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IDEAL AMENABILITY OF FRÉCHET ALGEBRAS

A. Ranjbari[1], A. Rejali[2]

The concept of ideal amenability were first introduced and studied for Banach algebras. In this paper, this concept is generalized for Fréchet algebras. Some hereditary properties of ideally amenability of Fréchet algebras are investigated.

Keywords: Fréchet Algebras, Amenability, Locally convex algebra , Ideal amenability of Fréchet Algebras.

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1. Introduction

Some of the notions related to Banach algebras, have been introduced and studied for Fréchet algebras. For example, the notion of amenability of a Fréchet algebra was introduced by Helemskii [19] and studied by A. Yu. Pirkovskii [23]. He generalized some theorems about amenability of Banach algebras such as strictly flat Banach A-bimodule, virtual diagonal and approximate diagonal of Banach algebras, to Fréchet algebras. Also in [21], P. Lawson and C. J. Read introduced and studied some notions about approximate amenability and approximate contractibility of Fréchet algebras. Furthermore in [1], Abtahi and et al introduced and studied the notion of weak amenability of Fréchet algebras. Moreover, according to the basic definition of Segal algebras [24] and abstract Segal algebras [7], recently they introduced the notion of Segal- Fréchet algebra in the Fréchet algebra $(\mathcal{A}, p_\ell)$. They also showed that every continuous linear left multiplier of a Fréchet algebra $(\mathcal{A}, p_\ell)$ is also a continuous linear left multiplier of any Segal Fréchet algebra in $(\mathcal{A}, p_\ell)$. Furthermore they showed that if $\mathcal{A}$ is a commutative Fréchet $Q-$algebra , then the space of all modular maximal closed ideals of $\mathcal{A}$ and any Segal Fréchet algebra $(\mathcal{B}, q_m)$ in $(\mathcal{A}, p_\ell)$ are homeomorphic. Particularly, they obtained that $(\mathcal{A}, p_\ell)$ is semisimple if and only if $(\mathcal{B}, q_m)$ is semisimple; see [2] and [3] for more information.

In [15], Gorgi and Yazdanpanah, introduced and studied the notions of I-weak amenability and also ideal amenability of Banach algebras, where I is a closed two-sided ideal in a Banach algebra $\mathcal{A}$. Mainly, they showed that these concepts are different from amenability and weak amenability of Banach algebras.

In this paper we introduce and study the notion of $I-$weak amenability and also ideal amenability of Fréchet algebra. We also investigate some hereditary properties of ideally amenable Fréchet algebras.

2. Preliminaries

In this section, we recall and review some of the basic terminologies about Fréchet algebras. For further details, see [12], [19] and [25].

A locally convex space (LCS) is a topological vector space such that every open set containing the origin contains an open convex set containing the origin (every neighborhood $U(0)$ contains an open convex neighborhood $V(0)$ of 0). Throughout the paper, all locally

[1]Department of Mathematics, University of "ISFAHAN", Isfahan, Iran, E-mail: aranjbari1353@yahoo.com
[2]Department of Mathematics, University of "ISFAHAN", Isfahan, Iran, E-mail: rejali@sci.ui.ac.ir

convex spaces are assumed to be Hausdorff. A Fréchet space is a complete locally convex space, whose topology is given by a countable fundamental system of seminorms $\{p_n\}_{n\in\mathbb{N}}$; see [22] for more information. Note that by passing over to $(max_{1\leq j\leq n}p_j)_{n\in\mathbb{N}}$, one may assume that $\{p_n\}_{n\in\mathbb{N}}$ is a increasing sequence.

A set $\mathcal{P}$ of continuous seminorms on the locally convex space E is called fundamental system if for every continuous seminorm q there is $p\in\mathcal{P}$ and $C>0$ so that $q\leq C\cdot p$. By [22, Lemmas 22.4,22.5], every locally convex space E has a fundamental system of seminorms $(p_\alpha)_{\alpha\in A}$; equivalently a family of the seminorms satisfying the following properties:

(i) For every $x\in E$ with $x\neq 0$, there exists an $\alpha\in A$ with $p_\alpha(x)>0$;
(ii) For all $\alpha,\beta\in A$, there exists $\gamma\in A$ and $C>0$ such that

$$max(p_\alpha(x),p_\beta(x))\leq Cp_\gamma(x) \qquad (x\in E).$$

A topological algebra $\mathcal{A}$ is an algebra, which is a topological vector space and the multiplication

$$\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{A}, \quad (a,b)\mapsto ab$$

is a separately continuous mapping. A Fréchet algebra is a complete topological algebra, whose topology is given by the countable family of increasing submultiplicative seminorms.

