FIXED POINTS FOR $(\psi - \phi)$-WEAK CONTRACTIONS IN S-METRIC SPACES

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In this article, we introduce the class of $(\psi - \phi)$-weak contractions and establish some unique fixed point theorems in the setting of complete S-metric spaces. Also, we give some examples in support of our results. Our results extend the corresponding result of [9, 13] *and several other results from the current existing literature.*

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1. Introduction

Let (X, d) be a metric space and let $T\colon X \to X$ be a mapping.
(a) A point $x \in X$ is called a fixed point of T if $x = Tx$.
(b) T is called contraction if there exists a fixed constant $0 \le k < 1$ such that

$$d(T(x), T(y)) \le k\, d(x, y) \tag{1}$$

for all $x, y \in X$. If X is complete, then every contraction has a unique fixed point and that point can be obtained as a limit of repeated iteration of the mapping at any point of X (the Banach contraction principle). Obviously, every contraction is a continuous function.

The development of fixed point theory is based on the generalization of contraction conditions is one direction or/and generalization of underlying space of the operator under consideration on the other. The Banach contraction mapping principle is one of the pivotal results of analysis. It is a very famous tool for solving existence problems in various fields of mathematics. Banach contraction principle plays an important role in solving non linear equations, and it is one of the most useful results in metric fixed point theory. Banach contraction principle has been generalized in various ways either by using contractive conditions or by imposing some additional conditions on the underlying space. The Banach contraction mapping theorem and its several extensions have been generalized using recently developed notion of weakly contractive maps.

In 1997, Alber and Delabrieer [2] introduced the concept of weak contraction as follows.

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Definition 1.1. *([2]) (Weak Contraction Mapping) A mapping $T\colon X \to X$ where (X, d) is a complete metric space is said to be ϕ-weak contraction if*

$$d(T(x), T(y)) \leq d(x, y) - \phi(d(x, y)) \tag{2}$$

for all $x, y \in X$, where $\phi\colon [0, \infty) \to [0, \infty)$ is continuous and non-decreasing, $\phi(x) = 0$ if and only if $x = 0$ and $\lim_{x\to\infty} \phi(x) = \infty$.

They defined such mappings for single-valued maps on Hilbert spaces and proved a novel fixed point result for weak contraction in the said space. Rhoades [13] showed that most results of [2] are true for any Banach space. Also Rhoades proved the following generalization of the Banach contraction principle.

Theorem 1.1. *(Generalized Banach Contraction Principle) Let (X, d) be a nonempty complete metric space and let $T\colon X \to X$ be a ϕ-weak contraction on X. If ϕ is a continuous and nondecreasing function with $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$, then T has a unique fixed point.*

Remark 1.1. *Every contraction is a ϕ-weak contraction if we take $\phi(t) = kt$, where $0 < k < 1$.*

Weakly contractive mappings have been dealt with in a number of papers. Some of these works are noted in [3, 4, 13, 18].

Dutta and Choudhury [9] in 2008 introduced a generalized Banach contraction mapping principle which includes the generalization noted in Theorem 1.1 as follows.

Theorem 1.2. *([9]) Let (X, d) be a complete metric space and let $T\colon X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(T(x), T(y))) \leq \psi(d(x, y)) - \phi(d(x, y)) \tag{3}$$

for all $x, y \in X$, where $\psi, \phi\colon [0, \infty) \to [0, \infty)$ are both continuous and monotone non-decreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then T has a unique fixed point.

In 2009, Doric [8] extended $(\psi - \phi)$-contractions to a pair of maps which generalized the result of Dutta and Choudhury [9]. For more literature in this direction we refer the reader to Abbas and Doric [1], Choudhury et al. [5], Choudhury et al. [6], Karapinar and Pitea [10], Murthy et al. [11], Popescu [12] and Shatanawi and Postolache [16].

In 2012, Sedghi et al. [14] introduced the notion of S-metric space which is a generalization of a G-metric space and D^*-metric space. In [14] the authors proved some properties of S-metric spaces. Also, they obtained some fixed point theorems in the setting of S-metric spaces for a self-map.

The purpose of this paper is to generalize the result of Dutta and Choudhury [9] from complete metric space to the setting of complete S-metric spaces. First of all, we give the concept and basic properties related to S-metric spaces.

2. **Preliminaries**

We need the following definitions and lemmas in the sequel.

Definition 2.1. *([14]) Let X be a nonempty set and $S\colon X^3 \to [0, \infty)$ be a function satisfying the following conditions for all x, y, z, $t \in X$:*

(SM_1) $S(x, y, z) = 0$ if and only if $x = y = z$;

(SM_2) $S(x,y,z) \leq S(x,x,t) + S(y,y,t) + S(z,z,t)$.

Then the function S is called an S-metric on X and the pair (X,S) is called an S-metric space or simply SMS.

