}_2$ has constant first curvature $\tilde{\kappa}$ along this curve. Let $u_1, u_2 \in \Gamma(ker\mathcal{T}_*)^\perp$ be arbitrary orthonormal pair of vectors at $p \in \mathfrak{M}_1$. Suppose that $\alpha = \alpha(s)$, $s \in I$ be a circle with κ on $\mathfrak{M}_1$ with initial conditions $\alpha(0) = p$, $\dot{\alpha}(0) = u_1$ and $\overset{\mathfrak{M}_1}{\nabla}_{\dot{\alpha}} \dot{\alpha}(0) = \kappa u_2$. From (2), we have

$$\tilde{\kappa}^2 = \kappa^2 + \|(\nabla \mathcal{T}_*)(\dot{\alpha}, \dot{\alpha})\|^2.$$

Since $\tilde{\kappa}$ is a constant and α is a circle on $\mathfrak{M}_1$, $\|(\nabla \mathcal{T}_*)(\dot{\alpha}, \dot{\alpha})\|$ is a constant on I. Then, we have

$$0 = \nabla^{\mathcal{T}}_{\dot{\alpha}} \left(g_{\mathfrak{M}_2}((\nabla \mathcal{T}_*)(\dot{\alpha}, \dot{\alpha}), (\nabla \mathcal{T}_*)(\dot{\alpha}, \dot{\alpha})) \right) = 2 g_{\mathfrak{M}_2}((\nabla^{\mathcal{T}}_{\dot{\alpha}})^\perp (\nabla \mathcal{T}_*)(\dot{\alpha}, \dot{\alpha}), (\nabla \mathcal{T}_*)(\dot{\alpha}, \dot{\alpha})).$$

Using (16) and (6), we get

$$g_{\mathfrak{M}_2}(\tilde{\nabla}^{\mathcal{T}}_{\dot{\alpha}} (\nabla \mathcal{T}_*)(\dot{\alpha}, \dot{\alpha}), (\nabla \mathcal{T}_*)(\dot{\alpha}, \dot{\alpha})) + 2\kappa g_{\mathfrak{M}_2}((\nabla \mathcal{T}_*)(\dot{\alpha}, Y_s), (\nabla \mathcal{T}_*)(\dot{\alpha}, \dot{\alpha})) = 0. \tag{18}$$

Evaluating equation (18) at $s = 0$, we get

$$g_{\mathfrak{M}_2}(\tilde{\nabla}^{\mathcal{T}}_{u_1} (\nabla \mathcal{T}_*)(u_1, u_1), (\nabla \mathcal{T}_*)(u_1, u_1)) + 2\kappa g_{\mathfrak{M}_2}((\nabla \mathcal{T}_*)(u_1, u_2), (\nabla \mathcal{T}_*)(u_1, u_2)) = 0. \tag{19}$$

Also, for another horizontal circle $\beta = \beta(s)$ of the same curvature κ on $\mathfrak{M}_1$ with initial conditions $\beta(0) = p$, $\dot{\beta}(0) = u_1$ and $\overset{1}{\nabla}_{\dot{\beta}}\dot{\beta}(0) = -\kappa u_2$, we have

$$g_{\mathfrak{M}_2}(\tilde{\nabla}^{\mathfrak{T}}_{u_1}(\nabla\mathfrak{T}_*)(u_1,u_1),(\nabla\mathfrak{T}_*)(u_1,u_1)) - 2\kappa g_{\mathfrak{M}_2}((\nabla\mathfrak{T}_*)(u_1,u_2),(\nabla\mathfrak{T}_*)(u_1,u_1)) = 0, \quad (20)$$

which corresponds to (19). Then from (19) and (20), we obtain

$$\kappa g_{\mathfrak{M}_2}((\nabla\mathfrak{T}_*)(u_1,u_2),(\nabla\mathfrak{T}_*)(u_1,u_1)) = 0.$$

Taking into consideration Lemma 3.1, we can see that $\mathfrak{T}$ is $h-$ isotropic Riemannian map. □

Corollary 3.1. *A totally umbilical Riemannian map $\mathfrak{T}$ is h-isotropic at the point p_1. Conversely a h-isotropic Riemannian map $\mathfrak{T}$ is a totally umbilical at p_1 if it satisfies*

$$(\nabla\mathfrak{T}_*)(X,Y) = 0$$

for two orthonormal vector fields X and Y at p_1 in $\Gamma((\ker\mathfrak{T}_{*p_1})^{\perp})$.