Let $\mathcal{A}$ be a Fréchet algebra , A Fréchet space X is called a Fréchet $\mathcal{A}$-bimodule, if X is an algebraic $\mathcal{A}$-bimodule and the actions on both sides are continuous; in the other words if $(\mathcal{A},(p_l))$ be a Fréchet algebra then a Fréchet space $(X,(q_m))$ is called a Fréchet $\mathcal{A}$-bimodule Whenever for all $m\in\mathbb{N}$ there exists $c_m>0$ and m_0; $l_m\in\mathbb{N}$ such that for all $a\in\mathcal{A}$ and $x\in X$:

$$q_m(a\cdot x)\leq c_m p_{l_m}(a)q_{m_0}(x)$$

and

$$q_m(x\cdot a)\leq c_m p_{l_m}(a)q_{m_0}(x)$$

Let $(\mathcal{A},(p_l))$ be a Fréchet algebra, and X be a locally convex $\mathcal{A}$-bimodule, then a *derivation* from $\mathcal{A}$ into X is a continuous linear mapping $D:\mathcal{A}\to X$ such that for all $a,b\in\mathcal{A}$, $D(ab)=a\cdot D(b)+D(a)\cdot b$.

If $x\in X$, we define $ad_x:\mathcal{A}\to X$ by $ad_x(a)=a\cdot x-x\cdot a$ for each $a\in\mathcal{A}$. Then ad_x is a derivation. Such derivations are called *inner derivations*.

A subset M of a linear space E is called *absorbant* if $U_{t>0}tM=E$.

A topological vector space E is called *barrelled* if every closed, absolutely convex, absorbant set (“*barrel*”) is a neighborhood of zero.

A Hausdorff LCS E is *quasinormable* if for each 0-neighborhood $U\subset E$ there exists a 0-neighborhood $V\subset U$ such that for each $\epsilon>0$ there exists a bounded set $B\subset E$ satisfying $V\subset B+\epsilon U$.

The Fréchet algebra $(\mathcal{A},(p_l))$ is called *amenable* (weak *amenable*) if for each locally convex A-bimodule X every continuous derivation $D:\mathcal{A}\to X^*$ $(D:\mathcal{A}\to\mathcal{A}^*)$ is inner.

We know from [17] that if X is a Fréchet space then the dual space X^* of X will be endowed with the strong topology, which means the topology of uniform convergence on bounded subsets of X, i.e.:

$$p_B(y)=\sup_{x\in B}|y(x)|$$

where B runs through the bounded subsets of X.

Note that this coincides with the usual norm topology of X, if X is a normed space. Moreover X^* is a locally convex space. Note that give a Fréchet $\mathcal{A}$-bimodule X, X^* is a locally convex $\mathcal{A}$-bimodule with the continuous actions in the usual way. In fact these actions are separately continuous. By [23], the action of $\mathcal{A}$ on X^* often fails to be jointly continuous. X^* is a locally convex space, and X^{**} is a Fréchet space. Also, X can be continuously embedded in X^{**} via the usual injection $\iota : X \longrightarrow X^{**}$, and $\iota(X)$ is weak*-dense in X^{**}. Also note that given a Fréchet $\mathcal{A}$-bimodule X, X^* is a locally convex bimodule with continuous actions in the usual way.

3. **Ideal Amenability of Fréchet Algebras**

Let $(\mathcal{A}, (p_\ell))$ be a Fréchet algebra and I be a closed (two-sided) ideal in $\mathcal{A}$. Similar to the Banach algebra case, we say that $\mathcal{A}$ is $I-$weakly amenable if every continuous derivation $D : \mathcal{A} \longrightarrow I^*$ is inner. Moreover we introduce the concept of ideal amenability for Frćhet algebras as the following:

Definition 3.1. *A Fréchet algebra $\mathcal{A}$ is called ideally amenable if it is $I-$weakly amenable, for every closed (two-sided) ideal I in $\mathcal{A}$.*

Recall that if $(\mathcal{A}, (p_\ell))$ is a Fréchet algebra and B and C are non-empty subsets of $\mathcal{A}$, then $B \cdot C = \{bc : b \in B, c \in C\}$, and BC is the linear span of $B \cdot C$ and we write B^2 for BB. Also $\mathcal{A}$ is called essential if $\overline{\mathcal{A}^2} = \mathcal{A}$. Moreover, for $\phi \in \sigma(\mathcal{A})$, the set consisting of all non-zero continuous characters on $\mathcal{A}$, a point derivation d at ϕ is a linear functional satisfying

$$d(xy) = d(x)\phi(y) + \phi(x)d(y),$$

where $x, y \in \mathcal{A}$. Indeed, d is a derivation into the $\mathcal{A}$-bimodule $\mathbb{C}$, where the module actions is defined by $x \cdot \lambda = \lambda \cdot x = \lambda\phi(x)$, $x \in \mathcal{A}$, $\lambda \in \mathbb{C}$.

We commence with the following result. Since ideal amenability implies weak amenability, thus this result is immediately obtained from [1, Theorem 2.3].