Example 2.1. *([14]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then* $\mathrm{S(x,y,z)} = \|\mathrm{y+z-2x}\| + \|\mathrm{y-z}\|$ *is an S-metric on X.*

Example 2.2. *([14]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then* $\mathrm{S(x,y,z)} = \|\mathrm{x-z}\| + \|\mathrm{y-z}\|$ *is an S-metric on X.*

Example 2.3. *([15]) Let $X = \mathbb{R}$ be the real line. Then* $\mathrm{S(x,y,z)} = |\mathrm{x-z}| + |\mathrm{y-z}|$ *for all* $\mathrm{x,y,z} \in \mathbb{R}$ *is an S-metric on X. This S-metric on X is called the usual S-metric on X.*

Lemma 2.1. *([14], Lemma 2.5) If (X,S) be an S-metric space, then we have $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$.*

Lemma 2.2. *([14], Lemma 2.12) Let (X,S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ then $S(x_n,x_n,y_n) \to S(x,x,y)$ as $n \to \infty$.*

Definition 2.2. *([14]) Let (X,S) be an S-metric space.*

(1) *A sequence $\{x_n\}$ in X converges to $x \in X$ if $S(x_n,x_n,x) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n,x_n,x) < \varepsilon$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.*

(2) *A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n,x_n,x_m) \to 0$ as $n,m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n,m \geq n_0$ we have $S(x_n,x_n,x_m) < \varepsilon$.*

(3) *The S-metric space (X,S) is called complete if every Cauchy sequence in X is convergent in X.*

Definition 2.3. *Let T be a self mapping on an S-metric space (X,S). Then T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in X with $x_n \to x$, we have $Tx_n \to Tx$ as $n \to \infty$.*

Definition 2.4. *([14]) Let (X,S) be an S-metric space. A mapping $T\colon X \to X$ is said to be a contraction if there exists a constant $0 \leq L < 1$ such that*

$$S(Tx,Tx,Ty) \leq L\,S(x,x,y) \tag{4}$$

for all $x,y \in X$. If the S-metric space (X,S) is complete then the mapping defined as above has a unique fixed point.

Now, we generalize the definitions of ϕ-weak contraction and $(\psi-\phi)$-weak contraction in S-metric spaces. The definitions are as follows.

Definition 2.5. *(Weak Contraction Mapping) Let (X,S) be an S-metric space. A mapping $T\colon X \to X$ is said to be ϕ-weak contraction if*

$$S(Tx,Tx,Ty) \leq S(x,x,y) - \phi(S(x,x,y)) \tag{5}$$

for all $x,y \in X$, where $\phi\colon [0,\infty) \to [0,\infty)$ is continuous and non-decreasing, $\phi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$.

Remark 2.1. *If we take $\phi(t) = Lt$ where $0 < L < 1$ then (5) reduces to (4).*

Definition 2.6. *Let (X, S) be an S-metric space. A mapping $T\colon X \to X$ is said to be $(\psi - \phi)$- weak contraction if for all $x, y \in X$*

$$\psi(S(Tx,Tx,Ty)) \quad \leq \quad \psi(S(x,x,y)) - \phi(S(x,x,y)) \tag{6}$$

where $\psi, \phi\colon [0,\infty) \to [0,\infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$.

Remark 2.2. *(i) If we take $\psi(t) = t$ for all $t \geq 0$ and $\phi(t) = (1-L)\psi(t)$ where $0 < L < 1$, then (6) reduces to (4).*

(ii) If we take $\psi(t) = t$ for all $t \geq 0$, then (6) reduces to (5).

3. Main Results

In this section, we shall prove some unique fixed point theorems in the setting of complete S-metric spaces for $(\psi - \phi)$-weak contraction condition (6).

Theorem 3.1. *Let (X, S) be a complete S-metric space and let $T\colon X \to X$ be a self mapping satisfying the inequality (6), where $\psi, \phi\colon [0,\infty) \to [0,\infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then T has a unique fixed point.*

Proof. For any $x_0 \in X$, we construct the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$. Putting $x = x_{n-1}$ and $y = x_n$ in inequality (6), we get

$$\begin{aligned} \psi(S(x_n,x_n,x_{n+1})) &= \psi(S(Tx_{n-1},Tx_{n-1},Tx_n)) \\ &\leq \psi(S(x_{n-1},x_{n-1},x_n)) \\ &\quad -\phi(S(x_{n-1},x_{n-1},x_n)), \end{aligned} \tag{7}$$

which implies

$$S(x_n,x_n,x_{n+1}) \leq S(x_{n-1},x_{n-1},x_n) \tag{8}$$

by using monotone property of ψ-function. It follows that the sequence $\{S(x_n,x_n,x_{n+1})\}$ is monotone decreasing and so there exists $r \geq 0$ such that

$$\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = r. \tag{9}$$

We next prove that $r = 0$. Letting $n \to \infty$ in (7), we obtain

$$\psi(r) \leq \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$.

