NEHARI TYPE GROUND STATE SOLUTION FOR SCHRÖDINGER-BOPP-PODOLSKY SYSTEM

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This paper is dedicated to study the following Schrödinger-Bopp-Podolsky system

$$\begin{cases} -\Delta u + u + \phi u = f(x,u), & in\ \mathbb{R}^3, \\ -\Delta\phi + \Delta^2\phi = 4\pi u^2, & in\ \mathbb{R}^3. \end{cases}$$

We use the non-Nehari manifold approach to establish the existence of the Nehari type ground state solutions by introducing the conditions $\lim_{|t|\to\infty}\left(\int_0^t f(x,s)\mathrm{d}s\right)/|t|^3 = \infty$ *uniformly in* $x \in \mathbb{R}^3$ *and*

$$\left[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,t\tau)}{(t\tau)^3}\right]\mathrm{sign}(1-t) + \theta_0\frac{|1-t^2|}{(t\tau)^2} \geq 0, \quad \forall x \in \mathbb{R}^3, t > 0, \tau \neq 0$$

with constant $\theta_0 \in (0,1)$.

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1. Introduction

In [1], d'Avenia and Siciliano have attracted the attention on a new kind of elliptic system which, to the best of our knowledge, was never been considered before in the mathematical literature, although the problem was known among the physicists. It is named Schrödinger-Bopp-Podolsky system. Such system appears when we couple a Schrödinger field $\psi = \psi(t,x)$ with its electromagnetic field in the Bopp-Podolsky electromagnetic theory, and, in particular, in the electrostatic case for standing waves $\psi(t,x) = e^{i\omega t}u(x)$. The Bopp-Podolsky theory, developed by Bopp [2], and independently by Podolsky [3], is a second order gauge theory for the electromagnetic field. As the

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Mie theory [4] and its generalizations given by Born and Infeld [5, 6], it was introduced to solve the so called infinity problem that appears in the classical Maxwell theory.

Let us consider the nonlinear Schrödinger Lagrangian density

$$\mathcal{L}_{\mathrm{Sc}} = i\hbar\bar{\psi}\partial_t\psi - \frac{\hbar^2}{2m}|\nabla\psi|^2 + \frac{2}{p}|\psi|^p,$$

where $\psi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}, \hbar, m, p > 0$ and let $(\phi, \mathbf{A})$ be the gauge potential of the electromagnetic field $(\mathbf{E}, \mathbf{H})$, namely $\phi : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\partial_t\mathbf{A}, \quad \mathbf{H} = \nabla \times \mathbf{A}.$$

The coupling of the field ψ with the electromagnetic field $(\mathbf{E}, \mathbf{H})$ through the minimal coupling rule. If we consider standing waves $\psi(t, x) = e^{i\omega t/\hbar}u(x)$ in the purely electrostatic case ($\phi = \phi(x)$ and $\mathbf{A} = \mathbf{0}$) and normalize the constant $\hbar$ and m, we will face the following problem (see [1])

$$\begin{cases} -\Delta u + \omega u + q^2\phi u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1}$$

We point out that few papers are known to treat this type of system.

In the recent paper [1], d'Avenia and Siciliano deal with problem (1) and studied the existence, nonexistence and the behavior of the solution as $a \to 0$. Actually, they proved the following results. If $a, \omega > 0$ and $p \in (2, 6)$, then, there exists $q_* > 0$ such that, for all $q \in (-q_*, q_*)\setminus\{0\}$, problem (1) admits a nontrivial solution. If $a, \omega > 0$ and $p \in (3, 6)$, then, for all $q \neq 0$ problem (1) admits a nontrivial solution. Siciliano and Silva in [7] considered system (1) where $p \in (2, 3], \omega > 0, a \geq 0$ are fixed. They proved, by means of the fibering approach, that the system has no solutions at all for large values of q, and has two radial solutions for small q. They give also qualitative properties about the energy level of the solutions and a variational characterization of these extremal values of q.

As we see, from a variational point of view, system (1) can be obtained by means of a suitable "interaction" between the Schrödinger Lagrangian density and the Lagrangian density of the electromagnetic field according to the Bopp-Podolsky theory, and not the Maxwell theory. In the paper [8] of d'Avenia and Pisani, the Born-Infeld Lagrangian density interacting with the Klein-Gordon equation is considered. They found infinitely many radial solutions in the subcritical case via the symmetric mountain pass theorem. We cite this paper because the use of the Born-Infeld Lagrangian density for the electromagnetic field (in place of the classical Maxwell Lagrangian density) gives rise to the equation for the electrostatic field. And this type of system is studied for a couple of years, see [9, 10, 11, 12, 13, 14, 15]. This can be sees also as a consequence of the fact that a different (actually a better) Lagrangian of the electromagnetic field is considered in such a way that the classical Maxwell Lagrangian is a first approximation of this new one.