Proposition 3.1. *Let $(\mathcal{A}, p_\ell)$ be an ideally amenable Fréchet algebra. Then the following assertions hold:*

(i) *$\mathcal{A}$ is essential.*
(ii) *There are no non-zero, continuous point derivations on $\mathcal{A}$.*
(iii) *In the case where $\mathcal{A}$ is commutative, each continuous derivation from $\mathcal{A}$ into E is zero, for each commutative complete barrelled locally convex $\mathcal{A}$-bimodule E.*

The following example shows that the converse of part (i) in the previous proposition is not necessarily valid.

Example 3.1. *The space $C^\infty([a, b])$ of infinitely differentiable complex-valued functions on a (finite) interval $[a, b]$ in $\mathbb{R}$ is not a Banach space. However, with seminorms*

$$\mu_k(f) = \sup_{0 \le i \le k} \sup_{x \in [a,b]} |f^{(i)}(x)|$$

and metric

$$d(f, g) = \sum_{k=0}^{\infty} 2^{-k} \frac{\mu_k(f - g)}{\mu_k(f - g) + 1}$$

$C^\infty([a, b])$ is a commutative Fréchet algebra. As $C^\infty([a, b])$ is unital, it follows that it is essential. However $C^\infty([a, b])$ is not ideally amenable. In fact the continuous map $D : C^\infty([a, b]) \longrightarrow \mathbb{C}$, defined as

$$D(f) = f'(x_0) \qquad (x_0 \in (a, b))$$

is non-zero. However, D is point derivation at $\phi = \delta_{x_0}$. Thus D is a non-zero, continuous point derivation on $C^\infty([a, b])$. Now [1, Theorem 2.3] *implies that $C^\infty([a, b])$ is not weakly amenable and so it is not ideally amenable.*

The following Example show that if I is a closed (two-sided) ideal in $\mathcal{A}$ and $\mathcal{A}$ be I-weakly amenable Fréchet algebra then $\mathcal{A}$ does not have to be weakly amenabe.

Example 3.2. *Let $\mathcal{A} = C^{\infty}([a,b])$ and $I := \{0\}$. Then I is a closed (two-sided) ideal in $\mathcal{A}$. It is clear that every continuous derivation $D : \mathcal{A} \longrightarrow I^*$ is zero and so it is inner. It follows that $\mathcal{A}$ is I-weakly amenable. However, by Example 3.1 $\mathcal{A} = C^{\infty}([a,b])$ is not weakly amenable.*

The following result is a generalization of [15, Lemma 2.1] to the Frćhet case.

Proposition 3.2. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra such that every closed (two-sided) ideal in $\mathcal{A}$ be weakly amenable. Then $\mathcal{A}$ is a ideally amenable.*

Proof. Suppose that I is a closed (two-sided) ideal in $\mathcal{A}$, $D : \mathcal{A} \longrightarrow I^*$ is a derivation and $i : I \longrightarrow \mathcal{A}$ is the embedding map. Then $d = D|_I : I \longrightarrow I^*$ is a derivation. By the hypothesis, there exists $m \in I^*$ such that $d = ad_m$. On the other hand for $i, j \in I$ and $a \in \mathcal{A}$ we have,

$$\begin{aligned}
< ij, D(a) > &=< i, jD(a) > \\
&=< i, D(ja) - D(j)a > \\
&=< i, ja \cdot m - m \cdot ja > - < ai, j \cdot m - m \cdot j > \\
&=< ij, am > - < ai, mj > - < ai, jm > + < ai, mj > \\
&=< ij, am > - < aij, m > \\
&=< ij, a \cdot m - m \cdot a > \\
&=< ij, ad_m(a) >
\end{aligned}$$

Since I is weakly amenable, [1, Theorem 2.3] implies that $\overline{I^2} = I$. Therefore $D = ad_m$, and so D is inner. □

We generalize [15, Theorem 1.5] to Frćhet algebras. The proof is similar and so will be omitted.

Proposition 3.3. *Let $(\mathcal{A}, p_\ell)$ be a weakly amenable Fréchet algebra such that for each closed (two-sided) ideal I with $I = \overline{\mathcal{A}I \cup I\mathcal{A}}$, it is I-weakly amenable. Then $\mathcal{A}$ is ideally amenable.*

Remark 3.1. *In general, the equality $I = \overline{\mathcal{A}I \cup I\mathcal{A}}$ is not occurred for an arbitrary ideal I of a Frćhet algebra. For example let $(\mathcal{A}, (P_l))$ be a Fréchet algebra with $xy = 0$, for all $x, y \in \mathcal{A}$. Let I be any non-zero closed ideal of $\mathcal{A}$.Then $I\mathcal{A} = \{0\}$, whereas $I \neq \overline{\mathcal{A}I \cup I\mathcal{A}}$.*

The following result is a generalization of [15, Theorem 1.4], for Fréchet algebras.