Hence,

$$S(x_n,x_n,x_{n+1}) \to 0 \text{ as } n \to \infty. \tag{10}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. If otherwise, then there exists $\varepsilon > 0$ and increasing sequences of integers $\{m(k)\}$ and $\{n(k)\}$ such that for all integers k,

$$n(k) > m(k) > k, \tag{11}$$

$$S(x_{m(k)},x_{m(k)},x_{n(k)}) \geq \varepsilon. \tag{12}$$

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (11). Then

$$S(x_{m(k)},x_{m(k)},x_{n(k)-1}) < \varepsilon. \tag{13}$$

Now, using (12), (SM_2) and Lemma 2.1, we have

$$\begin{aligned}
\varepsilon &\leq S(x_{m(k)}, x_{m(k)}, x_{n(k)}) \\
&= S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\
&\leq 2S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) + S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) \\
&= 2S(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)}) + S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) \\
&\leq \varepsilon + 2S(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)}). \text{ (by (13))} \qquad (14)
\end{aligned}$$

Letting $k \to \infty$ in equation (14) and using (10), we get

$$\lim_{k\to\infty} S(x_{m(k)}, x_{m(k)}, x_{n(k)}) = \varepsilon. \qquad (15)$$

Again, with the help of (SM_2) and Lemma 2.1, we have

$$\begin{aligned}
S(x_{m(k)}, x_{m(k)}, x_{n(k)}) &\leq 2S(x_{m(k)}, x_{m(k)}, x_{m(k)-1}) \\
&\quad + S(x_{n(k)}, x_{n(k)}, x_{m(k)-1}) \\
&\leq 2S(x_{m(k)}, x_{m(k)}, x_{m(k)-1}) \\
&\quad + 2S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) \\
&\quad + S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}). \qquad (16)
\end{aligned}$$

Also, with the help of (SM_2) and Lemma 2.1, we have

$$\begin{aligned}
S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}) &\leq 2S(x_{m(k)-1}, x_{m(k)-1}, x_{m(k)}) \\
&\quad + S(x_{n(k)-1}, x_{n(k)-1}, x_{m(k)}) \\
&= 2S(x_{m(k)-1}, x_{m(k)-1}, x_{m(k)}) \\
&\quad + S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}). \qquad (17)
\end{aligned}$$

Letting $k \to \infty$ in equation (17) and using (10), (13) and (16), we get

$$\lim_{k\to\infty} S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \qquad (18)$$

Now consider inequality (6) and putting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$, we obtain

$$\begin{aligned}
\psi(S(x_{m(k)}, x_{m(k)}, x_{n(k)})) &= \psi(S(Tx_{m(k)-1}, Tx_{m(k)-1}, Tx_{n(k)-1})) \\
&\leq \psi(S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})) \\
&\quad - \phi(S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})). \qquad (19)
\end{aligned}$$

Next, letting $k \to \infty$ in equation (19) and using (15) and (18), we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon) < \psi(\varepsilon),$$

which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and therefore it is convergent in the complete S-metric space (X, S). So, suppose $x_n \to u$ as $n \to \infty$.

Now, putting $x = x_{n-1}$ and $y = u$ in equation (6), we obtain

$$\begin{aligned}
\psi(S(x_n, x_n, Tu)) &= \psi(S(Tx_{n-1}, Tx_{n-1}, Tu)) \\
&\leq \psi(S(x_{n-1}, x_{n-1}, u)) - \phi(S(x_{n-1}, x_{n-1}, u)).
\end{aligned}$$

Letting $n \to \infty$, using $\lim_{n\to\infty} x_n = u$ and the continuity of ψ and ϕ in the above inequality, we obtain

$$\psi(S(u, u, Tu)) \leq \psi(0) - \phi(0) = 0, \qquad (20)$$

which implies that $\psi(S(u,u,Tu)) = 0$, that is,

$$S(u,u,Tu) = 0 \text{ or } u = Tu. \tag{21}$$

This shows that u is a fixed point of T.

Next, to show that the fixed point of T is unique. For this, suppose that v is another fixed point of T such that $v = Tv$ with $v \neq u$. Then using equation (6), we have

$$\begin{aligned}\psi(S(u,u,v)) &= \psi(S(Tu,Tu,Tv)) \\ &\leq \psi(S(u,u,v)) - \phi(S(u,u,v)),\end{aligned}$$

or

$$\phi(S(u,u,v)) = 0, \tag{22}$$

by the property of ϕ, we have $S(u,u,v) = 0$, that is, $u = v$. This shows that the fixed point of T is unique. This completes the proof. □

Remark 3.1. *Theorem 3.1 extends Theorem 2.1 of Dutta and Choudhury* [9] *to the setting of complete S-metric space considered in this paper.*

If we take $\psi(t) = t$ for all $t \geq 0$ and $\phi(t) = (1-L)\psi(t)$ in Theorem 3.1, then we obtain the following result as corollary.