Coming back to the present paper, our aim is to study the Nehari type ground state solution for the Schrödinger-Bopp-Podolsky system with the nonlinear term has *asymptotically cubic or super-cubic growth*. For simplicity, we consider the parameters ω, q and a all equals 1. More specifically, we concern the following system

$$\begin{cases} -\Delta u + u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \Delta^2\phi = 4\pi u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{$\mathcal{SBP}$}$$

As described in Section 2, to solve problem ($\mathcal{SBP}$) is equivalent to solve

$$-\Delta u + u + \left(\frac{1-e^{-|x|/a}}{|x|} * u^2\right) u = f(x,u) \quad \text{in } \mathbb{R}^3, \tag{2}$$

whose solutions correspond to critical points of the energy functional defined in $H^1(\mathbb{R}^3)$ by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \left(\frac{1-e^{-|x|/a}}{|x|} * u^2\right) u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\mathrm{d}x, \tag{3}$$

where $F(x,u) = \int_0^u f(x,t)\mathrm{d}t$. Define

$$\mathcal{N} := \left\{u \in H^1(\mathbb{R}^3) : \langle \Phi'(u), u\rangle = 0, u \neq 0\right\},$$

which is the Nehari manifold of Φ. A solution is called a Nehari type ground state solution if its energy is minimal among all nontrivial solutions in $\mathcal{N}$, that is, a solution $u_0 \in H^1(\mathbb{R}^3)$ such that $\Phi(u_0) = \inf_{\mathcal{N}} \Phi > 0$.

Now, the nonlinearity $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ satisfy the following basic assumptions:

(f_0) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $f(x,t) = o(|t|)$ as $t \to 0$, uniformly in $x \in \mathbb{R}^3$, and there exist constants $C_0 > 0$ and $\kappa \in (2,6)$ such that,

$$|f(x,t)| \le C_0\left(1 + |t|^{\kappa-1}\right), \quad \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$

(f_1) there exists $\theta_0 \in (0,1)$ such that

$$\left[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,t\tau)}{(t\tau)^3}\right] \mathrm{sign}(1-t) + \theta_0 \frac{|1-t^2|}{(t\tau)^2} \ge 0, \quad \forall x \in \mathbb{R}^3, t > 0, \tau \neq 0, \tag{4}$$

(f_2) $\lim_{|t|\to\infty} F(x,t)/|t|^3 = \infty$ uniformly in $x \in \mathbb{R}^3$.

Since system ($\mathcal{SBP}$) is set on $\mathbb{R}^3$, it is well-known that the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ $(2 \le s \le 6)$ is not compact, and then it is usually difficult to prove that a minimizing sequence or a Palais-Smale sequence is strongly convergent if we seek solutions of problem ($\mathcal{SBP}$) by variational methods. To overcome this difficulty we restrict ourselves to radial functions

$u = u(r),\ r = |x|$. More precisely, we shall consider the problem on the space of the radial functions

$$H_r^1(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) : u = u(r),\ r = |x|\}.$$

We recall (see [16]) that, for $2 < s < 6$, $H_r^1(\mathbb{R}^3)$ is compactly embedded into $L^s(\mathbb{R}^3)$.

The main result of this paper is the following.

Theorem 1.1. *Assume that conditions* (f_0)-(f_2) *hold. Then problem* $(\mathcal{SBP})$ *has a radial Nehari type ground solution.*

Remark 1.1. *Our paper is motivated by the work* [17]*. We will use the non-Nehari manifold approach developed by Tang* [18, 19]*. Unlike the Nehari manifold method, our approach lies on finding a minimizing Cerami sequence for* Φ *outside* $\mathcal{N}$ *by using the diagonal method, see Lemma 3.4.*

Example 1.1. *Now, we give functions which satisfies all the conditions* (f_0)–(f_2)*. Let* $f(x,\tau) = |\tau|^3\tau + |\tau|\tau/2$ *for all* $(x,\tau) \in \mathbb{R}^3 \times \mathbb{R}$*. It is easy to see that* f *satisfies* (f_0) *and* (f_2)*. Next, we show that* f *satisfies* (f_1)*. By elementary computations, one has*

$$\begin{aligned}\left[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,t\tau)}{(t\tau)^3}\right]&\operatorname{sign}(1-t) + \theta_0\frac{|1-t^2|}{(t\tau)^2}\\ &= |1-t||\tau| - \frac{|1-t|}{2t|\tau|} + \theta_0\frac{|1-t^2|}{(t\tau)^2}\\ &= \frac{|1-t|}{(t\tau)^2}\left[|\tau|^3t^2 - \frac{1}{2}|\tau|t + \theta_0(1+t)\right],\end{aligned} \tag{5}$$

for all $x \in \mathbb{R}^3$*,* $t > 0$*,* $\tau \neq 0$*. Note that*

$$\begin{cases} |\tau|^3t^2 - \frac{1}{2}|\tau|t + \theta_0(1+t) \geq \left(\theta_0 - \frac{1}{2}\right)t, & |\tau| \leq 1, \forall t > 0,\\ |\tau|^3t^2 - \frac{1}{2}|\tau|t + \theta_0(1+t) \geq \left(t|\tau| - \frac{1}{4}\right)^2 + \theta_0 - \frac{1}{16}, & |\tau| > 1, \forall t > 0,\end{cases}$$

then (5) *implies that* f *satisfies* (4) *with* $\theta_0 = 1/2$*. The another function is* $f(x,\tau) = \tau^3 - |\tau|^{3/2}\tau + |\tau|\tau$ *for all* $(x,\tau) \in \mathbb{R}^3 \times \mathbb{R}$*. Clearly,* f *satisfies* (f_0) *and* (f_2)*. Next, we show that* f *satisfies* (f_1)*. It is easy to check that*

