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APPLICATIONS AIMING BÜHLMANN'S CREDIBILITY MODEL

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Lucrarea prezintă anumite (unele) probleme practice de asigurare, care pot fi rezolvate prin mijloacele (metodele) teoriei credibilității. Toate rezultatele numerice din această lucrare au fost obținute utilizând modelul original al lui Bühlmann. Exemplele practice (aplicațiile) au fost prezentate pentru a ilustra posibilitățile teoriei credibilității. Aceasta a permis o aprofundare mai mare şi o înțelegere mai bună a aspectelor teoretice anterior discutate şi va evidenția calea spre posibilitățile practice ale unora dintre modelele originale ale lui Bühlmann. ***Principalele rezultate ale acestei lucrari sunt: 1) aplicațiile estimatorului optim de credibilitate Bühlmann****; 2)* ***estimarea parametrilor de structură din modelul clasic Bühlmann****, ca aplicație utilă în estimarea primelor de credibilitate pentru acest model clasic de credibilitate al lui* ***Bühlmann****; 3)* ***modelul de credibilitate recursivă - motivația noastră pentru a introduce acest model a fost aceea că am dorit ca noile pretenții (solicitări de despăgubire) să aibă mai multă valoare (greutate, pondere, importanță), decât cele mai vechi; 4) modelul de credibilitate care încorporează volumul riscului –*** *motivația noastra pentru a introduce acest model, constă în faptul că în modelul simplu de credibilitate am presupus că volumul riscului era acelaşi pentru toti anii, în timp ce, adesea, în special în reasigurări, se doreşte să se permită variația volumul riscului, iar pentru acest scop am introdus modelul care încorporează volumule riscului;***5)*** finalizăm această lucrare, prezentând ca aplicație a modelului simplu, modelul de regresie a credibilității, ce permite efecte precum inflația (în cadrul modelului simplu al lui Bühlmann am permis ca $EVar(X_j|\theta)$ să varieze; în modelul de regresie a credibilității permitem variația lui EX_j).*

The paper presents some practical insurance problems that can be solved by means of credibility theory. All numerical results in this paper were obtained using the original Bühlmann model. Practical examples (applications) will be given to illustrate the possibilities of credibility. This will give more insight and understanding of the previously discussed theoretical aspects and will point the way to the practical possibilities of some of the original Bühlmann model.

The main results *of the paper are:* ***1) the applications*** *of* ***the optimal credibility estimator*** *of* ***Bühlmann****;* ***2) estimation*** *of* ***the structural parameters*** *in* ***the classical Bühlmann model****, as* ***useful application from them when estimating the credibility premium*** *for* ***this classical Bühlmann model****;* ***3) the recursive credibility model****-our motivation for introducing this model was that we wanted new claims to have more weight than older claims;* ***4) the credibility model incorporating risk volumes****-our motivation for introducing this model is that in the simple model we assumed that the risk volume was the same for all years; often, especially in reinsurance, one wants to allow the risk volumes to vary and for that*

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purpose we will introduce the credibility model incorporating risk volumes; ***5)*** *we end this paper presenting as application of the simple model* ***the credibility regression model*** *allowing for effects like inflation (in the simple credibility model of Bühlmann we allowed $EVar(X_j|\theta)$ to vary; in the credibility regression model we allowed EX_j to vary).*

Key - words: the risk premium, the credibility calculations, Bühlmann's original model, Bühlmann's classical model.

Mathematics Subject Classifications: 62P05.

1. Introduction

In this article we first present the original Bühlmann model, which involves only one isolated contract**.** We derive the best linear credibility estimators for this model (see the applications of the optimal credibility estimator of Bühlmann) and we consider as applications of this result: 1) estimation of the structural parameters in the classical Bühlmann model, as useful applications from them when estimating the credibility premium for this classical Bühlmann model; 2) the recursive credibility model (our motivation for introducing this model was that we wanted new claims to have more weight than older claims); 3) the credibility model incorporating risk volumes (our motivation for introducing this model is the fact that in the simple model we assumed that the risk volume was the same for all the years; often, especially in reinsurance, one wants to allow for varying risk volumes, and for that purpose Bühlmann & Straub introduced the credibility model incorporating risk volumes; 4) we end this paper, giving as example of application of the simple model, the credibility regression model allowing the effects like inflation (in the simple model we allowed $EVar(X_i|\theta)$ to vary; in the credibility regression model we are going to allow EX_j to vary).

