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TIGHTENING FRAMES USING SOME ACTIONS ON THE FRAME OPERATOR

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In this paper, letting S be a frame operator, we show that there is a sequence of polynomials $\{p_n(S)\}_{n\in\mathbb{N}}$, without the constant term, converging to the identity operator. We can choose $n \in \mathbb{N}$ such that the reconstruction formula induced by $p_n(S)$ is as tight as desired. The first degree polynomial with optimal bounds and a second degree polynomial are introduced.

Keywords: frame, tight frame, spectral mapping theorem, frame algorithm.
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1. **Preliminaries**

Although Duffin and Schaeffer [8] introduced frames in studying nonharmonic Fourier series, their importance was recognized some decades later, when Young wrote his book entitled *Nonharmonic Fourier series* [15]. Thereafter, Daubechies, Grossmann and Meyer extended and expanded frames [7]. The interested readers are referred to [5] for studying about frame theory.

The art of frames theory [2] is its flexibility under perturbations and permutations (unconditional convergence) in presenting functions using a *single* summation($\sum$) (1.2). In the present paper, we slightly deviate from this old tradition, single summation (3.17), in order to obtain some reconstruction formulas with some better bounds (3.16).

In this paper, $\mathcal{H}$ denotes a separable Hilbert space, $\mathcal{B}(\mathcal{H})$ is the set of bounded linear operators on $\mathcal{H}$ and $\mathbb{N}$ is the set of natural numbers. For $T \in \mathcal{B}(\mathcal{H})$, T is said to be positive if $0 \leqslant \langle Tf, f\rangle$ for all $f \in \mathcal{H}$. A partial order on $\mathcal{B}(\mathcal{H})$ is defined as follows. We write $T \leqslant S$ if $\langle Tf, f\rangle \leqslant \langle Sf, f\rangle$ for all $f \in \mathcal{H}$. For $A, B \in \mathbb{R}$, by $A \leqslant S \leqslant B$, we mean that $AI \leqslant S \leqslant BI$, where I denotes the identity operator.

A sequence of vectors $\{f_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$ is called a ***frame*** if there are positive constants A,B such that

$$A\|f\|^2 \leqslant \sum_{i\in\mathbb{N}} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2, \quad f \in \mathcal{H}. \tag{1.1}$$

When $A = B$, $\{f_i\}_{i\in\mathbb{N}}$ is called a tight frame. If the right inequality is satisfied, $\{f_i\}_{i\in\mathbb{N}}$ is said to be a Bessel sequence with bound B. It is proven in [5] that the operator

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{i\in\mathbb{N}} \langle f, f_i\rangle f_i,$$

is well-defined, positive and invertible and

$$f = \sum_{i\in\mathbb{N}} \langle f, S^{-1}f_i\rangle f_i = \sum_{i\in\mathbb{N}} \langle f, f_i\rangle S^{-1}f_i. \tag{1.2}$$

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Also $A \leqslant S \leqslant B$ and consequently [5],

$$\left\| I - \frac{1}{B} S \right\| \leqslant 1 - \frac{A}{B} < 1. \tag{1.3}$$

Suppose that for some $\alpha, \beta \in (0, \infty)$,

$$\|I - \alpha S\| \leqslant \beta < 1.$$

Then we have $1 - \alpha S \leqslant \beta < 1$, $\alpha S - 1 \leqslant \beta < 1$ and $0 < 1 - \beta < 1$. So

$$0 < \frac{1-\beta}{\alpha} \leqslant S \leqslant \frac{1+\beta}{\alpha}.$$

Inverting conclusions above proves that for $\alpha, \beta \in (0, \infty)$,

$$\|I - \alpha S\| \leqslant \beta < 1 \quad \Longleftrightarrow \quad 0 < \frac{1-\beta}{\alpha} \leqslant S \leqslant \frac{1+\beta}{\alpha}. \tag{1.4}$$

Letting $\alpha = \frac{2}{B+A}$ and $\beta = 1 - \frac{2A}{B+A}$,

$$\|I - \frac{2}{B+A} S\| \leqslant 1 - \frac{2A}{B+A} < 1 \quad \Longleftrightarrow \quad 0 < A \leqslant S \leqslant B. \tag{1.5}$$

2. Tightening Frames

In practise, computing the inverse of frame operators is hard and we should find some efficient numerical methods to compute the inverse of the frame operators [3, 4]. When the frame is tight, the frame operator becomes $S = AI$ and every $f \in \mathcal{H}$ has the representation $f = \frac{1}{A} \sum_{i \in \mathbb{N}} \langle f, f_i \rangle f_i$.