$$\begin{aligned}&\left[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,t\tau)}{(t\tau)^3}\right]\operatorname{sign}(1-t) + \theta_0\frac{|1-t^2|}{(t\tau)^2}\\ =&\frac{\left|1-t^{1/2}\right|}{|t\tau|^{1/2}} - \frac{|1-t|}{|t\tau|} + \theta_0\frac{|1-t^2|}{(t\tau)^2}\\ =&\frac{\left|1-t^{1/2}\right|}{(t\tau)^2}\left[|t\tau|^{3/2} - \left(1+t^{1/2}\right)|t\tau| + \theta_0\left(1+t^{1/2}\right)(1+t)\right]\\ :=&\frac{\left|1-t^{1/2}\right|}{(t\tau)^2}h(t,|\tau|), \quad \forall x \in \mathbb{R}^3, t > 0, \tau \neq 0.\end{aligned}$$

By elementary computations, for any $t > 0$, *we have*

$$h(t, |\tau|) \geq \min_{\tau \neq 0} h(t, |\tau|) = h(t, \tau_0) \qquad \text{with } \tau_0 = \frac{4\left(1 + t^{1/2}\right)^2}{9t}$$

and so

$$\begin{aligned} h(t, |\tau|) &\geq -\frac{4}{27}\left(1 + t^{1/2}\right)^3 + \theta_0 \left(1 + t^{1/2}\right)(1 + t) \\ &= \left(1 + t^{1/2}\right)\left[\left(\theta_0 - \frac{4}{27}\right) t - \frac{8}{27} t^{1/2} + \left(\theta_0 - \frac{4}{27}\right)\right] \\ &= \left(1 + t^{1/2}\right) \frac{27\theta_0 - 4}{27}\left[\left(t^{1/2} - \frac{4}{27\theta_0 - 4}\right)^2 + 1 - \frac{16}{(27\theta_0 - 4)^2}\right]. \end{aligned}$$

Now, f *satisfies* (4) *with* $\theta_0 = 1/3$.

2. The variational framework

Now, we establish few basic standard notations. For $p \in [1, +\infty], L^p(\mathbb{R}^3)$ is the usual Lebesgue space with norm $\|u\|_p$. We denote with $H^1(\mathbb{R}^3)$ the usual Sobolev space endowed with scalar product and norm given by

$$(u, v) := \int_{\mathbb{R}^3} \nabla u \cdot \nabla v \mathrm{d}x + \int_{\mathbb{R}^3} uv \mathrm{d}x, \qquad \|u\| := (u, u)^{1/2}.$$

For $p \geq 2, D^{1,p}(\mathbb{R}^3)$ is the Banach space defined as the completion of the test functions $C_c^\infty(\mathbb{R}^3)$ with respect to the L^p-norm of the gradient. We define X the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|\phi\|_X := \|\nabla \phi\|_2 + \|\Delta \phi\|_2.$$

As a final convention, $\mathrm{d}x$ denotes the Lebesgue measure in integrals, it always omitted in the following paper, otherwise we will write explicitly the measure.

The natural functional spaces in which find the solutions of $(\mathcal{SBP})$ are:

$$u \in H^1(\mathbb{R}^3), \quad \phi \in X.$$

By a (weak) solution of $(\mathcal{SBP})$, we mean a pair $(u, \phi) \in H^1(\mathbb{R}^3) \times X$ such that

$$\forall v \in H^1(\mathbb{R}^3): \quad \int_{\mathbb{R}^3} \nabla u \cdot \nabla v + \int_{\mathbb{R}^3} uv + \int_{\mathbb{R}^3} \phi uv = \int_{\mathbb{R}^3} f(x, u)v, \tag{6}$$

$$\forall \xi \in X: \quad \int_{\mathbb{R}^3} \nabla \phi \cdot \nabla \xi + \int_{\mathbb{R}^3} \Delta \phi \Delta \xi = 4\pi \int_{\mathbb{R}^3} \phi u^2. \tag{7}$$

As we say before, we will consider this problem on the space of the radial functions $H_r^1(\mathbb{R}^3)$, and by Palais principle of symmetric criticality (see [20, p. 18]), if u is a critical point of Φ restricted to $H_r^1(\mathbb{R}^3)$ then u is a critical point of Φ. So, we will use the space $H_r^1(\mathbb{R}^3)$ in the following manuscript. We have now a first variational principle; indeed, it is easy to see that the critical points of the functional

$$\mathcal{J}(u,\phi)=\frac{1}{2}\|u\|^2+\frac{1}{2}\int_{\mathbb{R}^3}\phi u^2-\int_{\mathbb{R}^3}F(x,u)-\frac{1}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi|^2-\frac{1}{16\pi}\int_{\mathbb{R}^3}|\Delta\phi|^2 \quad (8)$$

on $H_r^1(\mathbb{R}^3)\times X$ are exactly the weak solutions of $(\mathcal{SBP})$, according to (6) and (7). However, since this functional $\mathcal{J}$ is strongly indefinite, we adopt a reduction procedure which is successfully used with the "classical" Schrödinger-Maxwell system.