Corollary 3.1. *Let (X,S) be a complete S-metric space and let $T\colon X \to X$ be a self mapping satisfying the inequality*

$$S(Tx,Tx,Ty) \leq L\, S(x,x,y) \tag{23}$$

for all $x,y \in X$, where $0 \leq L < 1$ is a constant. Then T has a unique fixed point in X.

Remark 3.2. *Corollary 3.1 extends the well known Banach contraction principle from complete metric space to the setting of complete S-metric spaces considered in this paper.*

If we take $\psi(t) = t$ for all $t \geq 0$ in Theorem 3.1, then we obtain the following result as corollary.

Corollary 3.2. *Let (X,S) be a complete S-metric space and let $T\colon X \to X$ be a self mapping satisfying the inequality*

$$S(Tx,Tx,Ty) \leq S(x,x,y) - \phi(S(x,x,y)) \tag{24}$$

for all $x,y \in X$, where $\phi\colon [0,\infty) \to [0,\infty)$ is a continuous and monotone nondecreasing function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Remark 3.3. *Corollary 3.2 extends the corresponding result of Rhoades* [13] *to the setting of complete S-metric spaces considered in this paper.*

Theorem 3.2. *Let (X,S) be a complete S-metric space and let $T\colon X \to X$ be a self mapping such that for all $x,y \in X$*

$$\psi(S(Tx,Tx,Ty)) \leq \psi(M(x,x,y)) - \phi(M(x,x,y)), \tag{25}$$

where

(a) $\psi\colon [0,\infty) \to [0,\infty)$ is a continuous monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$,

(b) $\phi\colon [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$,

(c) $M(x,x,y) = \max\Big\{S(x,x,y), S(x,x,Tx), S(y,y,Ty), \frac{1}{2}[S(x,x,Ty)+S(y,y,Tx)]\Big\}$.
Then T has a unique fixed point.

Proof. Let $x_0 \in X$. We define the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then trivially x_n is a fixed point of T. So, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Now using inequality (25) and putting $x = x_{n-1}$, $y = x_n$, we have

$$\begin{aligned} \psi(S(x_n, x_n, x_{n+1})) &= \psi(S(Tx_{n-1}, Tx_{n-1}, Tx_n)) \\ &\leq \psi(M(x_{n-1}, x_{n-1}, x_n)) \\ &\quad -\phi(M(x_{n-1}, x_{n-1}, x_n)), \end{aligned} \tag{26}$$

which implies

$$\psi(S(x_n, x_n, x_{n+1})) \leq \psi(M(x_{n-1}, x_{n-1}, x_n)). \tag{27}$$

Using the properties of ψ and ϕ functions in the above inequality, we obtain

$$S(x_n, x_n, x_{n+1}) \leq M(x_{n-1}, x_{n-1}, x_n). \tag{28}$$

Now, from condition (SM_2) and Lemma 2.1, we have

$$\begin{aligned} M(x_{n-1}, x_{n-1}, x_n) &= \max\Big\{S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, Tx_{n-1}), S(x_n, x_n, Tx_n), \\ &\qquad \frac{1}{2}[S(x_{n-1}, x_{n-1}, Tx_n) + S(x_n, x_n, Tx_{n-1})]\Big\} \\ &= \max\Big\{S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \\ &\qquad \frac{1}{2}[S(x_{n-1}, x_{n-1}, x_{n+1}) + S(x_n, x_n, x_n)]\Big\} \\ &= \max\Big\{S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \\ &\qquad \frac{1}{2}[S(x_{n-1}, x_{n-1}, x_{n+1})]\Big\} \\ &\leq \max\Big\{S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \\ &\qquad \frac{1}{2}[2S(x_{n-1}, x_{n-1}, x_n) + S(x_{n+1}, x_{n+1}, x_n)]\Big\} \\ &\leq \max\Big\{S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \\ &\qquad \frac{1}{2}[2S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})]\Big\}. \end{aligned}$$

If $S(x_n, x_n, x_{n+1}) > S(x_{n_1}, x_{n-1}, x_n)$, then $M(x_{n-1}, x_{n-1}, x_n) = S(x_n, x_n, x_{n+1}) > 0$. It furthermore implies that

$$\psi(S(x_n, x_n, x_{n+1})) \leq \psi(S(x_n, x_n, x_{n+1})) - \phi(S(x_n, x_n, x_{n+1})) \tag{29}$$

which is a contraction. So, we have

$$S(x_n, x_n, x_{n+1}) \leq M(x_{n-1}, x_{n-1}, x_n) \leq S(x_{n-1}, x_{n-1}, x_n). \tag{30}$$

Thus, the sequence $\{S(x_n, x_n, x_{n+1})\}$ is monotone nonincreasing and bounded. So there exists $r \geq 0$ such that

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = \lim_{n\to\infty} M(x_{n-1}, x_{n-1}, x_n) = r \geq 0. \tag{31}$$

Letting $n \to \infty$ in inequality (26), we obtain

$$\psi(r) \leq \psi(r) - \phi(r), \tag{32}$$

which is a contradiction unless $r = 0$. Hence,

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0. \tag{33}$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence. If we suppose that $\{x_n\}$ is not a Cauchy sequence, then there exists an $\varepsilon > 0$ and there exist subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that for all integers k

$$n(k) > m(k) > k, \tag{34}$$

$$S(x_{m(k)}, x_{m(k)}, x_{n(k)}) \geq \varepsilon. \tag{35}$$

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (34). Then, we have

$$S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{36}$$

Now, the following identities follow as in the proof of Theorem 3.1.