Proof. Assume that $\mathfrak{T}$ is a totally umbilical Riemannian map at the point p_1. Then for $X_1, X_2 \in \Gamma((\ker\mathfrak{T}_{*p_1})^{\perp})$, we have

$$(\nabla\mathfrak{T}_*)(X,Y) = g_{\mathfrak{M}_1}(X,Y)H_2.$$

Especially, we obtain

$$(\nabla\mathfrak{T}_*)(X,Y) = 0,$$

if X and Y are orthogonal. Then $\mathfrak{T}$ is h-isotropic. Conversely we suppose that $\mathfrak{T}$ is h-isotropic Riemannian map at p_1. Then $\nabla\mathfrak{T}_*$ satisfies (14) for an arbitrary orthogonal couple $X, Y \in \Gamma((\ker\mathfrak{T}_{*p_1})^{\perp})$. Since $(\nabla\mathfrak{T}_*)(X,Y) = 0$ and $\nabla\mathfrak{T}_*$ is linear, we have for the orthogonal couple $\frac{1}{\sqrt{2}}(X+Y)$ and $\frac{1}{\sqrt{2}}(X-Y)$

$$(\nabla\mathfrak{T}_*)(X,X) = (\nabla\mathfrak{T}_*)(Y,Y).$$

Let $\{X_1, X_2, ...X_n\}$ denote an orthonormal frame in $\Gamma((\ker\mathfrak{T}_{*p_1})^{\perp})$, then we get

$$(\nabla\mathfrak{T}_*)(X_1,X_1) = (\nabla\mathfrak{T}_*)(X_2,X_2) = ... = (\nabla\mathfrak{T}_*)(X_n,X_n).$$

Thus we have

$$H_2 = (\nabla\mathfrak{T}_*)(X_1,X_1),$$

where H_2 is the mean curvature vector field of distribution $range\mathfrak{T}_*$. Moreover, choosing $X = \sum_i a_i X_i$ and $Y = \sum_j b_j X_j$, we obtain

$$0 = (\nabla\mathfrak{T}_*)(X,Y) = \sum_{i,j} a_i b_j (\nabla\mathfrak{T}_*)(X_i,X_i) = g_{\mathfrak{M}_1}(X,Y)H_2,$$

which shows that $\mathfrak{T}$ is umbilical. □

4. A characterization of Riemannian maps in terms of helices

We prove the following theorem which shows the effect of transforming helices to the base manifold along Riemannian maps in this section.

Theorem 4.1. *Let $\mathfrak{T}$ be a Riemannian map from a connected Riemannian manifold $(\mathfrak{M}_1, g_{\mathfrak{M}_1})$, $dim\mathfrak{M}_1 \geq 2$ to a Riemannian manifold $(\mathfrak{M}_2, g_{\mathfrak{M}_2})$. Let α be a horizontal helix with curvature κ and torsion τ on $\mathfrak{M}_1$, then $\mathfrak{T}$ is umbilical and the mean curvature vector field H_2 satisfies the following equation*

$$\left(\nabla^{\mathfrak{T}^{\perp}}_{\xi_s}\right)^2 H_2 = -\tau^2 H_2,$$

if and only if for every horizontal helix α on the base manifold $\mathfrak{M}_1$, the corresponding curve $\mathfrak{T}\circ\alpha$ is a helix on $\mathfrak{M}_2$.

