A PREDICTOR-CORRECTOR SCHEME FOR THE NONLINEAR CHAOTIC VARIABLE-ORDER FRACTIONAL THREE-DIMENSIONAL SYSTEM

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This paper aims to study two-step fractional Adams-Bashforth-Moulton scheme for solving nonlinear fractional variable order differential equations. The fractional derivative used in this paper is the Hadamard fractional derivative. The predictor-corrector scheme is applied for the nu-merical solution of fractional variable-order chaotic systems. The illustrate scheme reduces these nonlinear fractional variable order differential equa-tions to a system of algebraic equations and then this system will be solved numerically by recursive scheme. Several numerical examples are reported to show the applicability and validity of the illustrated scheme.

Keywords: Fractional calculus, Adams-Bashforth-Moulton, Hadamard fractional derivative, Fractional-order chaotic systems.

1Introduction

Fractional operators involving fractional integrals and derivatives are as an extension of classical operators have been widely used in characterizing, integral equations, ordinary differential equations (ODEs) and partial differential equations (PDEs). Recently, subject of fractional calculus has attracted much attention, for this reason their plays an important and vital role in some fields of engineering and science, as ?uid mechanics, signal processing, diffusive transport, nonlinear biological systems, electrical networks, electrodynamics, nonlinear control theory, astrophysics, to name but a few [3–9]. Several various definitions of fractional integrals and derivatives are exist, for example, Coimbra, Riesz, Riemann-Liouville, Hadamard, Weyl, Grunwald-Letnikov, Marchaud, Liouville-Caputo, Caputo-Fabrizio and Atangana-Baleanu [10–12]. In the past decades, the solutions of fractional ODEs and PDEs have been widely

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employed in of physical phenomena as dynamical systems [13–20]. In most cases, these fractional differential equations (FDEs) do not have analytical solution or exact solution then we are interested to obtain their numerical solutions. Several numerical methods for finding the solution of FDEs use in the literature, for example, finite difference method [16], spectral methods [14, 15, 18], variation iteration method [21] and Adomian decomposition method [22].

A really strong numerical method to solve nonlinear fractional ODEs famous like fractional Adams–Bashforth–Moulton (FABM) scheme [23] has been applied in some examples to handle many chaotic models. This scheme is distinguished as a stable method for fractional ordinary differential equations and given chaotic models appearing in biology and other areas of science. This approach has already been developed for fractional models with Riemann-Liouville, Grunwald-Letnikov, Atangana-Baleanu and Caputo fractional operators. To the best of our knowledge, an extension to Hadamard fractional operators is new and no work has ever been performed in this direction. Consequently, we study a simple but effective numerical scheme that authorizes to deal with fractional order operators of Hadamard type. The Hadamard fractional integral of order $\alpha > 0$ is given in [23] as follows:

$$ {}_{H}J_{a}^{\alpha} f\left(x\right) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \left(ln\frac{x}{t} \right)^{\alpha - 1} \frac{f(t)}{t} dt. $$

Also, for $\alpha \in (0, 1)$, the Hadamard fractional derivative of order is introduced by:

$$ {}_{H}D_{a}^{\alpha} f\left(x\right) = \frac{1}{\Gamma\left(1-\alpha\right)} \frac{d}{dx} \int_{a}^{x} \left(ln\frac{x}{t} \right)^{-\alpha} \frac{f(t)}{t} dt. $$

Uniqueness and continuous dependence of solutions for fractional differential systems with Hadamard derivatives is proved in [24]. This work analyse the Adams-Bashforth- Moulton scheme based on the Hadamard fractional differential operators for fractional variable-order chaotic systems.

As an important concept in nonlinear science, chaos is considered by unstable dynamic behavior with sensitive dependence on initial conditions and includes infinite unstable periodic motions. Since the pioneering work by Pecora and Carroll [26], synchronization of chaotic systems has attracted a great deal of interest among scientists from various areas due to its potential applications in ecological and physical systems, modeling brain activity, system identification, pattern recognition phenomena, and secure communications [**?**,27,28,30]. Many schemes are suggested to realize chaotic synchronization such as adaptive control [31], sliding mode control [32], active control [33], optimal control [34], back stepping design [35] and etc.

Existence of chaos in fractional-order systems is conceived by Grigorenko in 2003 [36]. They have examined chaotic behavior in fractional-order Lorenz system and then some papers have been published dealing with the chaotic behavior in fractional-order systems [37–45]. Following this concept, many other fractional order chaotic systems have been established, such as fractional order Chuas [46], Lorenz [47], Chen [39], financial [48], Qi [49], Rabinovich-Fabrikant [50], Rossler [41], a chaotic fractional damped-driven pendulum. In [51] Hopf-bifurcation, chaos control and synchronization of a chaotic fractional- order system with chaos entanglement function. In [51] Review on smart grid control and reliability in presence of renewable energies:Challenges and prospects. In [53] a robust simulation- optimization approach for predisaster multi-period location-allocation-inventory planning [54]. Chen et al. [55] have proved a unified analysis for finite-time anti-synchronization of a class of integer-order and fractional-order chaotic systems. In [56], fractional-order chaotic and hyper-chaotic systems are proposed and their dynamics are investigated through numerical simulations. Li and Sun [57] have proposed an adaptive neural network back stepping control in solving fractional-order Chua–Hartley chaotic system. Atangana and Qureshi have introduced a new method based on a new fractional operator for solving nonlinear fractional equations [58]. In [59] a new fractional-order chaotic system and an adaptive synchronization of fractional order chaotic system are reported.

This paper concerns about solving fractional variable-order chaotic systems. From a specific point of view, we put forward a new method that is named fractional Adams-Bashforth-Moulton based on Hadamard fractional operator (FABMH), which is a predictor-corrector method based on Adams-Bashforth-Moulton scheme that is itself based on Hadamard fractional operator for solving fractional variable-order chaotic systems. This paper's proposed approach is a new one with no similar work being performed in this direction until this moment. Applying the approach for solving some novel chaotic systems with fractional order will justify the applicability and suitability of our scheme. For demonstrating the effectivity of our new numerical scheme, we will compare the Adams-Bashforth-Moulton method involving Hadamard fractional operator with the Bashforth forth-Moulton method involving Atangana-Baleanu-Caputo fractional operator [2] for a couple of fractional variable-order ordinary differential equations.