(i) $\lim_{k\to\infty} S(x_{m(k)}, x_{m(k)}, x_{n(k)}) = \varepsilon$.

(ii) $\lim_{k\to\infty} S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}) = \varepsilon$.

Also from the definition of M and from equation (33) and $(i) - (ii)$, we have

$$\lim_{n\to\infty} M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{37}$$

We now consider (25) and setting $x = x_{m(k)-1}$, $y = x_{n(k)-1}$, we have

$$\begin{aligned} \psi(S(x_{m(k)}, x_{m(k)}, x_{n(k)}) &= \psi(S(Tx_{m(k)-1}, Tx_{m(k)-1}, Tx_{n(k)-1})) \\ &\leq \psi(M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})) \\ &\quad -\phi(M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})). \end{aligned} \tag{38}$$

On letting $k \to \infty$ in equation (38) and using (37) and (i), we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon) < \psi(\varepsilon),$$

which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and therefore it is convergent in the complete S-metric space (X, S). So, suppose $x_n \to v$ as $n \to \infty$. Now we prove that $v = Tv$. Indeed, suppose $v \neq Tv$, then for $S(v, v, Tv) > 0$, there exists $N_1 \in \mathbb{N}$ such that for any $n > N_1$, we have

$$S(x_{n-1}, x_{n-1}, v) < \frac{1}{4} S(v, v, Tv), \tag{39}$$

$$S(x_{n-1}, x_{n-1}, x_n) < \frac{1}{4} S(v, v, Tv), \tag{40}$$

and

$$S(x_n, x_n, v) < \frac{1}{4} S(v, v, Tv). \tag{41}$$

Now, putting $x = x_{n-1}$ and $y = v$ in equation (25), we obtain

$$\begin{aligned} \psi(S(x_n, x_n, Tv)) &= \psi(S(Tx_{n-1}, Tx_{n-1}, Tv)) \\ &\leq \psi(M(x_{n-1}, x_{n-1}, v)) \\ &\quad -\phi(M(x_{n-1}, x_{n-1}, v)), \end{aligned} \tag{42}$$

where

$$\begin{aligned} M(x_{n-1}, x_{n-1}, v) &= \max\Big\{S(x_{n-1}, x_{n-1}, v), S(x_{n-1}, x_{n-1}, Tx_{n-1}), S(v, v, Tv), \\ &\qquad \frac{1}{2}[S(x_{n-1}, x_{n-1}, Tv) + S(v, v, Tx_{n-1})]\Big\} \\ &= \max\Big\{S(x_{n-1}, x_{n-1}, v), S(x_{n-1}, x_{n-1}, x_n), S(v, v, Tv), \\ &\qquad \frac{1}{2}[S(x_{n-1}, x_{n-1}, Tv) + S(v, v, x_n)]\Big\} \\ &\leq \max\Big\{S(x_{n-1}, x_{n-1}, v), S(x_{n-1}, x_{n-1}, x_n), S(v, v, Tv), \\ &\qquad \frac{1}{2}[2S(x_{n-1}, x_{n-1}, v) + S(Tv, Tv, v) + S(v, v, x_n)]\Big\} \\ &\quad \text{(by } (SM_2)) \\ &\leq \max\Big\{S(x_{n-1}, x_{n-1}, v), S(x_{n-1}, x_{n-1}, x_n), S(v, v, Tv), \\ &\qquad \frac{1}{2}[2S(x_{n-1}, x_{n-1}, v) + S(v, v, Tv) + S(x_n, x_n, v)]\Big\}. \\ &\quad \text{(by Lemma 2.1)} \end{aligned} \tag{43}$$

Using equation (39), (40) and (41) in (43), we obtain

$$\begin{aligned} M(x_{n-1}, x_{n-1}, v) &\leq \max\Big\{\frac{1}{4}S(v, v, Tv), \frac{1}{4}S(v, v, Tv), S(v, v, Tv), \\ &\qquad \frac{1}{2}[2.\frac{1}{4}S(v, v, Tv) + S(v, v, Tv) + \frac{1}{4}S(v, v, Tv)]\Big\}, \end{aligned}$$