2. The original credibility model of Bühlmann

In the original credibility model of Bühlmann, we consider one contract with unknown and fixed risk parameter θ, during a period of t years. The yearly claim amounts are noted by $X_1,\dots,X_t$. The risk parameter θ is supposed to be taken from some structure distribution $U(\cdot)$. It is assumed that, for given $\theta = \theta$, the claims are conditionally independent and identically distributed with known common distribution function $F_{X|\theta}(x,\theta)$. For this model we want to estimate the net premium $\mu(\theta) = E[X_r|\theta = \theta]$, $r = \overline{1,t}$ as well as X_{t+1} for a contract with risk parameter θ.

So in this section we first present Bühlmann’s original model, which implies only one isolated contract. The original Bühlmann model presents the optimal linear credibility estimate for the risk premium of this case. It turns out that this procedure does not provide us a statistic computable from the observations, since the result involves unknown parameters of the structure function. To obtain estimations for these structure parameters, for Bühlmann’s classical model, we embedded the contract in a group of contracts, all providing independent information about the structure distribution (see **3. The classical credibility model of Bühlmann**).

2.1 Bühlmann’s optimal credibility estimator

Suppose $X_1,\ldots,X_t$ are random variables with finite variation, which are, for given $\theta = \theta$, conditionally independent and identically distributed with already known common distribution function $F_{X|\theta}(x, \theta)$. The structure distribution function is $U(\theta) = P[\theta \le \theta]$. Let D represent the set of non-homogeneous linear combinations $g(\cdot)$ of the observable random variables $X_1, X_2, \ldots, X_t$:

$$g(\underline{X}') = c_0 + c_1X_1 + c_2X_2 + \ldots + c_tX_t \tag{2.1}$$

Then the solution of the problem:

$$\underset{g\in D}{Min}\, E\{[\mu(\theta) - g(X_1,\ldots,X_t)]^2\} \tag{2.2}$$

is:

$$g(X_1,\ldots,X_t) = z\overline{X} + (1-z)m \tag{2.3}$$

where $\underline{X}' = (X_1,\ldots,X_t)$ is the vector of observations, $z = at / (s^2+at)$, is the resulting credibility factor, $\overline{X} = \frac{1}{t}\sum_{i=1}^{t} X_i$ is the individual estimator, and a, s^2 and m are the structural parameters as defined in (2.4):

$$\left.\begin{aligned} m &= E[X_r] = E[\mu(\theta)],\ r = \overline{1,t}, \\ a &= Var\{E[X_r|\theta]\} = Var[\mu(\theta)],\ r = \overline{1,t}, \\ \sigma^2(\theta) &= Var[X_r|\theta = \theta],\ r = \overline{1,t}, \\ s^2 &= E\{Var[X_r|\theta]\} = E[\sigma^2(\theta)],\ r = \overline{1,t}. \end{aligned}\right\} \tag{2.4}$$

If $\mu(\theta)$ is replaced by X_{t+1} in (2.2), exactly the same solution (2.3) is obtained, since the co-variations with $\underline{X}$ are the same. To demonstrate the relationship (2.3), see [1], from references, pages 7-20.

2.2 Applications of the optimal credibility estimator of Bühlmann

Application 1: Recursive credibility estimation

We will analyze a little at the credibility estimator (2.3). This estimator has been criticized because it gives the claim amounts from all previous years the same weight; intuitively one should believe that new claims should have more weight than the old claims. However, as the claim amounts of different years were assumed to be exchangeable, it was only reasonable that the claim amounts should have equal weights. The following model (which is called "**Recursive credibility estimation**") is an attempt to amend this intuitive weakness, and then **an application** of **the original credibility model** of **Bühlmann**. We assume that $X_1, X_2, \ldots$ are conditionally independent given an unknown random sequence $\theta = \{\theta_i\}_{i=1}^{+\infty}$, and that for all i X_i depends on θ only through θ_i. This means that for each year i there is a separate risk parameter θ_i containing the risk characteristics of the policy in that year. The original credibility model of Bühlmann appears as a special case by assuming that $\theta_i = \theta_1$ for all i. We assume that: $E(X_i|\theta_i) = \mu(\theta_i)$ with the function μ independent of i. Assumption (2.5):