Proof. We consider that $p \in \mathfrak{M}_1$ and $\alpha(s)$ is a horizontal helix with curvature κ and torsion τ on the base manifold $\mathfrak{M}_1$, $\mathcal{T} \circ \alpha : I \to \mathfrak{M}_2$ is the corresponding curve and we can define a vector field $\mathcal{T}_*\xi$ along $\mathcal{T} \circ \alpha$ by

$$\mathcal{T}_*\xi(s)) = \mathcal{T}_{*\alpha(s)}\xi(s),$$

for each vector field ξ_s along α, where ξ_s is the unit tangent vector field along α and s is the arc length parameter. Now we assume that $\mathcal{T} \circ \alpha$ is a helix with the curvature $\tilde{\kappa}$ and torsion $\tilde{\tau}$ on $\mathfrak{M}_2$. From (1), we have

$$\left(\overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}_{\xi_s}\right)^3 \mathcal{T}_*(\xi_s) + \tilde{K}^2 \overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}_{\xi_s} \mathcal{T}_*(\xi_s) = 0, \tag{21}$$

where $\tilde{K}^2 = \tilde{\kappa}^2 + \tilde{\tau}^2$. Using (2), we get

$$\overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}_{\xi_s} \mathcal{T}_*(\xi_s) = \mathcal{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{\xi_s} \xi_s) + (\nabla \mathcal{T}_*)(\xi_s, \xi_s). \tag{22}$$

From (2) and (22), we obtain

$$\overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}_{\xi_s} \mathcal{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{\xi_s} \xi_s) = \mathcal{T}_*((\overset{\mathfrak{M}_1}{\nabla}_{\xi_s})^2 \xi_s) + (\nabla \mathcal{T}_*)(\xi_s, \overset{\mathfrak{M}_1}{\nabla}_{\xi_s} \xi_s).$$

Using (22) and (4) in the last equation, we find

$$\begin{aligned}\left(\overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}_{\xi_s}\right)^2 \mathcal{T}_*(\xi_s) \;=\; & -S_{(\nabla \mathcal{T}_*)(\xi_s,\xi_s)} \mathcal{T}_*(\xi_s) + \nabla^{\mathcal{T}^\perp}_{\xi_s} (\nabla \mathcal{T}_*)(\xi_s, \xi_s) + \mathcal{T}_*((\overset{\mathfrak{M}_1}{\nabla}_{\xi_s})^2 \xi_s) \\ & + (\nabla \mathcal{T}_*)(\xi_s, \overset{\mathfrak{M}_1}{\nabla}_{\xi_s} \xi_s).\end{aligned} \tag{23}$$

Using (2) and (23), we arrive at

$$\begin{aligned}\left(\overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}_{\xi_s}\right)^3 \mathcal{T}_*(\xi_s) \;=\; & \mathcal{T}_*(\overset{\mathfrak{M}_1}{\nabla}{}^3_{\xi_s} \xi_s) + (\nabla \mathcal{T}_*)(\xi_s, ((\overset{\mathfrak{M}_1}{\nabla}_{\xi_s})^2 \xi_s)) - \overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}_{\xi_s} (S_{(\nabla \mathcal{T}_*)(\xi_s,\xi_s)} \mathcal{T}_*(\xi_s)) \\ & + \overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}_{\xi_s} (\nabla^{\mathcal{T}^\perp}_{\xi_s} (\nabla \mathcal{T}_*)(\xi_s, \xi_s)) + \overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}_{\xi_s} ((\nabla \mathcal{T}_*)(\xi_s, \overset{\mathfrak{M}_1}{\nabla}_{\xi_s} \xi_s)).\end{aligned}$$