In comparison with (1.2), this reconstruction is more desirable since it is not necessary to calculate the inverse of the frame operator. Since $\|I - \frac{1}{B} S\| \leqslant 1 - \frac{A}{B} < 1$, in [6] and [5, Theorem A.5.3] it is shown that the inverse of the frame operator can be written in the form of Neumann series

$$S^{-1} = \frac{1}{B} \sum_{n=0}^{\infty} \left(I - \frac{1}{B} S \right)^n. \tag{2.1}$$

By closing A and B, S tends to the identity operator (up to constant), and hence the series (2.1) converges more quickly. For closing A and B some concepts have been introduced: *controlled frames* [1, 14] and *scalable frames* (*weighted frames*) [1, 9, 10]. The idea of controlled frames is multiplying the frame operator by a positive invertible operator C in order to obtain some better frame bounds A' and B', i.e.,

$$0 < A < A' \leqslant SC \leqslant B' < B < \infty.$$

The aim of studying scalable frames is to find a sequence of nonnegative scalars $\{c_i\}_{i \in \mathbb{N}}$ and some constants A' and B' such that

$$A\|f\|^2 < A'\|f\|^2 \leqslant \sum_{i \in \mathbb{N}} c_i |\langle f, f_i \rangle|^2 \leqslant B'\|f\|^2 < B\|f\|^2, \quad 0 \neq f \in \mathcal{H}.$$

In the methods described, letting S' be the new frame operator associated with a controlled or scalable frame, by (1.3) we have

$$\left\| I - \frac{1}{B'} S' \right\| \leqslant 1 - \frac{A'}{B'} \lneqq 1 - \frac{A}{B}.$$

For this purpose, we follow another way. We look for a polynomial p such that

$$0 < A < A' \leqslant p(S) \leqslant B' < B < \infty.$$

for some constant A' and B' (3.16). Our method strongly depends on some results in the operator theory which are established in the next section.

3. Polynomial Actions on the Frame Operator

Let $T \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator and h be a continuous function over

$$\sigma(T) := \{\lambda \in \mathbb{C} |\ \lambda I - T \text{ is not invertible}\},$$

the so-called spectrum of T. The spectral mapping theorem [6, VIII.2.7] states that $\sigma(h(T)) = h(\sigma(T))$. Let S be the frame operator of a frame with bounds A, B. Then $\sigma(S) \subset [A, B]$ and $\sup\{x \mid x \in \sigma(S)\} = \|S\| \leqslant B$, [13, Proposition 5.1.1]. Putting S^{-1} instead S we have $\sigma(S^{-1}) \subset [B^{-1}, A^{-1}]$, and by above equation

$$\begin{aligned}\|S^{-1}\| = \sup\{x \mid x \in \sigma(S^{-1})\} &= \sup\{x \mid x \in \sigma(S)^{-1}\}\\ &= (\inf\{x \mid x \in \sigma(S)\})^{-1} \leqslant A^{-1}.\end{aligned}$$

In the second equality we used the function $h(x) = x^{-1}$ in the formula $\sigma(h(T)) = h(\sigma(T))$, which h is continuous on $\sigma(S^{-1}) \subset [B^{-1}, A^{-1}]$. So $A \leqslant \|S^{-1}\|^{-1} = \inf\{x \mid x \in \sigma(S)\}$. Therefore,

$$A \leqslant \|S^{-1}\|^{-1} = \inf\{x \mid x \in \sigma(S)\} \leqslant \sup\{x \mid x \in \sigma(S)\} = \|S\| \leqslant B. \tag{3.1}$$

Assuming p is a polynomial with real coefficients so that $p([A, B]) \subset (0, \infty)$, we have

$$\sigma(p(S)) = p(\sigma(S)) \subset p([A, B]) \subset (0, \infty). \tag{3.2}$$

Since $[A, B]$ is compact and p is a continuous function, $p([A, B])$ is compact and there are $A', B' \in (0, \infty)$ such that $p([A, B]) \subset [A', B']$. Since by (3.2) $\sigma(p(S)) \subset [A', B']$, 0 is not in $\sigma(p(S))$. Hence $p(S)$ is invertible and using $p(S)$ instead S in (3.1),

$$A' \leqslant \left\|p(S)^{-1}\right\|^{-1}, \quad \|p(S)\| \leqslant B'.$$

In other words, $A' \leqslant p(S) \leqslant B'$. The arguments above motivate the use of polynomials p such that $p(S)$ induces some better frame bounds, i.e.,

$$A < A' \leqslant p(S) \leqslant B' < B, \qquad \left\|I - \frac{1}{B'}p(S)\right\| \leqslant 1 - \frac{A'}{B'} \lneqq 1 - \frac{A}{B}.$$