Theorem 3.1. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra and I be a closed (two-sided) ideal in $\mathcal{A}$ such that $\mathcal{A}$ be I-weakly amenable. Suppose that $\phi \in \sigma(\mathcal{A})$, such that $I \not\subseteq ker(\phi)$. Then there is no non-zero point derivation at ϕ.*

Proof. We follow the proof of [15, Theorem 1.4]. If $d : \mathcal{A} \longrightarrow \mathbb{C}_\phi$ is a non-zero continuous point derivation at ϕ, and let $\pi : \mathcal{A}^* \longrightarrow I^*$ be the adjoint of $i : I \longrightarrow \mathcal{A}$. Consider the map $D : \mathcal{A} \longrightarrow I^*$, defined by $D(a) = d(a)\pi(\phi)$. It is easy to see that D is a derivation. We show that D is continuous. Let $a \in \mathcal{A}$ and $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ such that $a_n \longrightarrow a$, in the topology of $\mathcal{A}$. Since $d \in \mathcal{A}^*$ is continuous, $d(a_n) \longrightarrow d(a)$, in the topology of $\mathbb{C}$. Since I^* is a $\mathbb{C}$-bimodule and $\pi(\phi) \in I^*$, thus $d(a_n)\pi(\phi) \longrightarrow d(a)\pi(\phi)$. It follows that $D(a_n) \longrightarrow D(a)$, and so D is continuous. By the hypothesis there exists $\lambda \in I^*$ such that $D(a) = a \cdot \lambda - \lambda \cdot a$ $(a \in \mathcal{A})$. Since $I \not\subseteq ker(\phi)$, take $i \in I$ with $\phi(i) = 1$. If $ker(\phi) \subseteq ker(d)$, then there exists $0 \neq \alpha \in \mathbb{C}$ such that $d = \alpha\phi$. Thus

$$2\alpha = 2\alpha\phi(i) = 2d(i) = 2d(i)\phi(i) = d(i^2) = \alpha\phi(i^2) = \alpha.$$

It follows that $\alpha = 0$, which is a contradiction. Consequently $ker(\phi) \not\subseteq ker(d)$. Thus there exists $a \in ker(\phi)$ with $d(a) = 1$. Set $i' = i + (1 - d(i))ia = i + ia - d(i)ia$. Then

$\phi(i') = d(i') = 1$ and so

$$1 = (D(i'))(i') =< i', i' \cdot \lambda > - < i', \lambda \cdot i' >=< {i'}^2, \lambda > - < {i'}^2, \lambda >= 0,$$

is a contradiction. □

The following proposition provides us a sufficient condition, under which ideal amenability of $\mathcal{A}$ (commutative Fréchet algebra) is obtained. This is similar to [1, Proposition 2.4]. The proof is similar and so is left to the readers. First, recall that an element p of $\mathcal{A}$ is called an idempotent if $p^2 = p$.

Proposition 3.4. *Let $(\mathcal{A}, p_\ell)$ be a commutative Fréchet algebra. Suppose that $\mathcal{A}$ is spanned by its idempotents. Then $\mathcal{A}$ is ideally amenable.*

We give an example of a Fréchet algebra which is ideally amenable, but not quasinormable. First, recall that a Hausdorff LCS is called quasinormable for every zero neighborhood U, there exists a zero neighborhood $V \subseteq U$ such that for every $\epsilon > 0$ there exists a bounded set B satisfying $V \subseteq B + \epsilon V$. Moreover we say that a topological algebra $\mathcal{A}$

(i) has a right (respectively, left) locally bounded approximate identity if For each 0-neighborhood $U \subset \mathcal{A}$ there exists $C > 0$ such that for each finite subset $F \subset \mathcal{A}$ there exists $b \in CU$ with $a - ab \in U$ (respectively, $a - ba \in U$) for all $a \in F$;

(ii) has a two-sided locally bounded approximate identity (or just a locally bounded approximate identity) if for each 0-neighborhood $U \subset \mathcal{A}$ there exists $C > 0$ such that for each finite subset $F \subset \mathcal{A}$ there exists $b \in CU$ with $a - ab \in U$ and $a - ba \in U$ for all $a \in F$.