So, we obtain from (2) and (4)

$$\begin{aligned}\left(\overset{\mathfrak{M}_2}{\nabla}{}^{\mathcal{T}}_{\xi_s}\right)^3 \mathcal{T}_*(\xi_s) = \; & \mathcal{T}_*(\overset{\mathfrak{M}_1}{\nabla}{}^3_{\xi_s} \xi_s) + (\nabla \mathcal{T}_*)(\xi_s, (\overset{\mathfrak{M}_1}{\nabla}_{\xi_s})^2 \xi_s) - S_{\nabla^{\mathcal{T}^\perp}_{\xi_s} (\nabla \mathcal{T}_*)(\xi_s,\xi_s)} \mathcal{T}_*(\xi_s) \\ & - (\nabla \mathcal{T}_*)(\xi_s, S_{(\nabla \mathcal{T}_*)(\xi_s,\xi_s)} \mathcal{T}_*(\xi_s)) - \mathcal{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{\xi_s} {}^*\mathcal{T}_* S_{(\nabla \mathcal{T}_*)(\xi_s,\xi_s)} \mathcal{T}_*(\xi_s)) \\ & + \left(\nabla^{\mathcal{T}^\perp}_{\xi_s}\right)^2 (\nabla \mathcal{T}_*)(\xi_s, \xi_s) - S_{(\nabla \mathcal{T}_*)(\xi_s, \overset{\mathfrak{M}_1}{\nabla}_{\xi_s} \xi_s)} \mathcal{T}_*(\xi_s) + \nabla^{\mathcal{T}^\perp}_{\xi_s} (\nabla \mathcal{T}_*)(\xi_s, \overset{\mathfrak{M}_1}{\nabla}_{\xi_s} \xi_s).\end{aligned} \tag{24}$$

Substituting (24) and (22) into (21), we obtain

$$\begin{aligned}& \mathcal{T}_*(\overset{\mathfrak{M}_1}{\nabla}{}^3_{\xi_s} \xi_s) + (\nabla \mathcal{T}_*)(\xi_s, \overset{\mathfrak{M}_1}{\nabla}{}^2_{\xi_s} \xi_s) - \mathcal{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{\xi_s} {}^*\mathcal{T}_* S_{(\nabla \mathcal{T}_*)(\xi_s,\xi_s)} \mathcal{T}_*(\xi_s)) \\ & - (\nabla \mathcal{T}_*)(\xi_s, S_{(\nabla \mathcal{T}_*)(\xi_s,\xi_s)} \mathcal{T}_*(\xi_s)) - S_{\nabla^{\mathcal{T}^\perp}_{\xi_s} (\nabla \mathcal{T}_*)(\xi_s,\xi_s)} \mathcal{T}_*(\xi_s) \\ & + \left(\nabla^{\mathcal{T}^\perp}_{\xi_s}\right)^2 (\nabla \mathcal{T}_*)(\xi_s, \xi_s) - S_{(\nabla \mathcal{T}_*)(\xi_s, \overset{\mathfrak{M}_1}{\nabla}_{\xi_s} \xi_s)} \mathcal{T}_*(\xi_s) \\ & + \nabla^{\mathcal{T}^\perp}_{\xi_s} (\nabla \mathcal{T}_*)(\xi_s, \overset{\mathfrak{M}_1}{\nabla}_{\xi_s} \xi_s) + \tilde{K}^2 \mathcal{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{\xi_s} \xi_s) + \tilde{K}^2 (\nabla \mathcal{T}_*)(\xi_s, \xi_s) = 0.\end{aligned} \tag{25}$$

By looking at $range\mathfrak{T}_*$ and $(range\mathfrak{T}_*)^\perp$ components of (25), we have

$$\begin{aligned}&\mathfrak{T}_*(\overset{\mathfrak{M}_1}{\nabla^3_{\xi_s}}\xi_s)+\tilde{K}^2\mathfrak{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{\xi_s}\xi_s)-\mathfrak{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{\xi_s}{}^*\mathfrak{T}_*S_{(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}\mathfrak{T}_*(\xi_s))\\&-S_{\nabla^{\mathfrak{T}\perp}_{\xi_s}(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}\mathfrak{T}_*(\xi_s)-S_{(\nabla\mathfrak{T}_*)(\xi_s,\overset{\mathfrak{M}_1}{\nabla}_{\xi_s}\xi_s)}\mathfrak{T}_*(\xi_s)=0\end{aligned}$$