First, we consider the simplest polynomials of the form $p(x) = x^n, x \in \mathbb{R}, n \in \mathbb{N}$. For every $n \in \mathbb{N}$, $A^n \leqslant S^n \leqslant B^n$, and hence for $f \in \mathcal{H}$,

$$A^n\|f\|^2 \leqslant \sum_{i_1\in\mathbb{N}}\sum_{i_2\in\mathbb{N}}\cdots\sum_{i_n\in\mathbb{N}}\langle f, f_{i_1}\rangle\langle f_{i_1}, f_{i_2}\rangle\cdots\langle f_{i_{n-1}}, f_{i_n}\rangle\langle f_{i_n}, f\rangle \leqslant B^n\|f\|^2. \tag{3.3}$$

The reconstruction formulas

$$\begin{aligned} f &= \sum_{i_1\in\mathbb{N}}\sum_{i_2\in\mathbb{N}}\cdots\sum_{i_n\in\mathbb{N}}\langle f, S^{-n}f_{i_1}\rangle\langle f_{i_1}, f_{i_2}\rangle\cdots\langle f_{i_{n-1}}, f_{i_n}\rangle f_{i_n},\\ &= \sum_{i_1\in\mathbb{N}}\sum_{i_2\in\mathbb{N}}\cdots\sum_{i_n\in\mathbb{N}}\langle f, f_{i_1}\rangle\langle f_{i_1}, f_{i_2}\rangle\cdots\langle f_{i_{n-1}}, f_{i_n}\rangle S^{-n}f_{i_n}, \end{aligned} \tag{3.4}$$

hold for all $f \in \mathcal{H}$. But, $\left\|I - \frac{1}{B^n}S^n\right\| \leqslant 1 - \frac{A^n}{B^n} \longrightarrow 1$; the situation even became worse. We try the n-root of $S, \sqrt[n]{S}$; then $\sqrt[n]{A} \leqslant \sqrt[n]{S} \leqslant \sqrt[n]{B}$, and by (1.3)

$$\left\|1 - \frac{1}{\sqrt[n]{B}}\sqrt[n]{S}\right\| \leqslant 1 - \frac{\sqrt[n]{A}}{\sqrt[n]{B}} \longrightarrow 0.$$

For sufficiently large values of n, $\sqrt[n]{B}I \cong \sqrt[n]{S}$. Apparently, everything is good, but, the basic problem is finding a well-known representation for $\sqrt[n]{S}$ by summations ($\sum$). Therefore, again we have to return to the polynomials.

Theorem 3.1. *Let $\{f_i\}_{i\in\mathbb{N}}$ be a frame for $\mathcal{H}$ with frame operator S and frame bounds A,B. Then there is a sequence of real polynomials $\{p_n\}_{n\in\mathbb{N}}$, without the constant term, such that $p_n(S) \to I$, in the strong operator topology.*

Proof. Since $\mathcal{B}(\mathcal{H})$ is a C^*-algebra for $T \in \mathcal{B}(\mathcal{H})$, $\|TT^*\| = \|T\|^2$ [6]. If $T = T^*$ then $\left\|T^2\right\| = \|T\|^2$ and consequently for $n \in \mathbb{N}$,

$$\left\|T^{2n}\right\| = \|T\|^{2n}. \tag{3.5}$$

Now, by putting $T = I - \frac{1}{B}S$,

$$\left\|\left(I - \frac{1}{B}S\right)^{2n}\right\| = \left\|I - \frac{1}{B}S\right\|^{2n} \leqslant \left(1 - \frac{A}{B}\right)^{2n} \lneqq 1. \tag{3.6}$$

Obviously $\left(I - \frac{1}{B}S\right)^{2n}$ can be written in the form of $I - p_n(S)$, where p_n is a polynomial without the constant term. Therefore, (3.6) implies that

$$\|I - p_n(S)\| = \left\|\left(I - \frac{1}{B}S\right)^{2n}\right\| \leqslant \left(1 - \frac{A}{B}\right)^{2n} \to 0.$$

□

Notice that the theorem above is valid for all positive operators S whenever $\|I - S\| < 1$. The polynomial p_n in the proceeding theorem is a polynomial of degree $2n$.

As seen in the relations (3.3),(3.4) the reconstruction formulas involve computing $2n$ nested summations. For this, from now on, we will focus on polynomials of degree 2 (or 1) in the form of $p(x) = bx - ax^2$, $a, b \in \mathbb{R}$, satisfying $\|I - p(S)\| < 1$. We need to prove a key lemma.