Let $u\in H_r^1(\mathbb{R}^3)$ be fixed, there exists a unique element in X, that we denote with ϕ_u, such that

$$-\Delta\phi_u+\Delta^2\phi_u=4\pi u^2 \quad \text{in } \mathbb{R}^3. \quad (9)$$

Furthermore, the unique solution in X of the second equation in $(\mathcal{SBP})$ is

$$\phi_u:=\frac{1-e^{-|x|}}{|x|}*u^2.$$

Actually, when we consider the operator $-\Delta+\Delta^2$, we have that $\mathcal{K}(x-x_0)$, with $\mathcal{K}(x):=\frac{1-e^{-|x|}}{|x|}$, is the fundamental solution of the equation

$$-\Delta\phi+\Delta^2\phi=4\pi\delta_{x_0},$$

where δ_{x_0} is the delta function (see [1, Lemma 3.3]). For every fixed $u\in H^1(\mathbb{R}^3)$, the solutions of second equation of $(\mathcal{SBP})$ are critical points of the functional

$$E(\phi)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\phi|^2+\frac{1}{2}\int_{\mathbb{R}^3}|\Delta\phi|^2-\int\phi u^2 \quad (10)$$

defined on X. This functional is coercive; indeed, by the continuous embedding of X in $L^\infty(\mathbb{R}^3)$ (see [1, Lemma 3.1]),

$$E(\phi)\geq\frac{1}{2}\|\nabla\phi\|_2^2+\frac{1}{2}\|\Delta\phi\|_2^2-c\left\|u^2\right\|_1\|\nabla\phi\|_2.$$

Furthermore, E is weakly lower semicontinuous since each term in (10) is continuous and convex. Therefore, E admits a global minimum. The solution is unique because the operator $\mathcal{A}=-\Delta+\Delta^2+4\pi u^2$ is strictly monotone. Now, for ϕ_u the following useful properties hold.

Lemma 2.1 ([1])**.** *For every $u\in H_r^1(\mathbb{R}^3)$ we have:*

(i) *for every $y\in\mathbb{R}^3, \phi_{u(\cdot+y)}=\phi_u(\cdot+y)$;*
(ii) *$\phi_u\geq 0$;*
(iii) *for every $s\in(3,+\infty], \phi_u\in L^s(\mathbb{R}^3)\cap C^0(\mathbb{R}^3)$;*
(iv) *$\|\phi_u\|_6\leq C\|u\|^2$;*
(v) *if $u_n\rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$, then $\phi_{u_n}\to\phi_u$ in X;*
(vi) *ϕ_u is the unique minimizer of the functional*

$$E(\phi)=\frac{1}{2}\|\nabla\phi\|_2^2+\frac{1}{2}\|\Delta\phi\|_2^2-\int\phi u^2, \quad \phi\in X.$$

We introduce the map

$$\Gamma : u \in H^1_r(\mathbb{R}^3) \mapsto \phi_u \in X.$$

The next result is a consequence of the fact that $\mathcal{J}$ is C^1 and the implicit function theorem. The arguments used to prove Lemma 2.2 and Lemma 2.3 are exactly the same as in [21] for the Schrödinger-Maxwell system, or [22] for the Klein-Gordon-Maxwell system.

Lemma 2.2. *Let G_Γ be the graph of the map $\Gamma : u \in H^1_r(\mathbb{R}^3) \mapsto \phi_u \in X$. Then*

$$G_\Gamma = \left\{(u,\phi) \in H^1_r(\mathbb{R}^3) \times X : \partial_\phi \mathcal{J}(u,\phi) = 0\right\}.$$

Moreover

$$\Gamma \in C^1(H^1_r(\mathbb{R}^3); X).$$

In view of this, the functional

$$\begin{aligned} \Phi(u) &:= \mathcal{J}(u,\phi_u) \qquad (11)\\ &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} F(x,u) \qquad (12)\end{aligned}$$

is of class C^1 and in particular we have

$$\begin{aligned} \langle \Phi'(u), v\rangle &= \partial_u \mathcal{J}(u,\phi_u)[v] + \partial_\phi \mathcal{J}(u,\phi_u) \circ \Phi'(u)[v]\\ &= \partial_u \mathcal{J}(u,\phi_u)[v].\end{aligned}$$

Then by (8) we have

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v + \int_{\mathbb{R}^3} uv + \int_{\mathbb{R}^3} \phi_u uv - \int_{\mathbb{R}^3} f(x,u)v. \qquad (13)$$

Furthermore, we have the following result.

Lemma 2.3. *The following statements are equivalent:*

(i) *the pair $(u,\phi) \in H^1_r(\mathbb{R}^3) \times X$ is a critical point of $\mathcal{J}$ (i.e. (u,ϕ) is a solution of $(\mathcal{SBP})$),*

(ii) *u is a critical point of Φ and $\phi = \phi_u$.*

The functional Φ of the unique variable u obtained by $\mathcal{J}$ is usually called the *reduced functional*.

In view of Lemma 2.3, the critical points of Φ satisfy the equation

$$-\Delta u + u + \phi_u u = f(x,u) \quad \text{in} \quad \mathbb{R}^3, \qquad (14)$$

which is the equation we are going to consider in the following.