and

$$\begin{aligned}&(\nabla\mathfrak{T}_*)(\xi_s,(\overset{\mathfrak{M}_1}{\nabla}_{\xi_s})^2\xi_s)-(\nabla\mathfrak{T}_*)(\xi_s,S_{(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}\mathfrak{T}_*(\xi_s))+\left(\nabla^{\mathfrak{T}^\perp}_{\xi_s}\right)^2(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)\\&+\nabla\xi_s^{\mathfrak{T}^\perp}(\nabla\mathfrak{T}_*)(\xi_s,\overset{\mathfrak{M}_1}{\nabla}_{\xi_s}\xi_s)+\tilde{K}^2(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)=0.\end{aligned}$$

Considering (7), we arrive at

$$\begin{aligned}\mathfrak{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{\xi_s}{}^*\mathfrak{T}_*S_{(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}\mathfrak{T}_*(\xi_s))&=\left(\tilde{\nabla}_{\xi_s}S\right)_{(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}\mathfrak{T}_*(\xi_s)\\&+S_{\nabla^{\mathfrak{T}\perp}_{\xi_s}(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}\mathfrak{T}_*(\xi_s)+S_{(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}P\overset{\mathfrak{M}_2}{\nabla^{\mathfrak{T}}_{\xi_s}}\mathfrak{T}_*(\xi_s).\end{aligned}\tag{26}$$

Using (26) and Frenet formulas in $range\mathfrak{T}_*$ components of (25) we get

$$\begin{aligned}&(\tilde{K}^2-\kappa^2-\tau^2)\kappa\mathfrak{T}_*(V_2)-\kappa S_{(\nabla\mathfrak{T}_*)(\xi_s,V_2)}\mathfrak{T}_*(\xi_s)-\left(\tilde{\nabla}_{\xi_s}S\right)_{(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}\mathfrak{T}_*(\xi_s)\\&-2S_{\nabla^{\mathfrak{T}\perp}_{\xi_s}(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}\mathfrak{T}_*(\xi_s)-S_{(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}P\overset{\mathfrak{M}_2}{\nabla^{\mathfrak{T}}_{\xi_s}}\mathfrak{T}_*(\xi_s)=0.\end{aligned}\tag{27}$$

From (6), we have

$$2\nabla^{\mathfrak{T}^\perp}_{\xi_s}(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)=2\left(\tilde{\nabla}_{\xi_s}(\nabla\mathfrak{T}_*)\right)(\xi_s,\xi_s)+4(\nabla\mathfrak{T}_*)(\xi_s,\overset{\mathfrak{M}_1}{\nabla}_{\xi_s}\xi_s).\tag{28}$$

Substituting (28) into (27), we find

$$\begin{aligned}&(\tilde{K}^2-\kappa^2-\tau^2)\kappa\mathfrak{T}_*(V_2)-5\kappa S_{(\nabla\mathfrak{T}_*)(\xi_s,V_2)}\mathfrak{T}_*(\xi_s)-S_{(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}P\overset{\mathfrak{M}_2}{\nabla^{\mathfrak{T}}_{\xi_s}}\mathfrak{T}_*(\xi_s)\\&=2S_{(\tilde{\nabla}_{\xi_s}(\nabla\mathfrak{T}_*))(\xi_s,\xi_s)}\mathfrak{T}_*(\xi_s)+\left(\tilde{\nabla}_{\xi_s}S\right)_{(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}\mathfrak{T}_*(\xi_s).\end{aligned}\tag{29}$$