Lemma 3.1. *Let T be a self-adjoint operator and $p(x)$ be a real polynomial. Then*

$$\|p(T)\| = \sup\{|p(x)| \; |x \in \sigma(T)\} =: \|p(x)\|_{\infty,\sigma(T)},$$

and

$$\|I - p(T)\| = \sup\{|1 - p(x)| \; |x \in \sigma(T)\} =: \|1 - p(x)\|_{\infty,\sigma(T)}.$$

Furthermore, if S is a frame operator with bounds A, B, then

$$\|I - p(S)\| \leqslant \|1 - p(x)\|_{\infty,[A,B]}. \tag{3.7}$$

Proof. We know that [6],

$$\sup\{|x| \; |x \in \sigma(T)\} = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|T^{2n}\|^{\frac{1}{2n}} = \lim_{n\to\infty} \|T\|^{2n\frac{1}{2n}} = \|T\|;$$

the fact that T is a self-adjoint operator was used in the third equation. So $\|T\| = \sup\{|x| \; |x \in \sigma(T)\}$. Since $p(x)$ is real $\sigma(p(T)) = p(\sigma(T)) \subset \mathbb{R}$. Therefore, $p(T)$ is self-adjoint and thus

$$\|p(T)\| = \sup\{|x| \; |x \in \sigma(p(T))\} = \sup\{|x| \; |x \in p(\sigma(T))\}.$$

If we put $1 - p(x)$ instead of $p(x)$ in the previous equation, we get

$$\begin{aligned}\|I - p(T)\| &= \sup\{|x| \; |x \in (1 - p(.))(\sigma(T))\} \\ &= \sup\{|x| \; |x \in (1 - p(\sigma(T))\} \\ &= \sup\{|1 - p(x)| \; |x \in \sigma(T)\} \\ &= \|1 - p(x)\|_{\infty,\sigma(T)}.\end{aligned}$$

The last assertion of the theorem follows from the fact that $\sigma(S) \subset [A, B]$:

$$\|I - p(S)\| \leqslant \|1 - p(x)\|_{\infty,\sigma(T)} \leqslant \|1 - p(x)\|_{\infty,[A,B]}.$$

□

Proposition 3.2. *Let $\{f_i\}_{i\in\mathbb{N}}$ be a frame for $\mathcal{H}$ with the frame operator S and the frame bounds A and B. The polynomial $p_{\frac{2}{B+A}}(x) := \frac{2}{B+A}x$, minimizes $\|1-p(x)\|_{\infty,[A,B]}$, where p ranges over the family of polynomials of the form $p(x) = bx$, $b \in (0,\infty)$. In this situation,*

$$\left\| I - \frac{2}{B+A}S \right\| \leqslant \frac{B-A}{B+A}. \tag{3.8}$$

Proof. See the following figure: It is clear that $\inf_{b\in(0,\infty)} \|1-bx\|_{\infty,[A,B]}$, occurs when

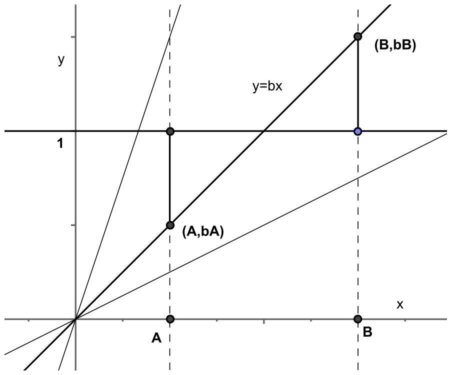

FIGURE 1

$\frac{1}{B} < b < \frac{1}{A}$. Thus

$$\inf_{b\in(0,\infty)} \|1-bx\|_{\infty,[A,B]} = \inf_{\frac{1}{B}\leqslant b\leqslant\frac{1}{A}} \|1-bx\|_{\infty,[A,B]}. \tag{3.9}$$

Define

$$\psi(b) := \|1-bx\|_{\infty,[A,B]}. \tag{3.10}$$

Observe that

$$\psi(b) = \max\{bB-1, 1-bA\}. \tag{3.11}$$

From Figure 2 it is obvious that $\psi(b)$ gets its infimum when $bB-1 = 1-bA$. So $b = \frac{2}{B+A}$ and

$$\inf_{\frac{1}{B}\leqslant b\leqslant\frac{1}{A}} \psi(b) = \psi\left(\frac{2}{B+A}\right) = \max\left\{\frac{2}{B+A}B-1, 1-\frac{2}{B+A}A\right\} = \frac{B-A}{B+A}. \tag{3.12}$$