3. **Proof of the main result**

Lemma 3.1. *Under assumptions* (f_0) *and* (f_1), *there results*

$$\Phi(u) \geq \Phi(tu) + \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{(1-\theta_0)\left(1-t^2\right)^2}{4}\|u\|^2, \tag{15}$$

for all $u \in H_r^1(\mathbb{R}^3)$, $t \geq 0$.

Proof. For any $x \in \mathbb{R}^3$, $t \geq 0$, $\tau \neq 0$, (f_1) yields

$$\begin{aligned}&\frac{1-t^4}{4}\tau f(x,\tau) + F(x,t\tau) - F(x,\tau) + \frac{\theta_0}{4}\left(1-t^2\right)^2\tau^2\\ &= \int_t^1 \left[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,s\tau)}{(s\tau)^3} + \theta_0\frac{(1-s^2)}{(s\tau)^2}\right] s^3\tau^4 \mathrm{d}s \\ &\geq 0.\end{aligned} \tag{16}$$

Note that

$$\Phi(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 - \int_{\mathbb{R}^3} F(x,u) \tag{17}$$

and

$$\langle \Phi'(u), u\rangle = \|u\|^2 + \int_{\mathbb{R}^3}\phi_u u^2 - \int_{\mathbb{R}^3} f(x,u)u. \tag{18}$$

Thus, by (16)–(18), one has

$$\begin{aligned}\Phi(u) - \Phi(tu) =& \frac{1-t^2}{2}\|u\|^2 + \frac{1-t^4}{4}\int_{\mathbb{R}^3}\phi_u u^2 + \int_{\mathbb{R}^3}[F(x,tu) - F(x,u)]\\ =& \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{\left(1-t^2\right)^2}{4}\|u\|^2\\ &+ \int_{\mathbb{R}^3}\left[\frac{1-t^4}{4}f(x,u)u + F(x,tu) - F(x,u)\right]\\ \geq& \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{(1-\theta_0)\left(1-t^2\right)^2}{4}\|u\|^2, \quad t \geq 0.\end{aligned}$$

This shows that (15) holds. □

Corollary 3.1. *Define*

$$\mathcal{N} := \left\{u \in H_r^1(\mathbb{R}^3) : \langle \Phi'(u), u\rangle = 0, u \neq 0\right\},$$

which is the Nehari manifold of Φ. *Now, under assumptions* (f_0) *and* (f_1), *for* $u \in \mathcal{N}$,

$$\Phi(u) = \max_{t\geq 0}\Phi(tu). \tag{19}$$

Unlike the super-cubic case, to show $\mathcal{N} \neq \emptyset$ in our situation, we have to overcome the competing effect of the nonlocal term. To this end, we define a set Λ as follows:

$$\Lambda = \left\{u \in H_r^1(\mathbb{R}^3) : \int_{\mathbb{R}^3}\left[u^2 + \phi_u u^2 - f(x,u)u\right] < 0\right\}.$$

Lemma 3.2. *Under assumptions* (f_0)*–*(f_2)*,* $\Lambda \neq \emptyset$ *and* $\mathcal{N} \subset \Lambda$*. Then, for any* $u \in \Lambda$*, there exists a unique* $t_u > 0$ *such that* $t_u u \in \mathcal{N}$*.*

Proof. First, we show that $\Lambda \neq \emptyset$. By using the Hardy-Littlewood-Sobolev inequality (see [23, p. 98]), we have the following inequality:

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)v(y)|}{|x-y|}\mathrm{d}x\mathrm{d}y \leq \frac{8\sqrt[3]{2}}{3\sqrt[3]{\pi}}\|u\|_{6/5}\|v\|_{6/5}, \quad u,v \in L^{6/5}\left(\mathbb{R}^3\right). \tag{20}$$

From (20) and Sobolev embedding theorem, there exists $C_1 > 0$ such that

$$\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \leq C_1\|u\|^4$$

for all $u \in H_r^1(\mathbb{R}^3)$. For any fixed $u \in H_r^1(\mathbb{R}^3)$ with $u \neq 0$, set $u_t(x) = u(tx)$ for $t > 0$. One has

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left[(tu_t)^2 + \phi_{(tu_t)}(tu_t)^2 - f(x,tu_t)\,tu_t\right]\mathrm{d}x \\
=&t^{-1}\int_{\mathbb{R}^3} u^2\mathrm{d}x + t^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1-e^{-\frac{|x-y|}{t}}}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^3}\frac{f\left(t^{-1}x,tu\right)tu}{t^3}\mathrm{d}x \\
\leq&t^{-1}\int_{\mathbb{R}^3} u^2\mathrm{d}x + t^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^3}\frac{f\left(t^{-1}x,tu\right)tu}{t^3}\mathrm{d}x \\
\leq&t^{-1}\|u\|_2^2 + C_1 t^{-1}\|u\|^4 - \int_{\mathbb{R}^3}\frac{f\left(t^{-1}x,tu\right)tu}{t^3}\mathrm{d}x.
\end{aligned} \tag{21}$$