$$Cov[\mu(\theta_i), \mu(\theta_j)] = \rho^{|i-j|}\lambda \qquad (2.5),$$

with $0<\rho<1$ and $\lambda>0$ (λ bigger than zero), means that the correlation between claim amounts from different years decreases when the time distance between the years increases, which is intuitively appealing. Furthermore we suppose that : $\mu = E[\mu(\theta_i)]$, $\varphi = E[Var(X_i|\theta_i)]$, $\lambda = Var[\mu(\theta_i)]$ for all i. Our motivation for introducing the present model was that we wanted new claims to have more weight than older claims. The following result (see (2.6)) shows that this desire has been satisfied. Suppose the coefficients $\alpha_{t0}, \alpha_{t1}, \ldots, \alpha_{tt}$ are defined by $\hat{\mu}(\theta_{t+1}) = \alpha_{t0} + \sum_{j=1}^{t} \alpha_{tj} X_j$ and assume that $\rho<1$. Then we obtain:

$$0 < \alpha_{t1} < \alpha_{t2} < \ldots < \alpha_{tt} < 1 \qquad (2.6).$$

To demonstrate above relationship, i.e. $\hat{\mu}(\theta_{t+1}) = \alpha_{t0} + \sum_{j=1}^{t} \alpha_{tj} X_j$, see [1], from references, pages 63-94.

Application 2: The credibility model incorporating risk volumes

In the simple model of Bühlmann, we assumed that the risk volume was the same for all years. Often, especially in reinsurance, one wants to allow for varying risk volumes, and for that purpose we will introduce **the credibility model incorporating risk volumes**, which is **an application** of **the simple credibility model** of **Bühlmann**. We consider a ceded insurance portfolio.

Suppose S_j represents the total claim amount of year j and P_j some measure of the risk volume in year j. By the loss ratio of year j we mean $X_j = S_j / P_j$. We assume that $X_1, X_2, \ldots$ are conditionally independent given an unknown random risk parameter θ, that $E(X_j|\theta) = \mu(\theta)$ is independent of j, and we obtain:

$$Var(X_j|\theta) = \frac{s^2(\theta)}{P_j},\ j = \overline{1,t} \qquad (2.7).$$

We introduce the structural parameters: $\mu = E[\mu(\theta)]$, $\varphi = E[s^2(\theta)]$, $\lambda = Var[\mu(\theta)]$. The assumption (2.7) is perhaps most reasonable if P_j is the number of risks in the portfolio in year j. If we assume that the claim amounts $Y_{j1}, \ldots, Y_{jP_j}$ of the P_j risks in year j are conditionally independent and identically distributed given θ, then: $Var(X_j|\theta) = Var\left(\frac{1}{P_j}\sum_{k=1}^{P_j} Y_{jk} \middle| \theta\right) = \frac{Var(Y_{j1}|\theta)}{P_j}$, and (2.7) arrives to the assumption that $Var(Y_{j1}|\theta) = s^2(\theta)$ independent of j. We have the following result (see (2.8)). The credibility estimator $\hat{\mu}(\theta)$ of $\mu(\theta)$ based on $\underline{X}' = (X_1, X_2, \ldots, X_t)$ is given by:

$$\hat{\mu}(\theta) = \frac{P}{P+K}\overline{X}_t + \frac{K}{P+K}\mu \qquad (2.8),$$

with $P = \sum_{j=1}^{t} P_j$, $\overline{X}_t = \frac{1}{P}\sum_{j=1}^{t} P_j X_j$, $K = \frac{\varphi}{\lambda}$. To demonstrate relationship (2.8), see [1], from references, pages 95-104.