Relations (3.9)-(3.12) imply that

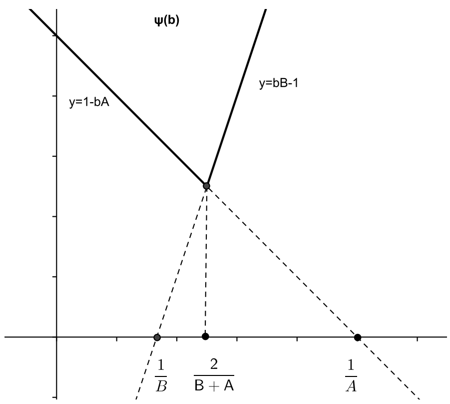

FIGURE 2

$$\inf_{b\in(0,\infty)} \|1-bx\|_{\infty,[A,B]} = \inf_{\frac{1}{B}\leqslant b\leqslant \frac{1}{A}} \|1-bx\|_{\infty,[A,B]} = \inf_{\frac{1}{B}\leqslant b\leqslant \frac{1}{A}} \psi(b) = \frac{B-A}{B+A}.$$

□

In what follows, two families of polynomials of degree 2 are described. Let $p_{ab}(x) = bx - ax^2$ with $a, b \geqslant 0$. For polynomials of this form the points $x = \frac{b}{2a}$ and $x = \frac{b}{a}$ are optimal point and zero point of p_{ab}, respectively. Assume that $\mathcal{P}_1$ is the family of polynomials p_{ab} such that

(1) $0 \leqslant p_{ab}\left(\frac{b}{2a}\right) \leqslant 1$,
(2) $0 \leqslant p_{ab}(A) \leqslant p_{ab}(B) \leqslant 1$,

Also, let $\mathcal{P}_2$ be the family of polynomials $p_{ab}(x)$ such that

(1) $1 \leqslant p_{ab}\left(\frac{b}{2a}\right) \leqslant 2$,
(2) $p_{ab}(A) \leqslant p_{ab}(B)$,

The possible infimum value for $\|I - p(S)\|$ will be considered in Theorem 3.3, where p belongs to one of these families, $\mathcal{P}_1, \mathcal{P}_2$. The proof of the following theorem will be given in the last section.

Theorem 3.3. *Suppose that $\{f_i\}_{i\in\mathbb{N}}$ is a frame for $\mathcal{H}$ with the frame operator S and the frame bounds A and B. Then*

(1) *Let $a_1 = \frac{4}{B+A}$, $b_1 = \frac{4}{(B+A)^2}$. The polynomial $p_1(x) = p_{a_1b_1}(x)$ belongs to $\mathcal{P}_1$ and*

$$\|I - p_1(S)\| \leqslant \left(\frac{B-A}{B+A}\right)^2. \tag{3.13}$$

(2) *Let $a_2 = \frac{8}{(B+A)^2+4AB}$, $b_2 = \frac{8(B+A)}{(B+A)^2+4AB}$. The polynomial $p_2(x) = p_{a_2b_2}(x)$ belongs to $\mathcal{P}_2$ and*

$$\|I - p_2(S)\| \leqslant \frac{(B-A)^2}{(B+A)^2+4AB}. \tag{3.14}$$

For every $p \in \mathcal{P}_1 \cup \mathcal{P}_2$,

$$\begin{aligned} p(S)f &= \left(bS - aS^2\right) f \\ &= \sum_{i\in\mathbb{N}} b\langle f, f_i\rangle f_i - \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} a\langle f, f_j\rangle\langle f_j, f_i\rangle f_i, \quad f \in \mathcal{H}. \end{aligned} \tag{3.15}$$

The theorem above has an interesting corollary.

Corollary 3.1. *For $p_2 = p_{a_2b_2}$ in Theorem 3.3 the operator $p_2(S)$ is a positive invertible operator and there are positive constants A', B' such that*

$$A < A' \leqslant \frac{B+A}{2} p_2(S) \leqslant B' < B. \tag{3.16}$$

We can take $A' = \frac{B+A}{2}(1-\beta)$, $B' = \frac{B+A}{2}(1+\beta)$ where $\beta = \frac{(B-A)^2}{(B+A)^2+4AB}$. Also, the reconstruction formulas

$$\begin{aligned} f &= \sum_{i\in\mathbb{N}} b_2 \left\langle f, p(S)^{-1} f_i \right\rangle f_i - \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} a_2 \left\langle f, p(S)^{-1} f_i\rangle\langle f_i, f_j \right\rangle f_j, \\ &= \sum_{i\in\mathbb{N}} b_2 \langle f, f_i\rangle p(S)^{-1} f_i - \sum_{i\in\mathbb{N}}\sum_{j} a_2 \langle f, f_i\rangle\langle f_i, f_j\rangle p(S)^{-1} f_j, \;\; f \in \mathcal{H}, \end{aligned} \tag{3.17}$$

hold.