The given article is outlined as follows. Section 2 briefly presents a number of basic definitions and properties of fractional calculus. In section 3, the fractional Adams-Bashforth-Moulton method based on Hadamard fractional operator for fractional ODEs is put forward. Section 4 deals with the results of our extended numerical experiments. Finally in Section 5, brief conclusions and future work are presented.

2 . Preliminaries

In this section, we introduce some of the important and main concepts of fractional calculus which are applied in next section.

Definition 2.1. *Let $\alpha \in R_+$ and $f \in L_1[a,b]$. Then for $\alpha \in (0,1]$, the Riemann-Liouville fractional integral is defined as:*

$$J_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt,$$

where $a \le x \le b$ and $\Gamma(.)$ is the Gamma function which is defined as:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

Definition 2.2. *Let $\alpha \in R_+$ and $m = [\alpha]$. Then the Riemann-Liouville fractional operator D_a^{α} of fractional order α is defined as:*

$$D_a^{\alpha} f = D^m J_a^{m-\alpha} f.$$

Definition 2.3. *The Hadamard fractional integral of variable order $\alpha(x)$ for a continuous function f is defined as:*

$$_H D_a^{\alpha(x)} f(x) = \frac{1}{\Gamma(\alpha(x))} \int_a^x \left(ln\frac{x}{t}\right)^{\alpha(x)-1} \frac{f(t)}{t} dt, \ \alpha(x) > 0. \tag{1}$$

Definition 2.4. *Let $f : [1,\infty) \to R$ be a continuous function. Then the Hadamard derivative of fractional variable order $\alpha(x)$ is defined as:*

$$_H D_a^{\alpha(x)} f(x) = \frac{1}{\Gamma(m-\alpha(x))} \left(x\frac{d}{dx}\right)^m \int_a^x \left(ln\frac{x}{t}\right)^{m-\alpha(x)-1} \frac{f(t)}{t} dt,$$
$$m-1 < \alpha(x) < m, \ m = [\alpha(x)].$$

3 *Numerical scheme to solve the fractional variable-order equation involving Hadamard fractional operator*

In this section, we use numerical scheme based on one-step Adams-Bashforth-Moulton scheme to solve the following kind of fractional variable-order equation with initial condition:

$$\begin{cases} D^{\alpha(t)} y(t) = f(t, y(t)), \quad a < t < T, \quad a > 0, \\ y(a) = y_0. \end{cases} \tag{2}$$

Here, we assume that f has a unique solution in interval $[a,T]$. For solving Eq. (2), we use the predictor-corrector scheme. For this purpose, suppose that $\{t_j = a + jh : j = 0,1,\ldots,N\}$ and $h = \frac{T-a}{N}$, where$N \in Z$. By applying Hadamard fractional integral given by (2) on the interval $[t_0, t_{k+1}]$, we obtain:

$$y(t_{k+1}) = y_0 + \frac{1}{\Gamma(\alpha(t_{k+1}))} \int_{t_0}^{t_{k+1}} \left(ln\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})-1} \frac{f(u,y(u))}{u} du, \tag{3}$$

The product trapezoidal quadrature formula subject to the weight function $\left(In\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})-1}$ can be applied by replacing the integral, with nodes $\{t_j : j = 0,1,\ldots,N\}$. The *approximate solution for the right-hand side integral in Eq. (3) is calculated as:*

$$\int_{t_0}^{t_{k+1}} \frac{\left(ln\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})-1}}{u} f(u)\,du \approx \int_{t_0}^{t_{k+1}} \frac{\left(ln\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})-1}}{u} \tilde{f}_{k+1}(u)\,du, \tag{4}$$

where $\tilde{f}_{k+1}$ is a piecewise linear interpolant for f, that it is obtained from the trapezoidal rule. The integral introduced in Eq. (4) can be written as follows:

$$\int_{t_0}^{t_{k+1}} \frac{\left(ln\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})-1}}{u} \tilde{f}_{k+1}(u)\,du = \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \frac{\left(ln\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})-1}}{u} \tilde{f}_{k+1}(u)\,du$$

$$\sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \frac{\left(ln\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})-1}}{u} \left[f(t_j)\frac{u-t_{j+1}}{t_j-t_{j+1}} + f(t_{j+1})\frac{u-t_j}{t_{j+1}-t_j}\right] du$$

$$= \sum_{j=0}^{k} \left[\frac{f(t_j)}{t_j-t_{j+1}} \underbrace{\int_{t_j}^{t_{j+1}} \frac{\left(ln\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})-1}}{u} (u-t_{j+1})\,du}_{} \right.$$

$$\left. + \frac{f(t_{j+1})}{t_{j+1}-t_j} \underbrace{\int_{t_j}^{t_{j+1}} \frac{\left(ln\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})-1}}{u} (u-t_j)\,du.}_{II} \right] \tag{5}$$

Now, we can obtain any part separately as:

$$I = \left[\frac{-1}{\alpha(t_{k+1})}\left(ln\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})}(u-t_j)\right]_{t_j}^{t_{j+1}} + \int_{t_j}^{t_{j+1}} \frac{\left(ln\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})}}{\alpha(t_{k+1})} du$$

$$= \frac{-h}{\alpha(t_{k+1})}\left(ln\frac{t_{k+1}}{t_{j+1}}\right)^{\alpha(t_{k+1})} - \frac{t_{k+1}}{\alpha(t_{k+1})} \int_{ln\frac{t_{k+1}}{t_j}}^{ln\frac{t_{k+1}}{t_{j+1}}} e^{-u} du$$

$$= \frac{-h}{\alpha(t_{k+1})}\left(ln\frac{t_{k+1}}{t_{j+1}}\right)^{\alpha(t_{k+1})} - \frac{t_{k+1}}{\alpha(t_{k+1})}\left(\Gamma\left(\alpha(t_{k+1})+1, ln\frac{t_{k+1}}{t_j}\right)\right.$$

$$\left. -\Gamma\left(\alpha(t_{k+1})+1, ln\frac{t_{k+1}}{t_{j+1}}\right)\right)$$