Application 3: **The credibility regression model**

We introduce as **application** of **the simple credibility model** of **Bühlmann**, **the credibility regression model** allowing for effects like inflation (in the simple model we allowed $EVar(X_i|\theta)$ to vary; in the credibility regression model we are going to allow EX_j to vary). In t**he credibility model incorporating risk volumes** we allowed $E[Var(X_i|\theta)]$ to vary. In the present model we are going to allow $E(X_j)$ to vary. Suppose $\underline{X} = (X_1, \ldots, X_t)'$ is an observed random (t×1) vector and θ an unknown random risk parameter. Instead of assuming time independence in the net risk premium: $\mu(\theta) = E(X_j|\theta)$, $j = \overline{1,t}$ we assume that the conditional expectation of the claims on a contract changes in time, is: $\mu_j(\theta) = E(X_j|\theta)$, $j = \overline{1,t}$. This application contains a description of the credibility regression model allowing for effects like inflation. Often it is unrealistic to assume that, for a given θ, the $\underline{X}' = (X_1, \ldots, X_t)$ are i.i.d.. To avoid this restriction, we will introduce the regression technique. The variables describing the contract are $(\theta, \underline{X}^{'})$. Using the conventions for matrix and vector notation, we have as a direct generalization of the Bühlmann hypothesis: $\mu_j(\theta) = E(X_j|\theta)$, $j = \overline{1,t}$ or $\underline{\mu}^{(t,1)}(\theta) = E(\underline{X} \mid \theta) = (\mu_1(\theta),\ldots,\mu_t(\theta))'$. We restrict the class of admissible functions $\mu_j(\cdot)$ to: $\underline{\mu}^{(t,1)}(\theta) = x^{(t,n)}\underline{\beta}^{(n,1)}(\theta)$, where $x^{(t,n)}$ is a matrix given in advance, the so – called design matrix, having full rank $n \le t$ and where the $\underline{\beta}^{(n,1)}(\theta)$ is the unknown regression constant. It is assumed that the matrixes: $Cov[\underline{\beta}^{(n,1)}] = a = a^{(n,n)}$, $E[Cov(\underline{X} \mid \theta)] = \Phi = \Phi^{(t,t)}$ are positive definite. We finally introduce: $\underline{b} = \underline{b}^{(n,1)} = E[\underline{\beta}^{(n,1)}(\theta)]$. So, let: $\mu_j(\theta) = \underline{x}_j^{'}\underline{\beta}(\theta)$, where the non – random $(1\times q)$ vector $\underline{x}_j^{'}$ is known, and let $\hat{\mu}_j(\theta)$ be the credibility estimator of $\mu_j(\theta)$ based on $\underline{X}^{'}$., with $j = \overline{1,t}$. We have the following result: the credibility estimator $\hat{\mu}_j(\theta)$ is given by $\hat{\mu}_j(\theta) = \underline{x}_j^{'}\left[Z\hat{\underline{b}} + (I - Z)\underline{b}\right]$, $j = \overline{1,t}$, with: $\hat{\underline{b}} = \hat{\underline{b}}^{(q,1)} = (x'\Phi^{-1}x)x'\Phi^{-1}\underline{X} \leftarrow$ is the best linear θ – unbiased estimator of $\underline{\beta}(\theta)$, $Z = Z^{(q\times q)} = ax'\Phi^{-1}x(I + ax'\Phi^{-1}x)^{-1} \leftarrow$ the resulting credibility factor, and $Z\hat{\underline{b}} + (I - Z)\underline{b}$ is the credibility estimator of $\underline{\beta}(\theta)$. To demonstrate above relationship, see [1], from references, pages 105-130.

Application 4:

Suppose the claims are integer-valued and Poisson (θ) distributed, as bellow:

$$dF_{X|\theta}(x,\theta) = \theta^x e^{-\theta} / x!,\ x = 0, 1 \qquad (2.9),$$

and suppose that the structure distribution of θ is a Gamma distribution:

$$u(\theta) = \theta^{\beta-1}e^{-\alpha\theta}\alpha^{\beta} / \Gamma(\beta),\ \theta > 0 \qquad (2.10).$$

In this case the best linear credibility estimator for $\mu(\theta)$ can be written as :

$$z\overline{x} + (1 - z)m = (v + \beta) / (t + \alpha) \qquad (2.11).$$

Since in this case $m = E[X] = E\{E[X|\theta]\} = E[\theta] = \beta / \alpha$, and for the ratio of the structure parameters a and s^2 we have:
$s^2 / a = E\{Var[X|\theta]\} / Var\{E[X|\theta]\} = E[\theta] / Var[\theta] = (\beta / \alpha) / (\beta / \alpha^2) = \alpha$, we find $z = at / (s^2+at) = t / (t + \alpha)$, so the best linear credibility estimator (2.3) for $\mu(\theta)$ can

be written under the form (2.11), where v = $\sum_{i=1}^{t} x_i$. To demonstrate relationship (2.11), see [1], from references, pages 37-44.