Note that for $u(x) \neq 0$, $F\left(t^{-1}x,tu\right)/|tu|^3 \to +\infty$ as $t \to +\infty$ uniformly in $x \in \mathbb{R}^3$ by (f_2), and (16) with $t = 0$ yields

$$\frac{1}{4}f(x,\tau)\tau - F(x,\tau) + \frac{\theta_0}{4}\tau^2 \geq 0, \quad \forall x \in \mathbb{R}^3, \tau \in \mathbb{R}, \tag{22}$$

then we have

$$\frac{f\left(t^{-1}x,tu\right)tu}{|tu|^3} \to +\infty \quad \text{as } t \to +\infty \text{ uniformly in } x \in \mathbb{R}^3. \tag{23}$$

Thus, it follows from (21) and (23) that

$$\int_{\mathbb{R}^3} \left[(tu_t)^2 + \phi_{(tu_t)}(tu_t)^2 - f(x,tu_t)\,tu_t\right] \to -\infty \quad \text{as } t \to +\infty.$$

Thus, taking $v = Tu_T$ for T large, we have $v \in \Lambda$. Hence $\Lambda \neq \emptyset$. From (13), it is easy to see that $\mathcal{N} \subset \Lambda$.

Next, we prove the last part of lemma. Let $u \in \Lambda$ be fixed and define a function $g(t) := \langle \Phi'(tu), tu\rangle$ on $[0,\infty)$. By (f_1), one has

$$f(x,t\tau)t\tau \geq f(x,\tau)\tau t^4 - \theta_0\left(t^2-1\right)(t\tau)^2, \quad \forall x \in \mathbb{R}^3, t \geq 1, \tau \in \mathbb{R}, \tag{24}$$

which yields

$$\int_{\mathbb{R}^3}\left[\theta_0(t\tau)^2+\phi_{t\tau}(t\tau)^2-f(x,t\tau)t\tau\right]\le t^4\int_{\mathbb{R}^3}\left[\theta_0\tau^2+\phi_\tau\tau^2-f(x,\tau)\tau\right], \quad (25)$$

for all $t\ge 1$, $\tau\in\mathbb{R}$. From (13) and (25) it follows that

$$g(t)\le t^2\|u\|^2+t^4\int_{\mathbb{R}^3}\left[u^2+\phi_u u^2-f(x,u)u\right]-\theta_0 t^2\int_{\mathbb{R}^3}u^2,\quad \forall t\ge 1. \quad (26)$$

Using (f_0), (13) and (26), it is easy to verify that $g(0)=0$, $g(t)>0$ for $t>0$ small and $g(t)<0$ for t large due to $u\in\Lambda$. Therefore, there exist a $t_u>0$ so that $g(t_u)=0$ and $t_u u\in\mathcal{N}$. We claim that t_u is unique for any $u\in\Lambda$. In fact, for any given $u\in\Lambda$, let t_1, $t_2>0$ such that $g(t_1)=g(t_2)=0$. Jointly with (15), we have

$$\begin{aligned}\Phi\left(t_1u\right)&\ge\Phi\left(t_2u\right)+\frac{t_1^4-t_2^4}{4t_1^4}\left\langle\Phi'\left(t_1u\right),t_1u\right\rangle+\frac{\left(1-\theta_0\right)\left(t_1^2-t_2^2\right)^2}{4t_1^4}\|u\|^2\\&=\Phi\left(t_2u\right)+\frac{\left(1-\theta_0\right)\left(t_1^2-t_2^2\right)^2}{4t_1^4}\|u\|^2\end{aligned} \quad (27)$$

and

$$\begin{aligned}\Phi\left(t_2u\right)&\ge\Phi\left(t_1u\right)+\frac{t_2^4-t_1^4}{4t_2^4}\left\langle\Phi'\left(t_2u\right),t_2u\right\rangle+\frac{\left(1-\theta_0\right)\left(t_2^2-t_1^2\right)^2}{4t_2^4}\|u\|^2\\&=\Phi\left(t_1u\right)+\frac{\left(1-\theta_0\right)\left(t_2^2-t_1^2\right)^2}{4t_2^4}\|u\|^2\end{aligned} \quad (28)$$

(27) and (28) imply $t_1=t_2$. Hence, $t_u>0$ is unique for any $u\in\Lambda$. □

Lemma 3.3. *Under assumptions* (f_0)–(f_2), *then*

$$\inf_{u\in\mathcal{N}}\Phi(u):=c=\inf_{u\in\Lambda,u\neq 0}\max_{t\ge 0}\Phi(tu)>0.$$

Proof. Both Corollary 3.1 and Lemma 3.2 imply that

$$c=\inf_{u\in\Lambda,u\neq 0}\max_{t\ge 0}\Phi(tu).$$

Using Lemma 3.1, it is easy to see that $c>0$. □

Lemma 3.4. *Under assumptions* (f_0)–(f_2), *there exist a constant* $c_*\in(0,c]$ *and a sequence* $\{u_n\}\subset H_r^1(\mathbb{R}^3)$ *satisfying*

$$\Phi\left(u_n\right)\to c_*,\quad \left\|\Phi'\left(u_n\right)\right\|\left(1+\left\|u_n\right\|\right)\to 0 \quad (29)$$

as $n\to\infty$.