Example 3.3. *Let P be a family of real - valued sequences such that for each $i \in \mathbb{N}$ there exists $p = (p_n) \in P$ with $p_i > 0$. Suppose also that P is directed, i.e., for each $p, q \in P$, there exists $r \in P$ such that $r_i \geq max\{p_i, q_i\}$ for all $i \in \mathbb{N}$. Recall that the Köthe sequence space $\lambda(P)$ is defined as follows:*

$$\lambda(P) = \{a = (a_i)_{i \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|a\|_p = \sum_{i=1}^{\infty} |a_i| p_i < \infty \quad \forall p \in P\}.$$

In the sequel, for each $i \in \mathbb{N}$ we set $e_i = (0, \cdots, 0, 1, 0, \cdots)$, where the single nonzero entry is in the ith slot. The linear span of the e_i's is denoted by C_{00}. Now Suppose that $p_i \geq 1$ for eachi $\in \mathbb{N}$. Then there exists a unique continuous product on $\lambda(P)$ such that $e_i e_j = e_{min\{i,j\}}$ for all $i, j \in \mathbb{N}$. then by [21, Lemma 10.1] *Together with this product, $\lambda(P)$ becomes a commutative locally convex algebra. Now for each $k \in \mathbb{N}$ we define an infinite matrix $\alpha^{(k)} = (\alpha_{ij}^{(k)})_{i,j \in \mathbb{N}}$ by setting*

$$\alpha_{ij}^{(k)} = \begin{cases} j^k & i < k \\ i^k & i \geq k. \end{cases}$$

Fix a bijection $\phi : \mathbb{N}^2 \longrightarrow \mathbb{N}$ such that $\phi(i, j+1) < \phi(i, j)$ for all $(i, j \in \mathbb{N})$. For each $k \in \mathbb{N}$ define a sequence $p^{(k)} = (p_n^{(k)})_{n \in \mathbb{N}}$ by $p_n^{(k)} = \alpha_{\phi^{-1}(n)}^{(k)}$. Finally, set $P = \{p^{(k)} : k \in \mathbb{N}\}$. Since P is countable, $\lambda(P)$ is a Fréchet algebra, and it is produced by idempotent elements e_i then by Proposition 3.4 the Fréchet algebra $\lambda(P)$ is ideally amenable. By [23, Lemma 10.4] *the Fréchet algebra $\lambda(P)$ has a locally bounded approximate identity but does not have a bounded approximate identity. Therefore, by* [23, Theorem 8.4]*, $\lambda(p)$ is not quasinormable.*

The following examples show that ideal amenability is not equivalent to weak amenability or amenability.

Example 3.4. *Let $s \subseteq \mathbb{C}^{\mathbb{N}}$ be the vector space of all complex sequences which tend to zero faster than any polynomial, i.e let*

$$s = \{x = (x_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : k^n|x_k| \longrightarrow 0 \ \textit{ as } \ k \longrightarrow \infty, \textit{for all } n \in \mathbb{N}\}.$$

From now on, $(x_k)_{k\in\mathbb{N}}$ will be denoted x. The vector space s becomes a commutative algebra, equipped with pointwise multiplication, and it is a Fréchet algebra with respect to the seminorms $(p_n)_{n\in\mathbb{N}}$, defined as

$$p_n(x) = \sup\{k^n|x_k| : k > 0\} \qquad (n \in \mathbb{N}).$$

It is clear that s does not have a locally bounded approximate identity. Then by [23, Theorem 9.7]*, it is not amenable. However it is produced by its idempotent elements e_i then so by Proposition 3.4 s is ideally amenable.*

4. **Hereditary properties**

Similar to Banach algebras, there is a canonical way of adjoining an identity to a Fréchet algebra $(\mathcal{A}, p_\ell)$, which is without unit. It is the direct sum $\mathcal{A}^\sharp = \mathcal{A} \oplus \mathbb{C}$, which is an algebra under the following product

$$(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu) \qquad (a, b \in \mathcal{A};\ \lambda, \mu \in \mathbb{C}).$$

We endow $\mathcal{A}^\sharp$ with the topology which is generated by the multiplicative seminorms $(q_\ell)_\ell$, defined by

$$q_\ell(a, \lambda) = p_\ell(a) + |\lambda|.$$

Then $\mathcal{A}^\sharp$ becomes a Fréchet algebra with the identity $(0, 1)$. Moreover if we identify every element $a \in \mathcal{A}$ as $(a, 0) \in \mathcal{A}_e$, then one can consider $\mathcal{A}$ as a closed ideal of $\mathcal{A}^\sharp$.

Proposition 4.1. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra. Then $\mathcal{A}$ is ideally amenable if and only if $\mathcal{A}^\sharp$ is ideally amenable.*

Proof. The proof is very similar to [15, Proposition 1.14] and is left to the readers. □

Suppose that I is a proper closed ideal of the Fréchet algebra $(\mathcal{A}, p_\ell)$. Then $\frac{\mathcal{A}}{I}$ endowed with the quotient topology is a Fréchet space and the topology is defined by the seminorms

$$\hat{p}_\ell(a + I) = \inf\{p_\ell(a + b) : \quad b \in I\}.$$

It is easy to show that the multiplication is continuous, and $\frac{\mathcal{A}}{I}$ is a topological algebra. Moreover, each $\hat{p}_n$ is submultiplicative and so $(\frac{\mathcal{A}}{I}, \hat{p}_\ell)$ is a Fréchet algebra; see [12, 3.2.10]. Recall that a linear functional ϕ on the Fréchet algebra $\mathcal{A}$ is called a trace if $\phi(ab) = \phi(ba)$, for all $a, b \in \mathcal{A}$.