Application 5:

Suppose the claims are a Negative Binomial (θ) distribution, so:

$dF_{X|\theta}(x,\theta) = \theta^x (1-\theta)^{1-x}, x \in \{0,1\}$ (2.12)

and suppose the structure distribution of θ to be a Beta distribution:

$u(\theta) = \theta^{\alpha-1}(1-\theta)^{\beta-1} / \beta(\alpha,\beta),\ \theta \in (0,1)$ (2.13).

In this case the best linear credibility estimator (2.3) for μ(θ) can be written as:

$z\bar{x} + (1-z)m$ = [t /(t + α + β) $\bar{x}$] + [α / (t + α + β)] (2.14).

As in this case m = E[X] = E{E[X|θ]} = E[θ] = α / (α + β), and the ratio of the structure parameters a and s^2 we obtain:
s^2 / a = E{Var[X|θ]} / Var{E[X|θ]} = E[θ(1 - θ] / Var[θ] = [E(θ) – E(θ^2)] / Var(θ) = {[α /(α+β)] – [α(α+1) / (α+β+1)]} / {[(αβ) / (α + β)^2(α + β + 1)]} = [αβ / (α + β + 1)] / [αβ / (α + β)^2 (α + β + 1)] = α + β, we find z = = at / (s^2 + at) = at / /{a[(s^2 / t) + t]} = t / (t + α + β), so the best linear credibility estimator (2.3) for μ(θ) can be written under the form (2.14). To demonstrate relationship (2.14), see [1], from references, pages 37-44.

Application 6:

Suppose the claims are a Exponential (θ) distribution, so:

$dF_{X|\theta}(x,\theta) = \theta e^{-\theta x}$, x>0 (2.15).

and suppose the structure distribution of θ to be a Gamma distribution:

$u(\theta) = \theta^{\beta-1} e^{-\alpha\theta} \alpha^{\beta} / \Gamma(\beta),\ \theta > 0$ (2.16).

In this case the best linear credibility estimator (2.3) for μ(θ) can be written as follows:

$z\bar{x} + (1-z)m$ = (v + α) / (t + β - 1), if β > 2 (2.17).

As in this case m = E[X] = E{E[X|θ]} = E[1 / θ] = α / (β - 1), if β > 1, and the ratio of the structure parameters a and s^2 we have:
s^2 / a = E{Var[X|θ]} / Var{E[X|θ]} = E[1 / θ^2] / Var(1 / θ) = β – 1, if β > 2, we find z = at / (s^2 + at) = at / {a[(s^2 / t) + + t]} = [t / (t + β – 1)][v / t] = v / [t + β – 1], if β > 2,), so the best linear credibility estimator for μ(θ) can be written under the form (2.17), where v = $\sum_{i=1}^{t} x_i$. To demonstrate relationship (2.17), see [1], from references, pages 37-44.

Application 7:

Suppose the claims are a Normal (θ, σ^2) distribution, so:

$$dF_{X|\theta}(x,\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\theta}{\sigma}\right)^2}, \ x \in \mathbb{R} \quad (2.18)$$

and suppose the structure distribution of θ to be a Normal (μ_0, σ_0^2) distribution:

$$u(\theta) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\theta-\mu_0}{\sigma_0}\right)^2}, \ \theta \in \mathbb{R} \quad (2.19).$$

In this case the best linear credibility estimator (2.3) for $\mu(\theta)$ can be written as follows:

$$z\bar{x} + (1-z)m = \left[\frac{v}{\sigma^2} + \frac{\mu_0}{\sigma_0^2}\right] / \left[\frac{t}{\sigma^2} + \frac{1}{\sigma_0^2}\right] \quad (2.20).$$

As in this case m = E[X] = E{E[X|θ]} = E(θ) = μ_0 and for the ratio of the structure parameters a and s^2 we have:
s^2 / a = E{Var[X|θ]} / Var{E[X|θ]} = E(σ^2) / Var(θ) = σ^2 / σ_0^2, we find z = at / (s^2 + at) = at / {a[(s^2 / t) + t]} = t / [(σ^2 / σ_0^2) + t)], so the best linear credibility estimator (2.3) for $\mu(\theta)$ can be written under the form (2.20), where $v = \sum_{i=1}^{t} x_i$. To demonstrate relationship (2.20), see [1], from references, pages 37-44.