$$= \frac{1}{\alpha(t_{k+1})}\left[-h\left(ln\frac{t_{k+1}}{t_{j+1}}\right)^{\alpha(t_{k+1})} + t_{k+1}\Gamma\left(\alpha(t_{k+1})+1, ln\frac{t_{k+1}}{t_{j+1}}\right)\right.$$

$$\left. -t_{k+1}\Gamma\left(\alpha(t_{k+1})+1, ln\frac{t_{k+1}}{t_j}\right)\right],$$

and

$$II = \frac{1}{\alpha(t_{k+1})}\left[-h\left(ln\frac{t_{k+1}}{t_j}\right)^{\alpha(t_{k+1})}\right.$$

$$+t_{k+1}\Gamma\left(\alpha(t_{k+1})+1, ln\frac{t_{k+1}}{t_{j+1}}\right)-t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_j}\right),$$

By substituting the above equations into Eq. (5)*, we have:*

$$\sum_{j=0}^{k}\left[\frac{f(t_j)}{h\alpha(t_{k+1})}\left(h\left(ln\frac{t_{k+1}}{t_j}\right)^{\alpha(t_{k+1})}-t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_{j+1}}\right)\right.\right.$$
$$\left.+t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_j}\right)\right)$$
$$+\frac{f(t_{j+1})}{h\alpha\left(t_{k+1}\right)}\left(-h\left(ln\frac{t_{k+1}}{t_{j+1}}\right)^{\alpha(t_{k+1})}+t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_{j+1}}\right)\right.$$
$$\left.\left.-t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_j}\right)\right)\right]. \tag{6}$$

Then, we compute the summation in the following way:

$$=\frac{f(t_0)}{ha(t_{k+1})}\left[h\left(ln\frac{t_{k+1}}{t_0}\right)^{\alpha(t_{k+1})}-t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_1}\right)\right.$$
$$\left.+t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_0}\right)\right]$$
$$+\frac{f(t_1)}{ha(t_{k+1})}\left[-h\left(ln\frac{t_{k+1}}{t_1}\right)^{\alpha(t_{k+1})}+t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_1}\right)\right.$$
$$\left.-t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_0}\right)\right]$$
$$+\frac{f(t_1)}{ha(t_{k+1})}\left[h\left(ln\frac{t_{k+1}}{t_1}\right)^{\alpha(t_{k+1})}-t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_2}\right)\right.$$
$$\left.+t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_1}\right)\right]$$
$$+\frac{f(t_2)}{ha(t_{k+1})}\left[-h\left(ln\frac{t_{k+1}}{t_2}\right)^{\alpha(t_{k+1})}+t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_2}\right)\right.$$
$$\left.-t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_1}\right)\right]+\dots$$
$$+\frac{f(t_k)}{ha(t_{k+1})}\left[h\left(ln\frac{t_{k+1}}{t_k}\right)^{\alpha(t_{k+1})}-t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_{k+1}}\right)\right.$$
$$\left.+t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_k}\right)\right]$$
$$+\frac{f(t_{k+1})}{ha(t_{k+1})}\left[-h\left(ln\frac{t_{k+1}}{t_{k+1}}\right)^{\alpha(t_{k+1})}+t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_{k+1}}\right)\right.$$

$$-t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_k}\right).$$

Then, we have:

$$= \frac{f\left(t_0\right)}{ha\left(t_{k+1}\right)}\left[h\left(ln\frac{t_{k+1}}{t_1}\right)^{\alpha(t_{k+1})} - t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_1}\right)\right.$$

$$+t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_0}\right)$$

$$+\frac{f(t_1)}{ha(t_{k+1})}\left[-t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_0}\right) - 2t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_1}\right)\right.$$

$$-t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_2}\right)$$

$$+\cdots+\frac{f\left(t_{k+1}\right)}{ha\left(t_{k+1}\right)}\left[t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1\right) - t_{k+1}\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_k}\right)\right].$$

By substituting the above equations into Eq. (3)*, we obtain the following relation:*

$$y\left(t_{k+1}\right) = y_0 + \frac{1}{\Gamma\left(\alpha\left(t_{k+1}\right)\right)}\left(\sum_{j=0}^{k} a_{j,k+1} f\left(t_j, y_i\right) + a_{k+1,k+1} f\left(t_{k+1}, y_{k+1}^p\right)\right),$$

respect to the following weights:

$$a_{j,k+1} = \frac{1}{h\alpha(t_{k+1})} \times \left[\begin{array}{ll} h\left(ln\frac{t_{k+1}}{t_0}\right)^{\alpha(t_{k+1})} + t_{k+1}\left[\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_0}\right)\right. & \\ \quad -\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_1}\right) & j=0 \\ t_{k+1}\left[-\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_{j-1}}\right) + 2\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_j}\right)\right. & \\ \quad -\Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_{j+1}}\right) & 1 \le j \le k \\ t_{k+1}\left[\Gamma\left(\alpha\left(t_{k+1}\right)+1\right) - \Gamma\left(\alpha\left(t_{k+1}\right)+1, ln\frac{t_{k+1}}{t_k}\right)\right], & j=k+1. \end{array}\right.$$

We obtain the predictor formula for y_{k+1}^p *later. This can be found by applying the product rectangle rule to the last term of the right hand side of* (4) *in the* *same way as the corrector formula, to achieve,*

$$\int_{t_0}^{t_{k+1}} \frac{\left(ln\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})-1}}{u} \tilde{f}_{k+1}(u)\,du = \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \frac{\left(ln\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})-1}}{u} \tilde{f}_{k+1}(u)\,du$$

$$= \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \frac{\left(ln\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})-1}}{u} \left[f(t_j)\frac{u-t_{j+1}}{t_j - t_{j+1}} + f(t_j)\frac{u-t_j}{t_{j+1}-t_j} \right](u)\,du$$

$$= \sum_{j=0}^{k} \left[\frac{f(t_j)}{t_j - t_{j+1}} \int_{t_j}^{t_{j+1}} \frac{\left(ln\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})-1}}{u} (u - t_{j+1})\,du \right.$$