Definition 4.1. *Let I be a closed ideal in a Fréchet algebra $\mathcal{A}$. Then I has the trace extension property if, for each $\lambda \in I^*$ with $a \cdot \lambda = \lambda \cdot a$ $(a \in \mathcal{A})$, there exists a continuous trace, ϕ on $\mathcal{A}$ such that $\phi|_{\mathcal{A}} = \lambda$.*

Let I be a closed two-sided ideal in Fréchet algebra $(\mathcal{A}, p_\ell)$. Abtahi has et al. in [1] showed that, if $\frac{\mathcal{A}}{I}$ and I are weakly amenable, and I be a quasinormable then $\mathcal{A}$ is weakly amenable. Also if I has the trace extension property and $\mathcal{A}$ is weakly amenable, then $\frac{\mathcal{A}}{I}$ is weakly amenable. We prove a similar proposition for ideal amenability.

Proposition 4.2. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra and I be a quasinormable closed (two-sided) ideal in $\mathcal{A}$.*

(i) *Suppose that* $\dfrac{\mathcal{A}}{I}$ *is ideally amenable. Then* I *has the trace extension property.*

(ii) *Suppose that* $\mathcal{A}$ *is ideally amenable and* I *has the trace extension property. Then* $\dfrac{\mathcal{A}}{I}$ *is ideally amenable.*

Proof. (i). It is obtained from [1, Proposition 2.10].

(ii). Let K be a closed two-sided ideal in $\frac{\mathcal{A}}{I}$. Then there is a closed (two-sided) ideal J in $\mathcal{A}$ such that $K = \dfrac{J}{I}$. Let $D : \dfrac{\mathcal{A}}{I} \longrightarrow K^*$ be a derivation. We write $\pi : J \longrightarrow \dfrac{J}{I}$, $q : \mathcal{A} \longrightarrow \dfrac{\mathcal{A}}{I}$ for the natural quotient maps and $\pi^* : K^* \longrightarrow J^*$ for the adjoint of π. Similar to [22, Proposition 23.30], $\tilde{D} = \pi^* oDoq : \mathcal{A} \longrightarrow J^*$ is a continuous derivation and so there exists $\lambda \in J^*$ such that $\tilde{D}(a) = a \cdot \lambda - \lambda \cdot a$, $(a \in \mathcal{A})$. Let m be the restriction of λ to I, i.e. $m := \lambda|_I$. Then $m \in I^*$ and for each $i \in I$

$$\begin{aligned} < i, am - ma >=< ia - ai, m > &= < ia - ai, \lambda >=< i, a\lambda - \lambda a > \\ &= < i, ad_\lambda(a) >=< i, \tilde{D}(a) > \\ &= < i, (\pi^* oDo\pi)(a) >=< \pi(i), (Do\pi)(a) > \\ &= < I, D(I + a) >= 0. \end{aligned}$$

It follows that $am = ma$ $(a \in \mathcal{A})$, and since I has the trace extension property, there exists $\tau \in \mathcal{A}^*$ such that $a \cdot \tau = \tau \cdot a$, $(a \in \mathcal{A})$ and $\tau|_I = m$. Let μ be the restriction of τ to J. Then $\mu = \tau|_J \in J^*$ and $\lambda - \mu = 0$ on I. Thus $\lambda - \mu \in (\dfrac{J}{I})^*$. Moreover it is not hard to see that

$$D(a + I) = a \cdot (\lambda - \tau) - (\lambda - \tau) \cdot a \qquad (a \in \mathcal{A}),$$

and so D is inner. Therefore $\dfrac{\mathcal{A}}{I}$ is ideally amenable. □

The following result is immediately obtained from Proposition 4.2.

Corollary 4.1. *Let* $(\mathcal{A}, p_\ell)$ *be a commutative Fréchet algebra and* I *be a quasinormable closed (two-sided) ideal in* $\mathcal{A}$*. If* $\mathcal{A}$ *is ideally amenable, then* $\dfrac{\mathcal{A}}{I}$ *is also ideally amenable.*

Theorem 4.1. *Let* $(\mathcal{A}, p_\ell)$ *be a Fréchet algebra and* I *be a quasinormable closed ideal of* $\mathcal{A}$*. If* I *and* $\dfrac{\mathcal{A}}{I}$ *are ideally amenable, then* $\mathcal{A}$ *is also ideally amenable.*