Application 8: **Credibility estimator minimizes mean squared error for exponential family with natural parameterization and prior**

Consider the exponential family of distributions with natural **parameterization**:

$$f_{X|\theta}(x,\theta) = p(x)e^{-\theta x} / q(\theta), \ x > 0, \ \theta > 0 \quad (2.21)$$

together with the natural conjugate priors with density:

$$u(\theta) = q(\theta)^{-t_0} e^{-\theta x_0} / c(t_0, x_0), \ \theta > 0 \quad (2.22),$$

where p(x) is an arbitrary non – negative function, t_0 and x_0 are positive constants, and c(t_0, x_0) is a normalization constant. For this case, the linear credibility estimator is:

$$z\bar{x} + (1-z)m = \left(x_0 + \sum_{i=1}^{t} x_i\right) / (t_0 + t) \quad (2.23),$$

where m = E[$\mu(\theta)$] = x_0 / t_0, s^2 / a = t_0, z = t / (t +t_0). Indeed: -by subsection 2.1 (**Bühlmann’s optimal credibility estimator**), we only have to prove that the optimal estimator:

$$E[\mu(\theta)|\underline{X}] = \left[\int \mu(\theta)\prod_{i=1}^{t} f_{X|\theta}(x_i,\theta)dU(\theta)\right] / \left[\int \prod_{i=1}^{t} f_{X|\theta}(x_i,\theta)dU(\theta)\right] \quad (2.24)$$

is a non-homogeneous linear combination of $X_1,...,X_t$.

First we express E[μ(θ)] in the prior parameters x_0 and t_0, then the Application 8 follows because of the special form of the posterior distribution. Because q(θ) is the normalizing constant of the distribution (2.21) one has:

$$q(\theta) = \int_0^{+\infty} p(x)e^{-\theta x}dx \quad (2.25).$$

So:

$$q'(\theta) = -\int_0^{+\infty} xp(x)e^{-\theta x}dx = -q(\theta)E[X|\theta=\theta] \quad (2.26),$$

since $E[X|\theta=\theta] = \int_0^{+\infty} xf_{X|\theta}(x,\theta)dx = \left[\int_0^{+\infty} xp(x)e^{-\theta x}dx\right]/q(\theta)$. Therefore the risk premium when $\theta = \theta$ equals is:

$\mu(\theta) = E[X|\theta=\theta] = -q'(\theta) / q(\theta)$ (2.27).

Taking the first derivative of (2.22) taking in consideration that θ is given, using (2.27) we obtain:

$u'(\theta) = [-t_0 q(\theta)^{-t_0-1} q'(\theta) e^{-\theta x_0}] / c(t_0, x_0) + [q(\theta)^{-t_0} e^{-\theta x_0}(-x_0)] / c(t_0, x_0) = t_0[-q'(\theta) / q(\theta)]\cdot[q(\theta)^{-t_0} e^{-\theta x_0} / c(t_0, x_0)] - x_0 [q(\theta)^{-t_0} e^{-\theta x_0} / c(t_0, x_0)] = t_0 u(\theta) - x_0 u(\theta) = [t_0 \mu(\theta) - x_0]u(\theta)$. So:

$u'(\theta) = [t_0 \mu(\theta) - x_0]u(\theta)$ (2.28).

Integrating this derivative over θ gives zero for the left side, since:

$$\int_0^{+\infty} u'(\theta)d\theta = u(+\infty) - u(0) = 0 \quad (2.29).$$

So the right side of (2.28) will be:

$$m = E[\mu(\theta)] = \int_0^{+\infty} \mu(\theta)u(\theta)d\theta = x_0 / t_0 \quad (2.30),$$

as:

$(2.28) \wedge (2.29) \Rightarrow \int_0^{+\infty} [t_0\mu(\theta) - x_0]u(\theta)d\theta = 0 \Leftrightarrow t_0 E[\mu(\theta)] -$