$$\left. = \frac{f(t_j)}{t_{j+1}-t_j} \int_{t_j}^{t_{j+1}} \frac{\left(ln\frac{t_{k+1}}{u}\right)^{\alpha(t_{k+1})-1}}{u} (u - t_j)\,du \right]$$

$$= \sum_{j=0}^{k} \left[\frac{f(t_j)}{h\alpha(t_{k+1})} \left(h\left(ln\frac{t_{k+1}}{t_j} \right)^{\alpha(t_{k+1})} - t_{k+1}\Gamma\left(\alpha(t_{k+1})+1, ln\frac{t_{k+1}}{t_{j+1}} \right) \right.\right.$$

$$\left. + t_{k+1}\Gamma\left(\alpha(t_{k+1})+1, ln\frac{t_{k+1}}{t_{j+1}} \right) \right) + \frac{f(t_j)}{h\alpha(t_{k+1})} \left(-h\left(ln\frac{t_{k+1}}{t_j} \right)^{\alpha(t_{k+1})} \right.$$

$$\left.\left. + t_{k+1}\Gamma\left(\alpha(t_{k+1})+1, ln\frac{t_{k+1}}{t_{j+1}} \right) - t_{k+1}\Gamma\left(\alpha(t_{k+1})+1, ln\frac{t_{k+1}}{t_{j+1}} \right) \right) \right]$$

$$= \sum_{j=0}^{k} \frac{f(t_j)}{\alpha(t_{k+1})} \left[\left(ln\frac{t_{k+1}}{t_j} \right)^{\alpha(t_{k+1})} - \left(ln\frac{t_{k+1}}{t_{j+1}} \right)^{\alpha(t_{k+1})} \right].$$

Substituting the above equations into Eq. (3) *and the following predictor relation is obtained:*

$$y_{k+1}^{p} = y_0 + \frac{1}{\Gamma(\alpha(t_{k+1}))} \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j),$$

respect to the following weight function:

$$b_{j,k+1} = \frac{1}{\alpha(t_{k+1})} \left[\left(ln\frac{t_{k+1}}{t_j} \right)^{\alpha(t_{k+1})} - \left(ln\frac{t_{k+1}}{t_{j+1}} \right)^{\alpha(t_{k+1})} \right].$$

Finally, the fractional Adams-Bashforth-Moulton scheme based on Hadamard fractional derivative is obtained as:

$$\begin{cases} y_{k+1}^{p} = y_0 + \frac{1}{\Gamma(\alpha(t_{k+1}))} \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_i), \\ y_{k+1}^{p} = y_0 + \frac{1}{\Gamma(\alpha(t_{k+1}))} \left(\sum_{j=0}^{k} a_{j,k+1} f(t_j, y_i) + a_{k+1,k+1} f\left(t_{k+1}, y_{k+1}^{P}\right) \right), \end{cases} \quad (7)$$

respect to the following weight function:

$$a_{j,k+1}=\frac{1}{h\alpha\left(t_{k+1}\right)}\times\begin{cases} h\left(ln\frac{t_{k+1}}{t_0}\right)^{\alpha(t_{k+1})}+t_{k+1}\left[\Gamma\left(\alpha\left(t_{k+1}\right)+1,ln\frac{t_{k+1}}{t_0}\right)\right. \\ \left.-\Gamma\left(\alpha\left(t_{k+1}\right)+1,ln\frac{t_{k+1}}{t_1}\right),\right. & j=0 \\ t_{k+1}\left[-\Gamma\left(\alpha\left(t_{k+1}\right)+1,ln\frac{t_{k+1}}{t_{j-1}}\right)\right. \\ \left.+2\Gamma\left(\alpha\left(t_{k+1}\right)+1,ln\frac{t_{k+1}}{t_j}\right)-\Gamma\left(\alpha\left(t_{k+1}\right)+1,ln\frac{t_{k+1}}{t_{j-1}}\right),\right. & 1\leq j\leq k \\ t_{k+1}\left[-\Gamma\left(\alpha\left(t_{k+1}\right)+1\right)\right. \\ \left.-\Gamma\left(\alpha\left(t_{k+1}\right)+1,ln\frac{t_{k+1}}{t_k}\right),\right. & j=k+1 \end{cases} \tag{8}$$

and

$$b_{j,k+1}=\frac{1}{\alpha\left(t_{k+1}\right)}\left[\left(ln\frac{t_{k+1}}{t_j}\right)^{\alpha(t_{k+1})}-\left(ln\frac{t_{k+1}}{t_{j+1}}\right)^{\alpha(t_{k+1})}\right]. \tag{9}$$

Now, the fractional Adams-Bashforth-Moulton scheme based on Hadamard fractional derivative is extended for the fractional variable-order systems:

$$\begin{cases} D_t^{\alpha(t)}x\left(t\right)=f_1\left(t,x\left(t\right),y\left(t\right),z\left(t\right)\right), \\ D_t^{\alpha(t)}y\left(t\right)=f_2\left(t,x\left(t\right),y\left(t\right),z\left(t\right)\right), \\ D_t^{\alpha(t)}z\left(t\right)=f_3\left(t,x\left(t\right),y\left(t\right),z\left(t\right)\right), \end{cases} \tag{10}$$

Where $\alpha(t)$ is variable order in terms of function time and initial condition(x_0,y_0,z_0). By applying the above method, systems (10) can be approximated as follows:

$$\begin{cases} x_{k+1}=x_0+\frac{1}{\Gamma(\alpha(t_{k+1}))}\sum_{j=0}^{k}a_{j,k+1}f_1\left(t_j,x_j,y_j,z_j\right)+a_{k+1,k+1}f_1\left(t_{k+1},x_{k+1}^P,y_{k+1}^P,z_{k+1}^P\right), \\ y_{k+1}=y_0+\frac{1}{\Gamma(\alpha(t_{k+1}))}\sum_{j=0}^{k}a_{j,k+1}f_2\left(t_j,x_j,y_j,z_j\right)+a_{k+1,k+1}f_2\left(t_{k+1},x_{k+1}^P,y_{k+1}^P,z_{k+1}^P\right), \\ z_{k+1}=z_0+\frac{1}{\Gamma(\alpha(t_{k+1}))}\sum_{j=0}^{k}a_{j,k+1}f_3\left(t_j,x_j,y_j,z_j\right)+a_{k+1,k+1}f_3\left(t_{k+1},x_{k+1}^P,y_{k+1}^P,z_{k+1}^P\right), \end{cases} \tag{11}$$