Proof. By (f_0) and (11), we know that there exist $\delta_0>0$ and $\rho_0>0$ such that

$$\Phi(u)\ge\rho_0,\quad \|u\|=\delta_0. \quad (30)$$

In view of Lemmas 3.2 and 3.3, we may choose $v_k\in\mathcal{N}\subset\Lambda$ such that

$$c-\frac{1}{k}<\Phi\left(v_k\right)<c+\frac{1}{k},\quad k\in\mathbb{N}. \quad (31)$$

Using Lemma 3.1 and (30), it is easy to check that $\Phi(tv_k) \geq \rho_0$ for small $t > 0$ and $\Phi(tv_k) < 0$ for large $t > 0$ due to $v_k \in \Lambda$. Since $\Phi(0) = 0$, then the mountain pass lemma in [24] implies that there exists a sequence $\{u_{k,n}\}_{n\in\mathbb{N}} \subset H_r^1(\mathbb{R}^3)$ satisfying

$$\Phi(u_{k,n}) \to c_k, \quad \|\Phi'(u_{k,n})\|(1+\|u_{k,n}\|) \to 0, \text{ as } n\to\infty, \quad k \in \mathbb{N}, \tag{32}$$

where $c_k \in \left[\rho_0, \sup_{t\geq 0}\Phi(tv_k)\right]$. By virtue of Corollary 3.1, one has $\Phi(v_k) = \sup_{t\geq 0}\Phi(tv_k)$. Hence, by (31) and (32), one has

$$\Phi(u_{k,n}) \to c_k \in \left[\rho_0, c+\frac{1}{k}\right), \quad \|\Phi'(u_{k,n})\|(1+\|u_{k,n}\|) \to 0, \tag{33}$$

as $n\to\infty$, $k\in\mathbb{N}$. Now, we can choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\Phi(u_{k,n_k}) \in \left[\rho_0, c+\frac{1}{k}\right), \quad \|\Phi'(u_{k,n_k})\|(1+\|u_{k,n_k}\|) < \frac{1}{k}, \quad k\in\mathbb{N}. \tag{34}$$

Let $u_k = u_{k,n_k}$, $k\in\mathbb{N}$. Then, going if necessary to a subsequence, we have

$$\Phi(u_n) \to c_* \in [\rho_0, c], \quad \|\Phi'(u_n)\|(1+\|u_n\|) \to 0 \text{ as } n\to\infty.$$

□

Lemma 3.5. *Under assumptions* (f_0)*–*(f_2)*, any sequence* $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ *satisfying* (29) *is bounded in* $H_r^1(\mathbb{R}^3)$.

Proof. By Lemma 3.1, one has

$$c_* + o(1) = \Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n\rangle \geq \frac{1-\theta_0}{4}\|u_n\|^2.$$

This shows that sequence $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. □

Next, we prove the minimizer of the constrained problem is a critical point of the functional Φ, which plays a crucial role to get the solution for problem $(\mathcal{SBP})$.

Lemma 3.6. *Under assumptions* (f_0)*–*(f_2)*, if* $u_0 \in \mathcal{N}$ *and* $\Phi(u_0) = c$*, then* u_0 *is a critical point of* Φ.

Proof. Assume that $u_0 \in \mathcal{N}$, $\Phi(u_0) = c$ and $\Phi'(u_0) \neq 0$. Then there exist $\delta > 0$ and $\rho > 0$ such that

$$\|u - u_0\| \leq 3\delta \Longrightarrow \|\Phi'(u)\| \geq \varrho.$$

In view of Lemma 3.1, one has

$$\begin{aligned}\Phi(tu_0) &\leq \Phi(u_0) - \frac{(1-\theta_0)(1-t^2)^2}{4}\|u_0\|^2 \\ &= c - \frac{(1-\theta_0)(1-t^2)^2}{4}\|u_0\|^2, \quad \forall t \geq 0.\end{aligned} \tag{35}$$

For $\varepsilon := \min\left\{3(1-\theta_0)\|u_0\|^2/64, 1, \varrho\delta/8\right\}$, $S := B(u_0,\delta)$, [20, Lemma 2.3] yields a deformation $\eta \in C([0,1]\times H_r^1(\mathbb{R}^3), H_r^1(\mathbb{R}^3))$ such that

(i) $\eta(1, u)=u$ if $\Phi(u)<c-2\varepsilon$ or $\Phi(u)>c+2\varepsilon$;
(ii) $\eta\left(1, \Phi^{c+\varepsilon} \cap B\left(u_{0}, \delta\right)\right) \subset \Phi^{c-\varepsilon}$;
(iii) $\Phi(\eta(1, u)) \leq \Phi(u), \forall u \in H_{r}^{1}(\mathbb{R}^{3})$;
(iv) $\eta(1, u)$ is a homeomorphism of $H_{r}^{1}(\mathbb{R}^{3})$.