Proof. Consider relation (1.4). Let $\alpha = \frac{2}{B+A}$ in (1.4) and use $\frac{B+A}{2}p_2(S)$ instead of S and $\beta = \frac{(B-A)^2}{(B+A)^2+4AB}$ instead of β. Thus $0 < \beta < 1$ and

$$\frac{B+A}{2}(1-\beta) = \frac{1-\beta}{\alpha} \leqslant \frac{B+A}{2}p_2(S) \leqslant \frac{1+\beta}{\alpha} = \frac{B+A}{2}(1+\beta). \tag{3.18}$$

Let $\gamma = \frac{B-A}{B+A}$. Obviously, $0 < \gamma < 1$ and

$$\beta = \frac{(B-A)^2}{(B+A)^2+4AB} < \left(\frac{B-A}{B+A}\right)^2 = \gamma^2 < \gamma < 1. \tag{3.19}$$

Hence

$$1-\gamma < 1-\beta < 1+\beta < 1+\gamma.$$

Using (3.18) and (3.19), we compute

$$\begin{aligned} A &= \frac{B+A}{2}\frac{2A}{B+A} = \frac{B+A}{2}\left(1-\frac{B-A}{B+A}\right) \\ &= \frac{B+A}{2}(1-\gamma) < \frac{B+A}{2}(1-\beta) \\ &\leqslant \frac{B+A}{2}p_2(S) \\ &\leqslant \frac{B+A}{2}(1+\beta) < \frac{B+A}{2}(1+\gamma) \\ &= \frac{B+A}{2}\left(1+\frac{B-A}{B+A}\right) \\ &= \frac{B+A}{2}\left(\frac{2B}{B+A}\right) \\ &= B. \end{aligned}$$

In short,

$$A < \frac{B+A}{2}(1-\beta) \leqslant \frac{B+A}{2}p_2(S) \leqslant \frac{B+A}{2}(1+\beta) < B.$$

Since $0 < \beta < 1$, $\frac{B+A}{2}(1-\beta) > 0$. Then $\frac{B+A}{2}p_2(S)$ is a positive invertible operator. The reconstruction formulas are easily followed by (3.15) for $p = p_2$. □

For $p \in \mathcal{P}_1$, a similar reasoning implies that $p(S)$ is positive and invertible. For every $f \in \mathcal{H}$, the sequences

$$\{\langle f, f_i\rangle\}_{i\in\mathbb{N}}, \{\langle f_j, f_i\rangle\}_{i\in\mathbb{N}}, \left\{\left\langle f, p(S)^{-1}f_i\right\rangle\right\}_{i\in\mathbb{N}} \in \ell^2,$$

participate in the reconstructions introduced in Corollary. The sequence $\{\langle f_j, f_i\rangle\}_{i,j\in\mathbb{N}}$ is fixed and $\{\langle f, f_i\rangle\}_{i,j\in\mathbb{N}}$, $\left\{\left\langle f, p(S)^{-1}f_i\right\rangle\right\}_{i\in\mathbb{N}}$ needed to be computed.

In the following theorem we derive a generalized version of frame algorithm by using the method of polynomial actions on the frame operator.

Theorem 3.4. *(Frame Algorithm of degree 2) The frame $\{f_i\}_{i\in\mathbb{N}}$ with bounds A, B is given. Let $p(x) = bx - ax^2$ be the polynomial defined in Theorem 3.3(2). For $f \in \mathcal{H}$ define $g_0 = 0$ and*

$$g_n = g_{n-1} + bSf - bSg_{n-1} - aS^2f + aS^2g_{n-1}, \quad n > 0.$$

Then $\{g_n\}_{n=0}^{\infty}$ converges to f and

$$\|f - g_n\| \leqslant \left(\frac{(B-A)^2}{(B+A)^2+4AB}\right)^n \|f\|.$$

$\mathcal{P}_1$	$0 \leqslant p_{ab}\left(\frac{b}{2a}\right) \leqslant 1 \quad \Leftrightarrow \quad 0 \leqslant b \leqslant 2\sqrt{a}$
$\mathcal{P}_1$, $\mathcal{P}_2$	$p_{ab}(A) \leqslant p_{ab}(B) \Leftrightarrow bA - aA^2 \leqslant bB - aB^2 \Leftrightarrow (B+A)a \leqslant b,\ \frac{(B+A)}{2} \leqslant \frac{b}{2a}$
$\mathcal{P}_1$	$p_{ab}(B) \leqslant 1 \quad \Leftrightarrow \quad b \leqslant Ba + \frac{1}{B}$
$\mathcal{P}_2$	$1 \leqslant p_{ab}\left(\frac{b}{2a}\right) \leqslant 2 \quad \Leftrightarrow \quad 2\sqrt{a} \leqslant b \leqslant 2\sqrt{2a}$

TABLE 1. Conditions of $\mathcal{P}_1$, $\mathcal{P}_2$ and their equivalences.