where

$$\begin{cases} x_{k+1}^P=x_0+\frac{1}{\Gamma(\alpha(t_{k+1}))}\sum_{j=0}^{k}b_{j,k+1}f_1\left(t_j,x_j,y_j,z_j\right), \\ x_{k+1}^P=y_0+\frac{1}{\Gamma(\alpha(t_{k+1}))}\sum_{j=0}^{k}b_{j,k+1}f_2\left(t_j,x_j,y_j,z_j\right), \\ x_{k+1}^P=z_0+\frac{1}{\Gamma(\alpha(t_{k+1}))}\sum_{j=0}^{k}b_{j,k+1}f_3\left(t_j,x_j,y_j,z_j\right), \end{cases} \tag{12}$$

and

$$a_{j,k+1} = \frac{1}{h\alpha(t_{k+1})} \times \begin{cases} h\left(ln\frac{t_{k+1}}{t_0}\right)^{a(t_{k+1})} + t_{k+1}\left[\Gamma\left(\alpha(t_{k+1})+1, ln\frac{t_{k+1}}{t_0}\right)\right. \\ \quad \left.-\Gamma\left(\alpha(t_{k+1})+1, ln\frac{t_{k+1}}{t_1}\right)\right., & j=0 \\ t_{k+1}\left[-\Gamma\left(\alpha(t_{k+1})+1, ln\frac{t_{k+1}}{t_{j-1}}\right)\right. \\ \left.+2\Gamma\left(\alpha(t_{k+1})+1, ln\frac{t_{k+1}}{t_j}\right) - \Gamma\left(\alpha(t_{k+1})+1, ln\frac{t_{k+1}}{t_{j-1}}\right)\right., & 1 \le j \le k \\ t_{k+1}\left[-\Gamma(\alpha(t_{k+1})+1)\right. \\ \quad \left.-\Gamma\left(\alpha(t_{k+1})+1, ln\frac{t_{k+1}}{t_k}\right)\right., & j=k+1 \end{cases} \tag{13}$$

$$b_{j,k+1} = \frac{1}{\alpha(t_{k+1})}\left[\left(ln\frac{t_{k+1}}{t_j}\right)^{\alpha(t_{k+1})} - \left(ln\frac{t_{k+1}}{t_{j+1}}\right)^{\alpha(t_{k+1})}\right]. \tag{14}$$

4***Numerical Results***

The purpose of this section is to apply our method which is described earlier for obtaining the approximate solution of fractional variable-order differential equations. To do this aim, we show some equations to demonstrate the proposed method performance. The applicability of the proposed scheme is extended to solve some chaotic systems modeled by the Hadamard derivatives in terms of function time. Also, in this section, we compare proposed scheme with Adams-Bashforth-Moulton method based involving Atangana-Baleanu-Caputo fractional operator [2].

4.1**Fractional variable-order chaotic**

The Lorenz oscillator is a three-dimensional system that exhibits chaotic now. A new three dimensional system, which is similar to Lorenz and other chaotic attractors, but has a different topological structure from any existing chaotic attractors, which is presented in [45] and it is described by the following equations:

$$\begin{cases} D_t^{\alpha(t)}x(t) = -ax(t) + by(t)z(t), \\ D_t^{\alpha(t)}y(t) = cy(t) - dx(t)z(t), \\ D_t^{\alpha(t)}z(t) = -kz(t) + mx(t)y(t). \end{cases} \tag{15}$$

Simulation results at the instances of fractional variable-order $\alpha(t) = tanh\left(\frac{\pi}{2}+t\right)$, *for initial conditions and parameters choice* $(x_0, y_0, z_0) = (3,2,1), a = 4, b = 3, c = 1, d = 7, k = 1$ *and m=2 are represented in Fig. 1. Figs. 2 and 3 illustrates the time series solution at* $t = 200$ *with time step* $h = 0.02$ *for the proposed method in comparison with the method that was introduced in [2]. Moreover, Table 1 provides the results obtained from Adams-Bashforth-Moulton method based on Hadamard fractional operator in comparison with Adams-Bashforth-Moulton method based on Atangana-Baleanu-Caputo fractional operator [2].*