Proof. Let J be an arbitrary closed (two-sided) ideal in $\mathcal{A}$ and let $D : \mathcal{A} \longrightarrow J^*$ be a derivation. Consider $\lambda : I \cap J \longrightarrow J$ to be the inclusion map. Then $\lambda^* oD|_I : I \longrightarrow (I \cap J)^*$ is a derivation, obviously. By [22, Proposition 23.30], $\lambda^* oD|_I$ is continuous. Since I is ideally amenable, there exists $\lambda_1 \in (I \cap J)^*$ such that $\lambda^* oD|_I = ad_{\lambda_1}$. Let $\overline{\lambda_1}$ be a Hahn-Banach extension of λ_1 on J. Define $\tilde{D} = D - ad_{\overline{\lambda_1}}$. Thus $\tilde{D}$ is a derivation from $\mathcal{A}$ into J^*. We show that $\tilde{D}|_{I\cap J} = 0$. For all $i \in I \cap J$ and $k \in J$ we have

$$\begin{aligned} < k, \tilde{D}(i) > &= < k, D(i) > - < k, ad_{\overline{\lambda_1}}(i) > \\ &= < k, D(i) > - < k, i\overline{\lambda_1} - \overline{\lambda_1} i > \\ &= < k, D(i) > - < ki - ik, \overline{\lambda_1} > \\ &= < k, D(i) > - < ki - ik, \lambda_1 > \\ &= < k, D(i) > - < k, i \cdot \lambda_1 - \lambda_1 \cdot i > \\ &= < k, D(i) > - < k, ad_{\lambda_1}(i) > \\ &= < k, D(i) > - < k, \lambda^* o D(i) > \\ &= < k, D(i) > - < j, D(\lambda(i)) > \\ &= < k, D(i) > - < k, D(i) > \\ &= 0. \end{aligned}$$

This showes that $\tilde{D}|_{I\cap J} = 0$. For all $a, b \in I$ and $c \in j$ we have

$$\begin{aligned} < c, \tilde{D}(ab) > &= < c, a \cdot \tilde{D}(b) > + < c, \tilde{D}(a) \cdot b > \\ &= < ca, \tilde{D}(b) > + < cb, \tilde{D}(a) > \\ &= 0 \end{aligned}$$

and so $\tilde{D}|_{I^2} = 0$. By proposition 4.2 part (i), $\bar{I^2} = I$ and cosequently $D|_I = 0$ As ideal amenability implies weak amenability, the same proof similar to [1, Theorem 2.6], implies that $D : \mathcal{A} \longrightarrow J^*$ is inner. It follows that $\mathcal{A}$ is ideally amenable. □

Remark 4.1. *Let $(\mathcal{A}, p_\ell)$ Fréchet algebra, $\mathcal{B}$ be a Banach algebra and let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a continuous homomorphism with dense range. Then if $\mathcal{A}$ is weakly amenable and commutative, then $\mathcal{B}$ is weakly amenable; see* [1, Proposition 2.8].

Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach (Fréchet) algebras and let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a continuous homomorphism with dense range. In general, ideal amenability of $\mathcal{A}$ does not imply ideal amenability of $\mathcal{B}$. In fact, there are some Banach space E with the approximation property such that $A(E)$ is not weakly amenable and so it is not ideally amenable; see [6, Corollary 3.5]. *Moreover, the nuclear algebra $N(E)$ of E is biprojective* [25, Corollary 4.3.6] *and so it is weakly amenable by* [8]. *Since $N(E)$ is topologically simple* [25, Exercise 4.3.8], *then $N(E)$ is ideally amenable. Whereas the inclusion map $i : N(E) \longrightarrow A(E)$ is a continuous homomorphism with dense range. This completes our arguments.*

We recall projective tensor product of Fréchet algebras $(\mathcal{A}, p_\ell)$ and $(\mathcal{B}, q_n)$, which has been introduced in [26]. The construction is similar to the Banach algebra case. It will be denoted by $(\mathcal{A}\widehat{\otimes}\mathcal{B}, r_\ell)$, where

$$r_{\ell,n}(u) = \inf\{\sum_{n\in\mathbb{N}} p_\ell(a_m) q_n(b_m) : u = \sum_{m\in\mathbb{N}} a_m \otimes b_m\},$$

for each $u \in \mathcal{A}\widehat{\otimes}\mathcal{B}$. By [26] and [27, Theorem 2] and also [28, Theorem 45.1], $(\mathcal{A}\widehat{\otimes}\mathcal{B}, r_{\ell,n})$ is again a Fréchet algebra.

As the final result, we generalize [29, Theorem 2.1] to Fréchet algebras.