Taking inner product with $\mathfrak{T}_*(\xi_s)$, both of two sides of (29) we obtain

$$\begin{aligned}&(\tilde{K}^2-\kappa^2-\tau^2)\kappa g_{\mathfrak{M}_2}(\mathfrak{T}_*(V_2),\mathfrak{T}_*(\xi_s))-5\kappa g_{\mathfrak{M}_2}(S_{(\nabla\mathfrak{T}_*)(\xi_s,V_2)}\mathfrak{T}_*(\xi_s),\mathfrak{T}_*(\xi_s))\\&-g_{\mathfrak{M}_2}(S_{(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}P\overset{\mathfrak{M}_2}{\nabla^{\mathfrak{T}}_{\xi_s}}\mathfrak{T}_*(\xi_s),\mathfrak{T}_*(\xi_s))=g_{\mathfrak{M}_2}(2S_{(\tilde{\nabla}_{\xi_s}(\nabla\mathfrak{T}_*))(\xi_s,\xi_s)}\mathfrak{T}_*(\xi_s),\mathfrak{T}_*(\xi_s))\\&+g_{\mathfrak{M}_2}(\left(\tilde{\nabla}_{\xi_s}S\right)_{(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}\mathfrak{T}_*(\xi_s),\mathfrak{T}_*(\xi_s)).\end{aligned}\tag{30}$$

Since α is a horizontal curve, we have

$$g_{\mathfrak{M}_2}(\mathfrak{T}_*(\xi_s),\mathfrak{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{\xi_s}\xi_s))=g_{\mathfrak{M}_1}(\xi_s,\overset{\mathfrak{M}_1}{\nabla}_{\xi_s}\xi_s)=\kappa g_{\mathfrak{M}_1}(\xi_s,V_2)=0.\tag{31}$$

Taking into consideration (31) and (5), (30) reduces to

$$\begin{aligned}&-5\kappa g_{\mathfrak{M}_2}((\nabla\mathfrak{T}_*)(\xi_s,V_2),(\nabla\mathfrak{T}_*)(\xi_s,\xi_s))-g_{\mathfrak{M}_2}(S_{(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}P\overset{\mathfrak{M}_2}{\nabla^{\mathfrak{T}}_{\xi_s}}\mathfrak{T}_*(\xi_s),\mathfrak{T}_*(\xi_s))\\&=2g_{\mathfrak{M}_2}((\tilde{\nabla}_{\xi_s}(\nabla\mathfrak{T}_*))(\xi_s,\xi_s),(\nabla\mathfrak{T}_*)(\xi_s,\xi_s))\\&\quad+g_{\mathfrak{M}_2}(\left(\tilde{\nabla}_{\xi_s}S\right)_{(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)}\mathfrak{T}_*(\xi_s),\mathfrak{T}_*(\xi_s)).\end{aligned}$$

Using (5) and (8), we get

$$\begin{aligned}&-5\kappa g_{\mathfrak{M}_2}((\nabla\mathfrak{T}_*)(\xi_s,V_2),(\nabla\mathfrak{T}_*)(\xi_s,\xi_s))-g_{\mathfrak{M}_2}((\nabla\mathfrak{T}_*)(\xi_s,\overset{\mathfrak{M}_1}{\nabla}_{\xi_s}\xi_s),(\nabla\mathfrak{T}_*)(\xi_s,\xi_s))\\&=3g_{\mathfrak{M}_2}((\tilde{\nabla}_{\xi_s}(\nabla\mathfrak{T}_*))(\xi_s,\xi_s),(\nabla\mathfrak{T}_*)(\xi_s,\xi_s)).\end{aligned}$$

By using (28), we obtain

$$-6\kappa g_{\mathfrak{M}_2}((\nabla \mathfrak{T}_*)(\xi_s, V_2), (\nabla \mathfrak{T}_*)(\xi_s, \xi_s)) = 3g_{\mathfrak{M}_2}(\nabla^{\mathfrak{T}^\perp}_{\xi_s}(\nabla \mathfrak{T}_*)(\xi_s, \xi_s), (\nabla \mathfrak{T}_*)(\xi_s, \xi_s)) - 6\kappa g_{\mathfrak{M}_2}((\nabla \mathfrak{T}_*)(\xi_s, V_2), (\nabla \mathfrak{T}_*)(\xi_s, \xi_s)).$$