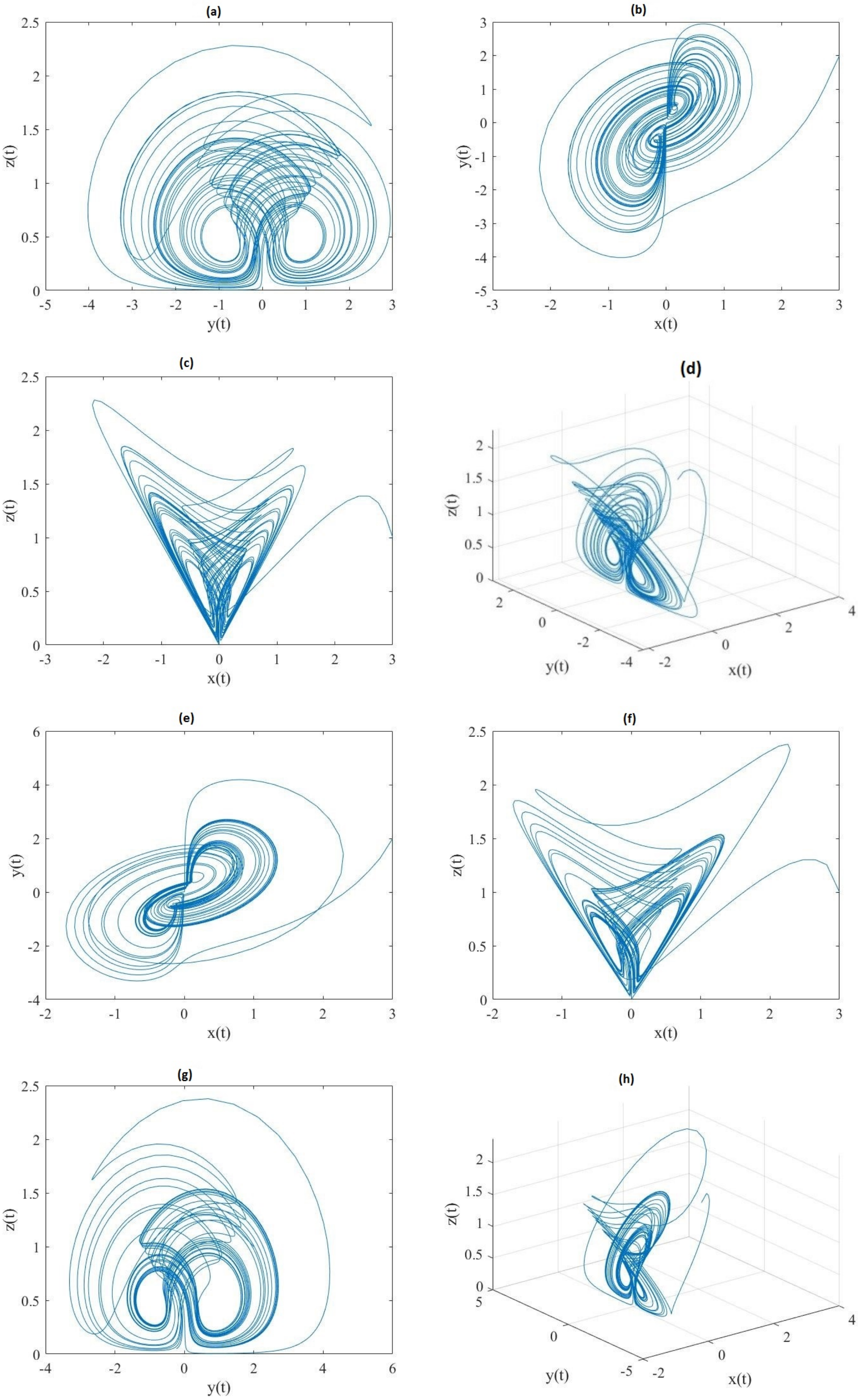

FIGURE 1. simulation results Adams-Bashforth-Moulton Hadamard of variables $(a), (b), (c), (d)$ and Adams-Bashforh-Multon-Atanganu $(e), (f), (g), (h)$ methods for the order $\alpha(t) = \tanh(\frac{\pi}{2} + t)$.

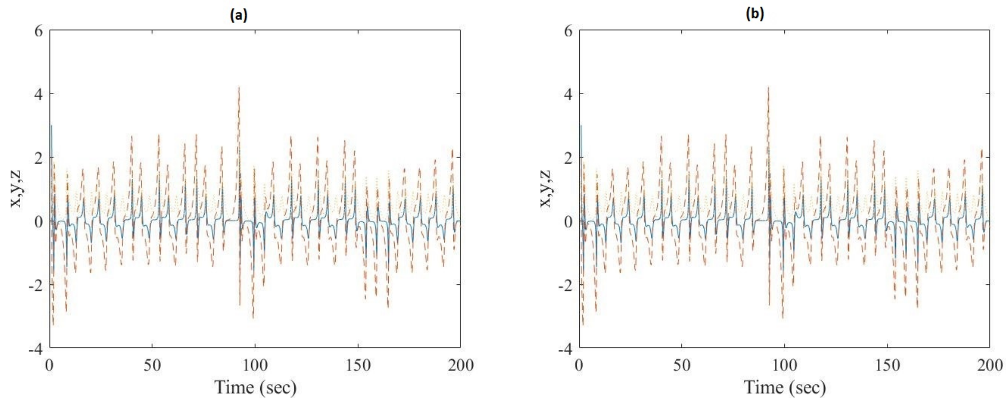

FIGURE 2. Time-series results reecting chaotic spatio-temporal oscillations at time $t = 200$ for Hadamard method (a) and Atangana method (b).

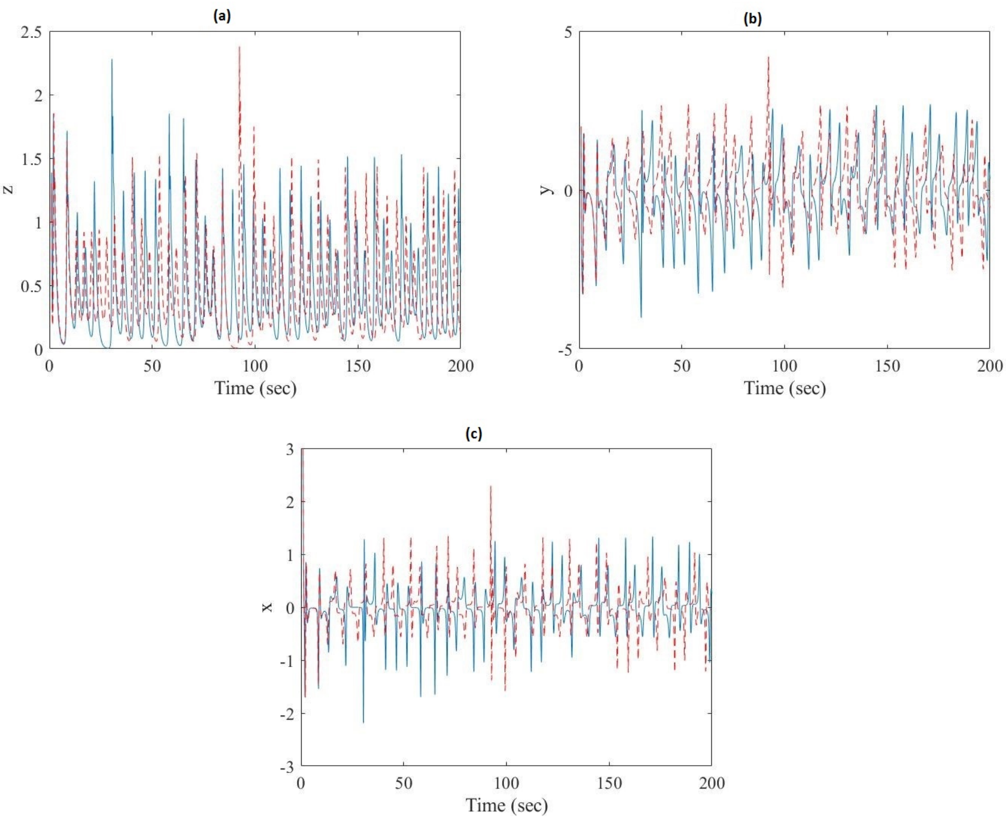

FIGURE 3. Comparison of time-series results between Adams-Bashforth-Moulton-Hadamard and Adams-Bashforth-Moulton-Atangana methods of variables $(a), (b), (c)$.