that is,

$$M(x_{n-1}, x_{n-1}, v) \leq S(v, v, Tv). \tag{44}$$

Now, using equation (44) in (42), we obtain

$$\psi(S(x_n, x_n, Tv)) \leq \psi(S(v, v, Tv)) - \phi(S(v, v, Tv)). \tag{45}$$

On letting $n \to \infty$ in inequality (45), we obtain

$$\psi(S(v, v, Tv)) \leq \psi(S(v, v, Tv)) - \phi(S(v, v, Tv)), \tag{46}$$

which is a contradiction unless $S(v, v, Tv) = 0$. Hence we conclude that $v = Tv$. This shows that v is a fixed point of T. Now, to show that the fixed point of T is unique. For this, suppose w is another fixed point of T such that $w = Tw$ with $w \neq v$. Now using equation (25) again, we have

$$\begin{aligned} \psi(S(v, v, w)) &= \psi(S(Tv, Tv, Tw)) \\ &\leq \psi(M(v, v, w)) - \phi(M(v, v, w)) \\ &\leq \psi(S(v, v, w)) - \phi(S(v, v, w)), \end{aligned}$$

which is a contradiction unless $S(v, v, w) = 0$. Thus we conclude that $v = w$. This shows that the fixed point of T is unique. This completes the proof. □

Remark 3.4. *Theorem 3.2 extends Theorem 2.2 of Doric* [8] *from complete metric space to the setting of complete S-metric spaces considered in this paper.*

Remark 3.5. *If we take* $\max\Big\{S(x, x, y), S(x, x, Tx), S(y, y, Ty), \frac{1}{2}[S(x, x, Ty)+S(y, y, Tx)]\Big\} = S(x, x, y)$*, then we obtain Theorem 3.1 of this paper.*

Now, we give some examples in support of our results.

Example 3.1. *Let* $X = [0,1]$. *We define* $S\colon X^3 \to [0,\infty)$ *by* $\mathrm{S}(\mathrm{x},\mathrm{y},\mathrm{z}) = |\mathrm{x}-\mathrm{z}| + |\mathrm{y}-\mathrm{z}|$ *for all* $\mathrm{x},\mathrm{y},\mathrm{z} \in X$, *then* S *is an* S*-metric on* X *called usual* S*-metric on* X. *Now, we define a map* $T\colon X \to X$ *by* $\mathrm{T}(\mathrm{x}) = \frac{1}{2}\sin \mathrm{x}$. *Then we have*

$$\begin{aligned} \mathrm{S}(\mathrm{Tx},\mathrm{Tx},\mathrm{Ty}) &= \Big|\mathrm{T}(\mathrm{x}) - \mathrm{T}(\mathrm{y})\Big| + \Big|\mathrm{T}(\mathrm{x}) - \mathrm{T}(\mathrm{y})\Big| \\ &= \Big|\frac{1}{2}(\sin \mathrm{x} - \sin \mathrm{y})\Big| + \Big|\frac{1}{2}(\sin \mathrm{x} - \sin \mathrm{y})\Big| \\ &\le \frac{1}{2}\big(|\mathrm{x}-\mathrm{y}| + |\mathrm{x}-\mathrm{y}|\big) = \frac{1}{2}\mathrm{S}(\mathrm{x},\mathrm{x},\mathrm{y}) \\ &= L\,\mathrm{S}(\mathrm{x},\mathrm{x},\mathrm{y}) \end{aligned}$$

where $L = \frac{1}{2} < 1$. *Thus,* T *satisfies all the conditions of Corollary 3.1. Hence, applying Corollary 3.1,* T *has a unique fixed point. Here it is seen that* $0 \in X$ *is the unique fixed point of* T.

Example 3.2. *Let* $X = [0,1]$. *We define* $S\colon X^3 \to \mathbb{R}_+$ *by*

$$\mathrm{S}(\mathrm{x},\mathrm{x},\mathrm{y}) = \begin{cases} 0 & \textit{if } \mathrm{x} = \mathrm{y}, \\ \max\{\mathrm{x},\mathrm{y}\} & \textit{if otherwise}, \end{cases}$$

for all $\mathrm{x},\mathrm{y} \in X$. *Then* (X,S) *is a complete* S*-metric space. Let* $T\colon X \to X$ *be a mapping defined as* $\mathrm{T}(\mathrm{x}) = \frac{\mathrm{x}^2}{2}$ *and* $\phi(\mathrm{t}) = \frac{\mathrm{t}^2}{4}$. *Without loss of generality, we assume that* $\mathrm{x} > \mathrm{y}$. *Then*

$$\mathrm{S}(\mathrm{Tx},\mathrm{Tx},\mathrm{Ty}) = \max\{\mathrm{Tx},\mathrm{Ty}\} = \frac{\mathrm{x}^2}{2},$$

$$\mathrm{S}(\mathrm{x},\mathrm{x},\mathrm{y}) = \max\{\mathrm{x},\mathrm{y}\} = \mathrm{x},$$