Hence, we yield

$$\nabla^{\mathfrak{T}}_{\xi_s}\left(g_{\mathfrak{M}_2}((\nabla \mathfrak{T}_*)(\xi_s, \xi_s), (\nabla \mathfrak{T}_*)(\xi_s, \xi_s))\right) = 0.$$

So, we can see that

$$\|(\nabla \mathfrak{T}_*)(\xi_s, \xi_s)\| = const..$$

On the other hand, from (4.9), we infer

$$(\tilde{K}^2 - \kappa^2 - \tau^2)\kappa \mathfrak{T}_*(V_2) = 5\kappa S_{(\nabla \mathfrak{T}_*)(\xi_s, V_2)} \mathfrak{T}_*(\xi_s) + S_{(\nabla \mathfrak{T}_*)(\xi_s, \xi_s)} \overset{\mathfrak{M}_2}{P} \nabla^{\mathfrak{T}}_{\xi_s} \mathfrak{T}_*(\xi_s) \tag{32}$$

so that

$$g_{\mathfrak{M}_2}((\nabla \mathfrak{T}_*)(\xi_s, \xi_s), (\nabla \mathfrak{T}_*)(\xi_s, V_2)) = 0. \tag{33}$$

Taking derivative of (28) and using $(range\mathfrak{T}_*)^\perp$ components of (25) and (7), we obtain

$$\begin{aligned} 5\kappa(\tilde{\nabla}_{\xi_s}(\nabla \mathfrak{T}_*))(\xi_s, V_2) + 3\kappa^2(\nabla \mathfrak{T}_*)(V_2, V_2) + 4\kappa\tau(\nabla \mathfrak{T}_*)(\xi_s, V_3) \\ = (\nabla \mathfrak{T}_*)(\xi_s, S_{(\nabla \mathfrak{T}_*)(\xi_s, \xi_s)} \mathfrak{T}_*(\xi_s)) + 4\kappa^2(\nabla \mathfrak{T}_*)(\xi_s, \xi_s) \\ -\tilde{\nabla}^2_{\xi_s}(\nabla \mathfrak{T}_*))(\xi_s, \xi_s) - \tilde{K}^2(\nabla \mathfrak{T}_*)(\xi_s, \xi_s). \end{aligned} \tag{34}$$

Changing V_3 into $-V_3$ into (34) we arrive at

$$(\nabla \mathfrak{T}_*)(\xi_s, V_3) = 0. \tag{35}$$

From (4.13), (35) and Corollary 3.1, we have that $\mathfrak{T}$ is umbilical map. If we change V_2 with $-V_2$ into (34), we get

$$5(\tilde{\nabla}_{\xi_s}(\nabla \mathfrak{T}_*))(\xi_s, V_2) = 0. \tag{36}$$

Taking inner product (32) with $\mathfrak{T}_*(V_2)$, we have

$$\begin{aligned} (\tilde{K}^2 - \kappa^2 - \tau^2)\kappa &= 5\kappa g_{\mathfrak{M}_2}((\nabla \mathfrak{T}_*)(\xi_s, V_2), (\nabla \mathfrak{T}_*)(\xi_s, V_2)) \\ &\quad +\kappa g_{\mathfrak{M}_2}((\nabla \mathfrak{T}_*)(\xi_s, \xi_s), (\nabla \mathfrak{T}_*)(V_2, V_2)), \end{aligned}$$

that is

$$(\tilde{K}^2 - \kappa^2 - \tau^2) = \|H_2\|^2. \tag{37}$$

Substituting (35) and (36) into (34), we get

$$\begin{aligned} 3\kappa^2(\nabla \mathfrak{T}_*)(V_2, V_2) = (\nabla \mathfrak{T}_*))(\xi_s, S_{(\nabla \mathfrak{T}_*)(\xi_s, \xi_s)} \mathfrak{T}_*(\xi_s)) \\ +4\kappa^2(\nabla \mathfrak{T}_*)(\xi_s, \xi_s) - \tilde{\nabla}^2_{\xi_s}(\nabla \mathfrak{T}_*))(\xi_s, \xi_s) - \tilde{K}^2(\nabla \mathfrak{T}_*)(\xi_s, \xi_s). \end{aligned}$$