TABLE 1. Comparing Adams-Bashforth-Moulton-Hadamard (a) and Adams-Bashforth-Moulton-Atangana (b) for $\alpha(t) = \tanh(\frac{\pi}{2} + t)$.

t	$Adams - Bashforth - Moulton - Hadamard$
	$(x(t), y(t), z(t))$
10	$(-2.13 \times 10^{-01}, -4.36 \times 10^{-1}, 5.69 \times 10^{-01})$
30	$(-1.76 \times 10^{-01}, -3.48 \times 10^{+00}, 1.72 \times 10^{-01})$
50	$(-4.76 \times 10^{-02}, -7.77 \times 10^{-01}, 9.18 \times 10^{-02})$
100	$(-2.76 \times 10^{-01}, -6.21 \times 10^{-01}, 6.25 \times 10^{-01})$
150	$(-2.62 \times 10^{-02}, -4.94 \times 10^{-01}, 6.35 \times 10^{-01})$
200	$(2.58 \times 10^{-01}, 3.45 \times 10^{-01}, 6.01 \times 10^{-01})$
	$Adams - Bashforth - Moulton - Atangana$
	$(x(t), y(t), z(t))$
10	$(-1.68 \times 10^{-01}, -4.37 \times 10^{-01}, 4.95 \times 10^{-01})$
30	$(8.98 \times 10^{-02}, 7.40 \times 10^{-01}, 1.76 \times 10^{-01})$
50	$(2.08 \times 10^{-02}, 1.48 \times 10^{-01}, 2.41 \times 10^{-01})$
100	$(6.79 \times 10^{-01}, 2.79 \times 10^{-01}, 1.21 \times 10^{+00})$
150	$(-1.78 \times 10^{-01}, -2.49 \times 10^{-01}, 5.06 \times 10^{-01})$
200	$(-4.84 \times 10^{-02}, -5.58 \times 10^{-01}, 1.21 \times 10^{-01})$

By then Hadamard derivatives in time. Chaos has been shown to be more useful in many engineering and scientific applications, and there has been strong and increasing demand for formulating chaos at will. Also, we compare our method with Adams-Bashforth-Moulton method based on Atangana-Baleanu-Caputo fractional operator [2].

5 ***Conclusions and future work***

This paper studied a new two-step Adams-Bashforth-Moulton method based on the Hadamard fractional derivative. In order for demonstrating the effectiveness of the new numerical scheme, we compared the proposed method with the method that is introduced in [2] for two fractional variable-order ordinary differential equations. The applicability and suitability of the scheme is proved when it is applied for solving two fractional variable-order chaotic systems. According to the diagrams and numerical results obtained, the two methods have shown good performance. The mathematical idea which is proposed in this study to solve fractional ordinary differential equations is extendable to fractional partial differential equations. Accordingly, developing this scheme for fractional PDEs will be our concern in our future research, in which we will also take into account higher-dimensional computational problems, theoretically investigating and analyzing the stability and convergence of our scheme.

REFERENCES

[1] *G. Bercu and M. Postolache*, Class of self-concordant functions on Riemannian manifolds, Balkan J. Geom. Appl., **14**(2009), No. 2, 13-20.

[2] *C.J. Zuniga-Aguilar, J.F. Gomez-Aguilar, R.F. Escobar-Jimenez, H.M. Romero-Ugalde*, Robust control for fractional variable-order chaotic systems with non-singular kernel, The European Physical Journal Plus., **133**(1) (2018) 13.

[3] *L. Liu, L. Zheng, F. Liu, X. Zhang*, Anomalous convection diffusion and wave coupling transport of cells on comb frame with fractional Cattaneo-Christov ux, Commun. Nonlinear Sci. Numer. Simul., **38**(2016) 45-58.

[4] *Z. Wang, X. Wang, Y. Li, X. Huang*, Stability and Hopf Bifurcation of Fractional-Order Complex-Valued Single Neuron Model with Time Delay, Int. J. Bifurc. Chaos., **27** (2017) 1-13.

[5] *Z. Wang, X. Huang, G. Shi*, Analysis of nonlinear dynamics and chaos in a fractional order nancial system with time delay, Comput. Math. Appl., **62** (2001) 1531-1539.

[6] *V.E. Tarasov*, Fractal electrodynamics via non-integer dimensional space approach, Phys. Lett. A., **379** (2015) 2055-2061.

[7] *M.S. Aslam, M.A.Z. Raja*, A new adaptive strategy to improve online secondary path modeling in active noise control systems using fractional signal processing approach, Signal Process., **107** (2015) 433-443.

[8] *E. Ziaei, M.H. Farahi, E. Safaie*, The approximate solution of nonlinear fractional optimal control problems by measure theory approach, Progr. Fract. Dier. Appl., **4** (2018) 1-13.

[9] *M. Yavuz, N. Ozdemir*, European Vanilla Option Pricing Model of Fractional Order without Singular Kernel, Fractal Fract. **2**(1) (2018).

[10] *I. Podlubny*, Fractional Dierential Equations: An Introduction to Fractional Derivatives, Fractional Dierential Equations, to Methods of Their Solution and some of Their Applications; Academic Press: San Diego, CA, USA, 1999.

[11] *M.F. Caputo*, A New Denition of Fractional Derivative without Singular Kernel, Prog. Fract. Dier. Appl., **1** (2015) 73-85.

[12] *A. Atangana, D. Baleanu*, New Fractional Derivatives with Nonlocal and Non-Singular Kernel: Theory and Application to Heat Transfer Model, Therm. Sci., **20** (2016) 763-769.