and

$$\phi(\mathrm{S}(\mathrm{x},\mathrm{x},\mathrm{y})) = \frac{\mathrm{x}^2}{4}.$$

Now, $\mathrm{S}(\mathrm{x},\mathrm{x},\mathrm{y}) - \phi(\mathrm{S}(\mathrm{x},\mathrm{x},\mathrm{y})) = \mathrm{x} - \frac{\mathrm{x}^2}{4}$. *Therefore* $\mathrm{S}(\mathrm{Tx},\mathrm{Tx},\mathrm{Ty}) = \frac{\mathrm{x}^2}{2} \le \mathrm{x} - \frac{\mathrm{x}^2}{4} = \mathrm{S}(\mathrm{x},\mathrm{x},\mathrm{y}) - \phi(\mathrm{S}(\mathrm{x},\mathrm{x},\mathrm{y}))$. *Hence* T *satisfies the inequality (24), so that* T *is a weakly contractive map. Thus, by Corollary 3.2,* T *has a unique fixed point and clearly it is "0" in* X.

Example 3.3. *Let* $X = [0,1] \cup \{2,3,4,\dots\}$ *and*

$$\mathrm{S}(\mathrm{x},\mathrm{x},\mathrm{y}) = \begin{cases} 2|\mathrm{x}-\mathrm{y}| & \textit{if } \mathrm{x},\mathrm{y} \in [0,1],\ \mathrm{x} \ne \mathrm{y}, \\ \\ 2\mathrm{x}+\mathrm{y} & \textit{if at least one of } \mathrm{x} \textit{ or } \mathrm{y} \notin [0,1] \textit{ and } \mathrm{x} \ne \mathrm{y}, \\ \\ 0 & \textit{if } \mathrm{x} = \mathrm{y}, \end{cases}$$

for all $\mathrm{x},\mathrm{y} \in X$. *Then* (X,S) *is a complete* S*-metric space.*

Let $\psi\colon [0,\infty) \to [0,\infty)$ *be defined as*

$$\psi(\mathrm{t}) = \begin{cases} \mathrm{t} & \textit{if } 0 \le \mathrm{t} \le 1, \\ \mathrm{t}^2 & \textit{if } \mathrm{t} > 1, \end{cases}$$

and let $\phi\colon [0,\infty) \to [0,\infty)$ *be defined as*

$$\phi(\mathrm{t}) = \begin{cases} \mathrm{t}^2 & \textit{if } 0 \le \mathrm{t} \le 1, \\ 2 & \textit{if } \mathrm{t} > 1. \end{cases}$$

Let $T\colon X \to X$ *be defined as*

$$\mathrm{T(x)} = \begin{cases} \mathrm{x} - 2\mathrm{x}^2 & \textit{if } 0 \le \mathrm{x} \le 1, \\ \mathrm{x} - 1 & \textit{if } \mathrm{x} \in \{2, 3, 4, \dots\}. \end{cases}$$

Without loss of generality, we assume that $\mathrm{x} > \mathrm{y}$ *and discuss the following cases.*

Case I If $\mathrm{x} \in [0, 1]$. *Then*

$$\begin{aligned} \psi(\mathrm{S(Tx, Tx, Ty)}) &= \mathrm{S(Tx, Tx, Ty)} \\ &= \mathrm{S}\Big(\mathrm{x} - 2\mathrm{x}^2, \mathrm{x} - 2\mathrm{x}^2, \mathrm{y} - 2\mathrm{y}^2\Big) \\ &= 2\Big[\big(\mathrm{x} - 2\mathrm{x}^2\big) - \big(\mathrm{y} - 2\mathrm{y}^2\big)\Big] \\ &= 2\Big[(\mathrm{x} - \mathrm{y}) - 2(\mathrm{x} - \mathrm{y})(\mathrm{x} + \mathrm{y})\Big] \\ &= 2(\mathrm{x} - \mathrm{y}) - 4(\mathrm{x} - \mathrm{y})(\mathrm{x} + \mathrm{y}) \\ &\le 2(\mathrm{x} - \mathrm{y}) - 4(\mathrm{x} - \mathrm{y})^2 \ \ (\textit{since}\ \mathrm{x} - \mathrm{y} \le \mathrm{x} + \mathrm{y}) \\ &= \mathrm{S(x, x, y)} - (\mathrm{S(x, x, y)})^2 \\ &= \psi(\mathrm{S(x, x, y)}) - \phi(\mathrm{S(x, x, y)}). \end{aligned}$$

Case II If $\mathrm{x} \in \{2, 3, 4, \dots\}$. *Then*

$$\mathrm{S(Tx, Tx, Ty)} = \mathrm{S}(\mathrm{x} - 1, \mathrm{x} - 1, \mathrm{y} - 2\mathrm{y}^2) \ \ \textit{if}\ \mathrm{y} \in [0, 1]$$

or

$$\mathrm{S(Tx, Tx, Ty)} = 2(\mathrm{x} - 1) + \mathrm{y} - 2\mathrm{y}^2 \le 2\mathrm{x} + \mathrm{y} - 2,$$

and

$$\mathrm{S(Tx, Tx, Ty)} = \mathrm{S}(\mathrm{x} - 1, \mathrm{x} - 1, \mathrm{y} - 1) \ \ \textit{if}\ \mathrm{y} \in \{2, 3, 4, \dots\}$$

or

$$\mathrm{S(Tx, Tx, Ty)} = 2(\mathrm{x} - 1) + \mathrm{y} - 1 \le 2\mathrm{x} + \mathrm{y} - 2.$$