From umbilicity and derivative of (28), we have

$$\begin{aligned} 3\kappa^2 H_2 = (\nabla \mathfrak{T}_*))(\xi_s, S_{(\nabla \mathfrak{T}_*)(\xi_s, \xi_s)} \mathfrak{T}_*(\xi_s)) + 4\kappa^2 H_2 - (\nabla^{\mathfrak{T}^\perp}_{\xi_s})^2 H_2 \\ +2\kappa^2 H_2 - 2\kappa^2 H_2 - \tilde{K}^2 H_2. \end{aligned}$$

From (5), we can see that

$$(\tilde{K}^2 - \kappa^2)H_2 = g_{\mathfrak{M}_2}((\nabla \mathfrak{T}_*)(\xi_s, \xi_s), (\nabla \mathfrak{T}_*)(\xi_s, \xi_s))H_2 - (\nabla^{\mathfrak{T}^\perp}_{\xi_s})^2 H_2.$$

So, we get

$$(\tilde{K}^2 - \kappa^2 - \|H_2\|^2)H_2 = -(\nabla^{\mathfrak{T}^\perp}_{\xi_s})^2 H_2.$$

Using (37), we have

$$(\nabla^{\mathfrak{T}^\perp}_{\xi_s})^2 H_2 = -\tau^2 H_2.$$

Conversely, we assume that $\mathfrak{T}$ is a umbilical map and mean curvature vector field satisfies $\left(\nabla^{\mathfrak{T}^{\perp}}_{\xi_s}\right)^2 H_2 = -\tau^2 H_2$. Then we calculate

$$\left(\overset{\mathfrak{M}_2}{\nabla}{}^{\mathfrak{T}}_{\xi_s}\right)^3 \mathfrak{T}_*(\xi_s) = \mathfrak{T}_*((\overset{\mathfrak{M}_1}{\nabla}_{\xi_s})^3\xi_s) - \mathfrak{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{\xi_s} {}^*\mathfrak{T}_*(S_{H_2}\mathfrak{T}_*(\xi_s))) \\ -(K^2 + \|H_2\|^2)H_2 - S_{\nabla^{\mathfrak{T}^{\perp}}_{\xi_s} H_2}\mathfrak{T}_*(\xi_s) + \left(\nabla^{\mathfrak{T}^{\perp}}_{\xi_s}\right)^2 H_2. \quad (38)$$

If we use (38), we have

$$\left(\overset{\mathfrak{M}_2}{\nabla}{}^{\mathfrak{T}}_{\xi_s}\right)^3 \mathfrak{T}_*(\xi_s) + (\kappa^2 + \tau^2 + \|H_2\|^2)\overset{\mathfrak{M}_2}{\nabla}{}^{\mathfrak{T}}_{\xi_s}\mathfrak{T}_*(\xi_s) = \mathfrak{T}_*((\overset{\mathfrak{M}_1}{\nabla}_{\xi_s})^3\xi_s) \\ -\mathfrak{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{\xi_s} {}^*\mathfrak{T}_*(S_{H_2}\mathfrak{T}_*(\xi_s))) - S_{\nabla^{\mathfrak{T}^{\perp}}_{\xi_s} H_2}\mathfrak{T}_*(\xi_s) + (\kappa^2 + \tau^2 + \|H_2\|^2)\mathfrak{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{\xi_s}\xi_s).$$

Then we get

$$\mathfrak{T}_*(\left(\overset{\mathfrak{M}_1}{\nabla}_{\xi_s}\right)^3 \xi_s) + (\kappa^2 + \tau^2)\mathfrak{T}_*(\overset{\mathfrak{M}_1}{\nabla}_{\xi_s}\xi_s) = 0.$$

Since α is a horizontal helix on $\mathfrak{M}_1$, then we find $\gamma = \mathfrak{T} \circ \alpha$ is a helix on $\mathfrak{M}_2$. □

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