Proof. Observe that

$$\begin{aligned} f - g_n &= \left(I - \left(bS - aS^2\right)\right)(f - g_{n-1}) \\ &= (I - p(S))(f - g_{n-1}). \end{aligned}$$

By induction,

$$f - g_n = (I - p(S))^n (f - g_0).$$

Thus (3.14) implies that

$$\begin{aligned} \|f - g_n\| &= \|(I - p(S))^n\| \, \|f - g_0\| \\ &\leqslant \|I - p(S)\|^n \, \|f - g_0\| \\ &\leqslant \left(\frac{8(B-A)}{(B+A)^2 + 4AB}\right)^n \|f\|. \end{aligned}$$

□

4. Proof of Theorem 3.3

Every polynomial p_{ab} defined on $[A, B]$ takes it's extermums at A,B and $\frac{b}{2a}$, because $p'_{ab}\left(\frac{b}{2a}\right) = 0$. We compute

$$p_{ab}\left(\frac{b}{2a}\right) = \frac{b^2}{4a}, \quad p_{ab}(A) = bA - aA^2, \quad p_{ab}(B) = bB - aB^2. \tag{4.1}$$

Fix $x_0, y_0 \in \mathbb{R}^+$. If $x_0 \leqslant p_{ab}\left(\frac{b}{2a}\right) \leqslant y_0$, then $2\sqrt{x_0 a} \leqslant b \leqslant 2\sqrt{y_0 a}$. The cases $x_0, y_0 = 0, 1, 2$ can be found in Table 1. When $p_{ab}(A) \leqslant p_{ab}(B)$ it can easily be seen that

$$\begin{aligned} bA - aA^2 \leqslant bB - aB^2 \quad &\Longleftrightarrow \quad (B^2 - A^2)a \leqslant b(B - A) \\ \Longleftrightarrow \quad (B+A)a \leqslant b \quad &\Longleftrightarrow \quad \frac{(B+A)}{2} \leqslant \frac{b}{2a}. \end{aligned}$$

The properties of classes $\mathcal{P}_1, \mathcal{P}_2$ and their equivalences are listed in Table 1. The reader can readily check them.

Proof of part (1). In Relation (3.6) put $n = 1$ and apply $\frac{2}{B+A}$ instead of $\frac{1}{B}$ to derive

$$\begin{aligned} \|I - p_1(S)\| &= \left\| I - \left(\frac{4}{B+A}S - \frac{4}{(B+A)^2}S^2\right)\right\| = \left\| \left(I - \frac{2}{B+A}S\right)^2 \right\| \leqslant \left\| I - \frac{2}{B+A}S \right\|^2 \\ &\leqslant \left(1 - \frac{2A}{B+A}\right)^2 = \left(\frac{B-A}{B+A}\right)^2. \end{aligned}$$

On the other hand, we prove that $p_{a_1 b_1}$ satisfies the conditions of $\mathcal{P}_1$:

$$p_{a_1 b_1}\left(\frac{b_1}{2a_1}\right) = \frac{\left(\frac{4}{B+A}\right)^2}{4\frac{4}{(B+A)^2}} = 1 \quad \Rightarrow \quad 0 \leqslant p_{a_1 b_1}\left(\frac{b_1}{2a_1}\right) \leqslant 1.$$

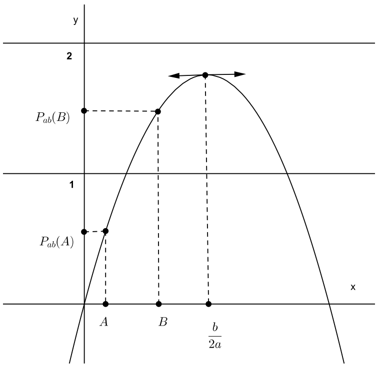

FIGURE 3. $\mathcal{P}_2 : A \leqslant B \leqslant \frac{b}{2a}$.

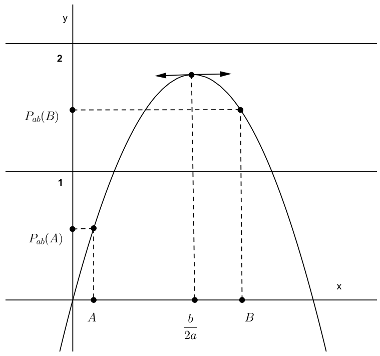

FIGURE 4. $\mathcal{P}_2 : A \leqslant \frac{b}{2a} \leqslant B$.