Theorem 4.2. *Let $(\mathcal{A}, p_\ell)$ and $(\mathcal{B}, q_n)$ be Fréchet algebras. If $\mathcal{A}\hat{\otimes}_\pi\mathcal{B}$ is ideally amenable, then $\mathcal{A}$ and $\mathcal{B}$ are essential.*

Proof. Suppose that $\mathcal{A}$ is not essential. By Hahn-Banach theorem there is a non-zero $\lambda_0 \in \mathcal{A}^*$ such that for all $a, a' \in \mathcal{A}$, $< aa', \lambda_0 >= 0$. Let λ_1 be a non-zero element of $\mathcal{B}^*$. The map $\lambda_0 \otimes \lambda_1$ is a linear functional on $\mathcal{A}\hat{\otimes}_\pi\mathcal{B}$ such that

$$< a \otimes b, \lambda_0 \otimes \lambda_1 >=< a, \lambda_0 >< b, \lambda_1 > .$$

Define $D : \mathcal{A}\hat{\otimes}_\pi\mathcal{B} \longrightarrow (\mathcal{A}\hat{\otimes}_\pi\mathcal{B})^*$ by

$$D(m) =< m, \lambda_0 \otimes \lambda_1 > \lambda_0 \otimes \lambda_1.$$

We show that D is a continuous derivation on $\mathcal{A}\hat{\otimes}_\pi\mathcal{B}$. Suppose that $u = \sum_{j=1}^{n} x_j \otimes y_j \in \mathcal{A}\hat{\otimes}_\pi\mathcal{B}$ and $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}\hat{\otimes}_\pi\mathcal{B}$ such that

$$\lim_{n\longrightarrow\infty} u_n = u,$$

in the topology of $\mathcal{A}$. Since $\lambda_0 \otimes \lambda_1 \in (\mathcal{A}\hat{\otimes}_\pi\mathcal{B})^*$, there exist $n_0, \ell_0 \in \mathbb{N}$ and $k > 0$ such that

$$| < v, \lambda_0 \otimes \lambda_1 > | \leq k r_{\ell_0,n_0}(v),$$

for all $v \in \mathcal{A}\hat{\otimes}_\pi\mathcal{B}$. By [18, Remark 23.2], $\sup_{u\in F} r_n(u) < \infty$ for each $n \in \mathbb{N}$, where $F \subset \mathcal{A}\hat{\otimes}_\pi\mathcal{B}$ is a bounded set. Therefore there exists $M_{n_0} > 0$ such that $\sup_{u\in F} r_{\ell,n}(u) \leq M_{n_0}$. Hence for each $v \in F$, we have

$$\begin{aligned} | < v, D(u_n - u) > | &= | < v, < u_n - u, \lambda_0 \otimes \lambda_1 > \lambda_0 \otimes \lambda_1 > | \\ &= | < u_n - u, \lambda_0 \otimes \lambda_1 > || < v, \lambda_0 \otimes \lambda_1 > | \\ &\leq k r_{\ell_0,n_0}(v) | < u_n - u, \lambda_0 \otimes \lambda_1 > |. \end{aligned}$$

Thus

$$\begin{aligned} \sup_{u\in F} | < v, D(u_n - u) > | &= \sup_{u\in F} | < u_n - u, \lambda_0 \otimes \lambda_1 > || < v, \lambda_0 \otimes \lambda_1 > | \\ &\leq k M_{n_0}(v) | < u_n - u, \lambda_0 \otimes \lambda_1 > |. \end{aligned}$$

The right hand side of the above inequality tends to zero, and therefore D is continuous. Moreover some easy calculations show that D is a derivation. By the hypothesis, D is inner and so there exists ϕ in $(\mathcal{A}\hat{\otimes}_\pi\mathcal{B})^*$ such that $D = ad_\phi$; indeed for each $u \in \mathcal{A}\hat{\otimes}_\pi\mathcal{B}$, $D(u) = u \cdot \phi - \phi \cdot u$. Thus

$$\begin{aligned} < u, D(u) > &= < u, u \cdot \phi - \phi \cdot u > \\ &= < u, u \cdot \phi > - < u, \phi \cdot u > \\ &= < u^2, \phi > - < u^2, \phi >= 0. \end{aligned}$$

On the other hand, since $\lambda_0 \in \mathcal{A}^*$ and $\lambda_1 \in \mathcal{B}^*$ are non-zero, there is $a_0 \in \mathcal{A}$ and $b_0 \in \mathcal{B}$ such that $\lambda_0(a_0) \neq 0$ and $\lambda_1(b_0) \neq 0$. Thus

$$\begin{aligned} < a_0 \otimes b_0 \cdot D(a_0 \otimes b_0) &= < a_0 \otimes b_0, < a_0 \otimes b_0, \lambda_0 \otimes \lambda_1 > \lambda_0 \otimes \lambda_1 > \\ &= < a_0 \otimes b_0, \lambda_0 \otimes \lambda_1 >< a_0 \otimes b_0, \lambda_0 \otimes \lambda_1 > \\ &= (< a_0, \lambda_0 >)^2 (< b_0, \lambda_1 >)^2 \neq 0, \end{aligned}$$

which is a contradiction. It follows that $\mathcal{A}$ is essential. Similarly $\mathcal{B}$ is essential. □

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