$x_0 \int_0^{+\infty} u(\theta)d\theta = 0 \Leftrightarrow t_0 E[\mu(\theta)] - x_0\cdot 1 = 0 \Leftrightarrow E[\mu(\theta)] = x_0/t_0$. The conditional density of θ, given $\underline{X} = \underline{x}$ (posterior density) is, apart from a normalizing function of x_1, …, x_t:

$$f_{\theta|\underline{X}}(\theta,\underline{x}) = f_{\underline{X}|\theta}(\underline{x},\theta)\, f_\theta(\theta) \,/\, f_{\underline{X}}(\underline{x}) :: u(\theta)\prod_{i=1}^{t}\left\{p(x_i)e^{-\theta x_i}/q(\theta)\right\} ::$$

$$q(\theta)^{-(t_0+t)}e^{-\left(x_0+\sum_i x_i\right)} \quad (2.31).$$

Density (2.31) is of the same type as the original structure density (2.22), with x_0 replaced by ($x_0 + \sum_i x_i$) and t_0 by (t_0 + t). So by using (2.30) the posterior mean (2.24), which is the mean squared error – optimal estimator for μ(θ), we obtain:

$$E[\mu(\theta)|X_1,\ldots,X_t] = (x_0 + \sum_i x_i) / (t_0 + t) \quad (2.32).$$

This is indeed a non – homogeneous linear combination of $X_1,\ldots,X_t$. By (2.30) we have $m = x_0 / t_0$, and comparing (2.32) with (2.3) we can observe that $t_0 = s^2 / a$ and $z = t / (t + t_0)$. The **parameterization** is called **natural** because the exponent part is a linear function of θ, and by taking a natural conjugate prior the posterior distribution is of the same type as the prior distribution (to demonstrate relationship (2.23) detailed, see [1], from references, pages 37-44). We restrict to x > 0 and θ>0, and suppose furthermore that at the final point of the intervals the densities are zero. These restrictions are not strictly necessary. It should be noted that the solution (2.3) of the linear credibility problem only yields a statistic computable from the observations, if the structure parameters m, s^2 and a are known. Generally, however, the structure function U(·) is not known. Then the 'estimator' as it stands is not a statistic. Its interest is merely theoretical, but it will be the basis for further results on credibility.

In the following section we consider different contracts, each with the same structure parameters a, m and s^2, so we can estimate these quantities using the statistics of the different contracts.

3. The classical credibility model of Bühlmann

In this section we will introduce the classical Bühlmann model, which consists of a portfolio of contracts satisfying the constraints of the original Bühlmann model. The classical credibility model of Bühlmann, presents the best linear credibility estimators for this case. The contract index j is a random structure parameter θ_j and observations $X_{j1},\ldots, X_{jt}$: $(\theta_j, X_{j1},\ldots,X_{jt}) = (\theta_j, \underline{X}'_j)$. The contracts j = 1,…, k are assumed to be i.i.d. Moreover, for every contract j = 1,…,k and for $\theta_j = \theta_j$ fixed, the variables $X_{j1},\ldots, X_{jt}$ are conditionally independent and identically distributed. In the classical model of Bühlmann, all contracts have in common the fact that their variances and expectations are represented by the same functions $\sigma^2(\cdot)$ and μ(·) of the risk parameter.

Nevertheless the portfolio cannot be considered to be homogeneous because of the different results of the risk parameter θ_j for each contract.

So for every contract, the following covariance matrix of the observations during the period r = 1,.., t results:

$$\mathrm{Cov}[\underline{X}_j|\theta_j] = \lfloor Cov \lfloor X_{jr}, X_{jr'} \mid \theta_j \rfloor\rfloor_{r,r'=\overline{1,t}} = I^{(t,t)}\sigma^2(\theta_j) \quad (3.1),$$

where $I^{(t,t)}$ represents the (t×t) identity matrix. Also:

$$E[X_{jr}|\theta_j] = \mu(\theta_j),\ r = 1,\dots, t \quad (3.2).$$

Note that the usual definitions of the structure parameters apply, with θ_j replacing θ and X_{jr} replacing X_r, so:

$$m = E[X_{jr}] = E[\mu(\theta_j)],\ a = \mathrm{Var}[\mu(\theta_j)],\ s^2 = E[\sigma^2(\theta_j)] \quad (3.3).$$

3.1 Bühlmann's classical model

Consider a portfolio as depicted in Diagram 1. If both assumptions (B_1) and (B_2) exists:

(B_1) $E[X_{jr}|\theta_j] = \mu(\theta_j)$, $\mathrm{Cov}[\underline{X}_j \mid \theta_j] = \sigma^2(\theta_j)I^{(t,t)}$, j = 1,…, k

and:

(B_2) the contracts j = 1,…,k are independent, the variables $\theta_1,\dots,\theta_k$ are identically distributed, and the observations X_{jr} have finite variants, then the optimal non-homogeneous linear estimators M_j^a for $\mu(\theta_j)$, j = 1,…,k, in the least squares sense read:

$$\hat{\mu}(\theta_j) = M_j^a = (1 - z)m + zM_j \quad (3.4).$$

Here $M_j = \overline{X}_j = \frac{1}{t}\sum_{s=1}^{t} X_{js}$ represents the individual estimator for $\mu(\theta_j)$. The resulting credibility factor z which appears in the credibility adjusted estimator M_j^a is found as:

$$z = at / (s^2 + at) \quad (3.5),$$

with the structural parameters a and s^2 as defined above.

Contract:			1…………j…………k
Structure parameter:			θ_j
Observable variables:	p	1	X_{j1}
	e	2	X_{j2}
	r	.	.
	i	.	.
	o	.	.
	d	t	X_{jt}

Diagram 1 Bühlmann's classical model

To demonstrate relationship (3.4), see [1], from references, pages 131-144.

3.2 Application: The **estimation of the credibility premium for the classical Bühlmann model**

The credibility premium for this classical Bühlmann model involves three parameters a, s^2 and m. Now that we embedded the separate contract j in a group of identical contracts, it is possible to express the unbiased estimators of these quantities. For this estimation, we assume that we have a portfolio of k identical and independent policies that have been observed for t (≥ 2) years, and let X_{jr} represent the total claim amount of policy j in year r. Let: $M_j = \overline{X}_j = \frac{1}{t}\sum_{s=1}^{t} X_{js}$, $M_0 = \overline{\overline{X}}_{..} = \frac{1}{k}\sum_{j=1}^{k} M_j$. For m we propose the unbiased estimator: $\hat{m} = M_0 = \overline{\overline{X}}_{..}$. For each policy j, the empirical variation: $\frac{1}{t-1}\sum_{r=1}^{t}\left(X_{jr} - M_j\right)^2$ is an unbiased estimator of $Var(X_{jr}|\theta_j)$, and thus: $\hat{s}^2 = \frac{1}{k(t-1)}\sum_{j=1}^{k}\sum_{r=1}^{t}\left(X_{jr} - M_j\right)^2$ is an unbiased estimator of s^2. The empirical variation: $\frac{1}{k-1}\sum_{j=1}^{k}\left(M_j - M_0\right)^2$ is an unbiased estimator of $Var(M_j)$, and as: $Var(M_j) = \frac{\hat{s}^2}{t} + a$, we introduce the unbiased estimator: $\hat{a} = \frac{1}{k-1}\sum_{j=1}^{k}\left(M_j - M_0\right)^2 - \frac{\hat{s}^2}{t}$ for a. This estimator has the weakness that it may take negative values whereas a is non – negative. Therefore, we replace a by the estimator: $a^* = \max(0, \hat{a})$, thus losing unbiasedness, but gaining admissibility. Note that $\hat{m}$, $\hat{s}^2$ and a^* are consistent as $k \to +\infty$ (see [1] from references, pages 21-36).

4. Conclusions

The main results of the paper are: **1**) **the applications** of **the optimal credibility estimator** of **Bühlmann**; **2**) **estimation** of **the structural parameters** in **the classical Bühlmann model**, as **useful application from them when estimating the credibility premium** for **this classical Bühlmann model**; **3**) **the recursive credibility model**-our motivation for introducing this model was that we wanted new claims to have more weight than older claims; **4**) **the credibility model incorporating risk volumes**-our motivation for introducing this model is that in the simple model we assumed that the risk volume was the same for all years; often, especially in reinsurance, one wants to allow the risk volumes to vary and for that purpose we will introduce the credibility model incorporating risk volumes; **5**) we end this paper presenting as application of the simple model **the credibility regression model** allowing for effects like inflation (in the simple credibility model of Bühlmann we allowed $EVar(X_j|\theta)$ to vary; in the credibility regression model we allowed EX_j to vary).

Practical examples (applications) will be given to illustrate the possibilities of credibility. This will give more insight and understanding of the previously discussed theoretical aspects and will point the way to the practical possibilities of the credibility theory. All numerical results in this paper were obtained using the original Bühlmann model. So, the paper presents some practical insurance problems that can be solved by means of credibility theory

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