Also,

$$p_{a_1b_1}(A) = b_1A - a_1A^2 = \frac{4}{(B+A)}A - \frac{4}{(B+A)^2}A^2 = \frac{4AB}{(B+A)^2},$$
$$p_{a_1b_1}(B) = b_1B - a_1B^2 = \frac{4}{(B+A)}B - \frac{4}{(B+A)^2}B^2 = \frac{4AB}{(B+A)^2}.$$

Since

$$0 \leqslant (B-A)^2 \Rightarrow 2AB \leqslant B^2 + A^2 \Rightarrow 4AB \leqslant B^2 + A^2 + 2AB \Rightarrow 4AB \leqslant (B+A)^2 \Rightarrow \frac{4AB}{(B+A)^2} \leqslant 1,$$

then

$$0 \leqslant p_{a_1b_1}(A) = p_{a_1b_1}(B) \leqslant 1. \tag{4.2}$$

Proof of part (2). Under conditions of $\mathcal{P}_2$, it is obvious that (see Figures 3 and 4)

$$\|1 - p_{ab}(x)\|_{\infty,[A,B]} = \max\{l(a,b), k(a,b)\}, \tag{4.3}$$

where

$$l(a,b) := 1 - p_{ab}(A), \quad k(a,b) := p_{ab}\left(\frac{b}{2a}\right) - 1. \tag{4.4}$$

Use (4.1) to get $l(a,b) = 1 - bA + aA^2$, $\quad k(a,b) = \frac{b^2}{4a} - 1$. We show that $p_2 = p_{a_2b_2}$ satisfies the conditions of $\mathcal{P}_2$:

$$p_{a_2b_2}\left(\frac{b_2}{2a_2}\right) = \frac{b_2^2}{4a_2} = \left(\frac{8(B+A)}{(B+A)^2 + 4AB}\right)^2 \frac{(B+A)^2 + 4AB}{4 \times 8} = 2\frac{(B+A)^2}{(B+A)^2 + 4AB} \leqslant 2.$$

Hence $0 \leqslant p_{a_2b_2}\left(\frac{b_2}{2a_2}\right) \leqslant 2$.

We see that

$$p_{a_2b_2}(A) = b_2A - a_2A^2 = \frac{8(B+A)}{(B+A)^2 + 4AB}A - \frac{8}{(B+A)^2 + 4AB}A^2 = \frac{8AB}{(B+A)^2 + 4AB},$$

and

$$p_{a_2b_2}(B) = b_2B - a_2B^2 = \frac{8(B+A)}{(B+A)^2 + 4AB}B - \frac{8}{(B+A)^2 + 4AB}B^2 = \frac{8AB}{(B+A)^2 + 4AB}.$$

Thus

$$\begin{aligned} 0 \leqslant (B-A)^2 \quad &\Rightarrow \quad 2AB \leqslant B^2 + A^2 \Rightarrow \quad 8AB \leqslant B^2 + A^2 + 2AB + 4AB \\ &\Rightarrow \quad 8AB \leqslant (B+A)^2 + 4AB \quad \Rightarrow \quad \frac{8AB}{(B+A)^2 + 4AB} \leqslant 1. \end{aligned}$$

Consequently, $0 \leqslant p_{a_2b_2}(A) = p_{a_2b_2}(B) \leqslant 1$. We compute

$$\begin{aligned} l(a_2,b_2) &= 1 - p_{a_2b_2}(A) = 1 - \left(b_2A - a_2A^2\right) \\ &= 1 - \left(\frac{8(B+A)}{(B+A)^2+4AB}A - \frac{8}{(B+A)^2+4AB}A^2\right) \\ &= \frac{(B+A)^2+4AB-8A(B+A)+8A^2}{(B+A)^2+4AB} \end{aligned}$$

and by (**??**),

$$k(a_2,b_2) = p_{a_2b_2}\left(\frac{b_2}{2a_2}\right) - 1 = 2\frac{(B+A)^2}{(B+A)^2+4AB} - 1 = \frac{(B-A)^2-4AB}{(B+A)^2+4AB}.$$

So $\max\{l(a_2,b_2), k(a_2,b_2)\} = \frac{(B-A)^2}{(B+A)^2+4AB}$.

Relations (3.7), (4.3) and the previous equation give

$$\begin{aligned} \|I - p_2(S)\| &\leqslant \|1 - p_2(x)\|_{\infty,[A,B]} = \|1 - p_{a_2b_2}(x)\|_{\infty,[A,B]} \\ &= \max\{l(a_2,b_2), k(a_2,b_2)\} = \frac{(B-A)^2}{(B+A)^2+4AB}. \end{aligned}$$

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