By Corollary 3.1, $\Phi\left(t u_{0}\right) \leq \Phi\left(u_{0}\right)=c$ for $t \geq 0$, then it follows from (ii) that

$$\Phi\left(\eta\left(1, t u_{0}\right)\right) \leq c-\varepsilon, \quad \forall t \geq 0,|t-1|<\delta /\left\|u_{0}\right\| . \tag{36}$$

On the other hand, by (iii) and (35), one has

$$\begin{aligned} \Phi\left(\eta\left(1, t u_{0}\right)\right) \leq \Phi\left(t u_{0}\right) & \\ & \leq c-\frac{\left(1-\theta_{0}\right)\left(1-t^{2}\right)^{2}}{4}\left\|u_{0}\right\|^{2} \\ & \leq c-\frac{\left(1-\theta_{0}\right) \delta^{2}}{4}, \quad \forall t \geq 0,|t-1| \geq \delta /\left\|u_{0}\right\| . \end{aligned} \tag{37}$$

Combining (36) with (37), we have

$$\max _{t \in[1 / 2, \sqrt{7} / 2]} \Phi\left(\eta\left(1, t u_{0}\right)\right)<c .$$

We prove that $\eta\left(1, t u_{0}\right) \cap \mathcal{N} \neq \emptyset$ for some $t \in[1 / 2, \sqrt{7} / 2]$, contradicting to the definition of c. Define

$$\Psi_{0}(t):=\left\langle\Phi^{\prime}\left(t u_{0}\right), t u_{0}\right\rangle, \quad \Psi_{1}(t):=\left\langle\Phi^{\prime}\left(\eta\left(1, t u_{0}\right)\right), \eta\left(1, t u_{0}\right)\right\rangle, \quad \forall t \geq 0 .$$

Since $u_{0} \neq 0$, it follows from (iv) that $\eta(1, t u_{0}) \neq 0$ for all $t>0$. By Lemma 3.2 and the degree theory (see [25]), one can derive that

$$\operatorname{deg}\left(\Psi_{0},(1 / 2, \sqrt{7} / 2), 0\right)=1 .$$

It follows from (35) and (i) that

$$\eta\left(1, t u_{0}\right)=t u_{0} \text { for } t=1 / 2 \text { and } t=\sqrt{7} / 2 .$$

Thus, $\operatorname{deg}\left(\Psi_{1},(1 / 2, \sqrt{7} / 2), 0\right)=\operatorname{deg}\left(\Psi_{0},(1 / 2, \sqrt{7} / 2), 0\right)=1$. Hence, $\Psi_{1}\left(t_{0}\right)=0$ for some $t_{0} \in(1 / 2, \sqrt{7} / 2)$, that is $\eta(1, t_{0} u_{0}) \in \mathcal{N}$, which is a contradiction. □

Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Lemma 3.4 implies the existence of a sequence $\{v_{n}\} \subset H_{r}^{1}(\mathbb{R}^{3})$ satisfying (29), then

$$\Phi\left(v_{n}\right) \rightarrow c_{*} \in(0, c], \quad\left\|\Phi^{\prime}\left(v_{n}\right)\right\|\left(1+\left\|v_{n}\right\|\right) \rightarrow 0 .$$

By Lemma 3.5, $\{v_{n}\}$ is bounded in $H_{r}^{1}(\mathbb{R}^{3})$. Passing to a subsequence, we have $v_{n} \rightharpoonup v$ in $H_{r}^{1}(\mathbb{R}^{3})$, $v_{n} \rightarrow v$ in $L^{s}(\mathbb{R}^{3})$, $2<s<6$ and $v_{n}(x) \rightarrow v(x)$ a.e. on $\mathbb{R}^{3}$. For every $\phi \in C_{0}^{\infty}\left(\mathbb{R}^{3}\right)$, we have

$$\left\langle\Phi^{\prime}(v), \phi\right\rangle=\lim _{n \rightarrow \infty}\left\langle\Phi^{\prime}\left(v_{n}\right), \phi\right\rangle=0 .$$

Hence, $\Phi'(v) = 0$. This shows that $v \in \mathcal{N}$ is a nontrivial solution of problem $(\mathcal{SBP})$ and $\Phi(v) \geq c$. It follows from (f_1) and Fatou's lemma that

$$\begin{aligned}
c \geq c_* &= \lim_{n\to\infty}\left[\Phi\left(v_n\right) - \frac{1}{4}\left\langle \Phi'\left(v_n\right), v_n\right\rangle\right] \\
&= \lim_{n\to\infty}\left\{\frac{1-\theta_0}{4}\left\|v_n\right\|^2 + \frac{\theta_0}{4}\left\|\nabla v_n\right\|_2^2 + \int_{\mathbb{R}^3}\left[\frac{1}{4}f\left(x,v_n\right)v_n - F\left(x,v_n\right) + \frac{\theta_0}{4}v_n^2\right]\mathrm{d}x\right\} \\
&\geq \frac{1}{4}\liminf_{n\to\infty}\left[\left(1-\theta_0\right)\left\|v_n\right\|^2 + \theta_0\left\|\nabla v_n\right\|_2^2\right] \\
&\quad + \liminf_{n\to\infty}\int_{\mathbb{R}^3}\left[\frac{1}{4}f\left(x,v_n\right)v_n - F\left(x,v_n\right) + \frac{\theta_0}{4}v_n^2\right]\mathrm{d}x \\
&\geq \frac{1}{4}\|v\|^2 + \int_{\mathbb{R}^3}\left[\frac{1}{4}f(x,v)v - F(x,v)\right]\mathrm{d}x \\
&= \Phi(v) - \frac{1}{4}\left\langle \Phi'(v), v\right\rangle = \Phi(v).
\end{aligned}$$

This shows that $\Phi(v) \leq c$ and so $\Phi(v) = c = \inf_{\mathcal{N}}\Phi > 0$. □

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