Consequently,

$$\begin{aligned} \psi(\mathrm{S(Tx, Tx, Ty)}) &= \mathrm{S(Tx, Tx, Ty)}^2 \\ &\le (2\mathrm{x} + \mathrm{y} - 2)^2 \\ &< (2\mathrm{x} + \mathrm{y} - 2)(2\mathrm{x} + \mathrm{y} + 2) \\ &= (2\mathrm{x} + \mathrm{y})^2 - 4 < (2\mathrm{x} + \mathrm{y})^2 - 2 \\ &= (\mathrm{S(x, x, y)})^2 - \phi(\mathrm{S(x, x, y)}) \\ &= \psi(\mathrm{S(x, x, y)}) - \phi(\mathrm{S(x, x, y)}). \end{aligned}$$

Case III If $x = 2$. *Then* $y \in [0,1]$, $T(x) = 1$ *and* $S(Tx, Tx, Ty) = 2[1 - (y - 2y^2)] \le 2$. *So, we have* $\psi(S(Tx, Tx, Ty)) \le \psi(2) = 4$. *Again* $S(x, x, y) = 4 + y$. *So,*

$$\begin{aligned} \psi(S(x,x,y)) - \phi(S(x,x,y)) &= (4+y)^2 - \phi((4+y)^2) \\ &= (4+y)^2 - 2 \\ &= 14 + y^2 + 8y > 4 \\ &= \psi(S(Tx, Tx, Ty)). \end{aligned}$$

Considering all the above cases, we conclude that the inequality used in Theorem 3.1 remains valid for ψ, ϕ *and* T *constructed in the above example and consequently by applying Theorem 3.1,* T *has a unique fixed point. It is seen that "0" is the unique fixed point of* T.

Example 3.4. *Let* $X = [0,1]$. *We define* $S: X^3 \to \mathbb{R}_+$ *by*

$$S(x,x,y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{x,y\} & \text{if otherwise.} \end{cases}$$

for all $x, y \in X$. *Then* (X, S) *is a complete* S*-metric space. Let* $T: X \to X$ *be a mapping defined as*

$$T(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0, \\ 2x & \text{if } 0 < x < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

We define ψ *and* ϕ *on* $\mathbb{R}_+$ *by* $\psi(t) = \frac{t^2}{3}$ *and* $\phi(t) = \frac{t^2}{2}$. *Then it is easy to verify that* T *satisfies the inequality (6), that is,* T *is a* $(\psi - \phi)$*-weakly contractive map. Thus,* T *satisfies all the hypothesis of Theorem 3.1 and clearly it is seen that "1" is the unique fixed point of* T.

Example 3.5. *Let* $X = [0,1]$ *and define* $S: X^3 \to \mathbb{R}_+$ *by* $S(x,y,z) = |x - z| + |y - z|$ *and is called usual* S*-metric on* X. *Let* $T: X \to X$ *be a mapping defined as* $T(x) = \frac{x}{2}$. *Now*

$$M(x,x,y) = \max\Big\{2|x-y|, x, y, \frac{1}{2}(|2x-y| + |x-2y|)\Big\}$$

$$= \begin{cases} 2|x-y| & \text{if } 0 \le y \le \frac{1}{2}x, \\ x & \text{if } \frac{1}{2}x \le y, \end{cases}$$

for all $x, y \in X$.

For $\psi(t) = 2t$ *and* $\phi(t) = \frac{t}{2}$, *we have* $\psi(S(Tx, Tx, Ty)) = 2|x - y|$ *and*

$$\psi(M(x,x,y)) - \phi(M(x,x,y))$$

$$= \begin{cases} 3|x-y| & \text{if } 0 \le y \le \frac{1}{2}x, \\ \frac{3}{2}x & \text{if } \frac{1}{2}x \le y, \end{cases}$$

for all $x, y \in X$.

Now, we can easily see that mapping T *satisfies inequality (25) in Theorem 3.2 and clearly* $0 \in X$ *is the unique fixed point of* T.

4. **Conclusion**

In this paper, we introduce $(\psi - \phi)$-weak contraction in S-metric space and establish some unique fixed point theorems in complete S-metric spaces. Also we give some examples in support of our results. Our results extend and generalize some known results from the existing literature. Especially Theorem 3.1 extends Theorem 2.1 of Dutta and Choudhury [9], Corollary 3.1 extends well known Banach contraction principle, Corollary 3.2 extends the corresponding result of Rhoades [13] and Theorem 3.2 extends Theorem 2.2 of Doric [8] from complete metric space to that in the setting of complete S-metric space.

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