[13] *M. Caputo, M. Fabrizio*, Applications of new time and spatial fractional derivatives with exponential kernels, Progr. Fract. Diff. Appl., **2** (2016) 1-11.

[14] *H. Aminikhah, A. H. Refahi Sheikhani and H. Rezazadeh*, Stability analysis of linear distributed order system with multiple time delays, UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, **77**(2):207-218, 2015.

[15] *A. H. Refahi Sheikhani and M. Mashoof*, Numerical solution of fractional differential equation by wavelets and hybrid functions, Boletim da Sociedade Paranaense de Matematica, **36**(2):231-244, 2018.

[16] *H. Aminikhah, A. H. Refahi Sheikhani and H. Rezazadeh*, Approximate analytical solutions of distributed order fractional Riccati differential equation, Ain Shams Engineering Journal, **9**(4):581-588, 2018.

[17] *F. Mehrdoust, A.H. Refahi Sheikhani, M. Mashoof, S. Hasanzadeh*, Block-pulse operational matrix method for solving fractional Black-Scholes equation, Journal of Economic Studies, **44**(3)(2017) 489-502.

[18] *M. Mashoof, A. H. Refahi Sheikhani and H. Saberi Najafi*, Stability Analysis of Distributed- Order Hilfer-Prabhakar Systems Based on Inertia Theory, Mathematical Notes, **104**(1-2):74-85,2018.

[19] *H. Aminikhah, A. H. Refahi Sheikhani, T. Houlari and H. Rezazadeh*, Numerical solution of the distributed-order fractional Bagley-Torvik equation, IEEE/CAA Journal of Automatica Sinica, **6**(3):760-765, 2019.

[20] *M. Mashoof, A. H. Refahi Sheikhani and H. Saberi Najafi*, Stability analysis of distributed order Hilfer-Prabhakar differential equations, Hacettepe Journal of Mathematics and Statistics, **47**(2):299-315, 2018.

[21] *Z. Odibat, S. Momani*, Application of the variation iteration method to nonlinear dierential equations of fractional order, Int. J. Nonlinear Sci. Numer. Simul., **7** (2006) 15-27.

[22] *S.S. Ray*, Analytical solution for the space fractional diusion equation by two-step Adomian Decomposition method, Commun. Nonlinear Sci. Numer. Simul., **14** (2009) 1295-1306.

[23] *K. Diethelm, N.J. Ford, A.D. Freed*, Detailed error analysis for a fractional Adams method, Numer. Algorithms, **36** (2004) 31-52.

[24] *J. Hadamard*, Essai sur letude des fonctions, donnees par leur developpement de Taylor. Gauthier-Villars, 1892.

[25] *Q. Ma, J. Wang, R. Wang, X. Ke*, Study on some qualitative properties for solutions of a certain two-dimensional fractional dierential system with Hadamard derivative, Appl. Math. Lett. **36** (2014) 7-13.

[26] *L.M. Pecora, T.L. Carroll*, Synchronization in chaotic systems, Phys. Rev. Lett. **64** (1990) 821-824.

[27] *G. Chen, X. Dong*, From Chaos to Order: Methodologies. Perspectives and Applications. Singapore: World Scientic; 1998.

[28] *E.W. Bai, K.E. Lonngren, A. Ucar*, Secure communication via multiple parameter modulation in a delayed chaotic system, Chaos Solitons Fractals, **23** (2005) 1071-1076.

[29] *H. Puebla, J. Alvarez-Ramirez*, More secure communication using chained chaotic oscillators, Phys. Lett. A, **283** (2001) 96-108.

[30] *P. Gholamin, A.H.R. Sheikhani*, A new three-dimensional chaotic system: Dynamical properties and simulation, Chinese j. phys., **55**(4) (2017) 1300-1309.

[31] *Y.W. Wang, C. Wen, M. Yang, J.W. Xiao*, Adaptive control and synchronization for chaotic systems with parametric uncertainties, Phys. Lett. A, **372** (2008) 2409-2414.

[32] *M. Falahpoor, M. Ataei, A. Kiyoumarsi*, A chattering-free sliding mode control design for uncertain chaotic systems, Chaos Solitons Fractals, **42** (2009) 1755-1765.

[33] *H.N. Agiza, M.T.Yassen*, Synchronization of Rossler and chaotic dynamical systems using active control, Phys. Lett. A, **278** (2001) 191-197.

[34] *M.L. Hung, J.S. Lin, J.J. Yan, T.L. Liao*, Optimal PID control design for synchronization of delayed discrete chaotic systems, Chaos Solitons Fractals, **35** (2008) 781-785.

[35] *J. Zhang, C. Li, H. Zhang, J. Yu*, Chaos synchronization using single variable feedback based on backstepping method, Chaos Solitons Fractals, **21** (2004) 1183-1193.

[36] *I. Grigorenko, E. Grigorenko*, Chaotic dynamics of the fractional Lorenz system, Phys. Rev. Lett., **91**(3) (2003) 034101.

[37] *A.E. Matouk*, Chaos, feedback control and synchronization of a fractional-order modied autonomous Van der Pol–Dung circuit, Commun. Nonlinear Sci. Numer. Simul., **16** (2011) 975-86.

[38] *Y. Yu, H.X. Li, S. Wang, J. Yu*, Dynamic analysis of a fractional-order Lorenz chaotic system, Chaos Solitons Fractals, **42** (2009) 1181-1189.

[39] *J.G. Lu, G. Chen*, A note on the fractional order Chen system, Chaos Solitons Fractals, **27** (2006) 685-688.

[40] *J.G. Lu*, Chaotic dynamics of the fractional order Lu system and its synchronization, Phys. Lett. A., **534** (2006) 305-311.

[41] *C. Li, G. Chen*, Chaos and hyperchaos in the fractional order Rossler equations, Physica A: Statis. Mech. Appl., **341** (2004) 55-61.

[42] *J.G. Lu*, Chaotic dynamics and synchronization of fractional order Arneodo's systems, Chaos Solitons Fractals, **26** (2005) 1125-1133.
[43] *L.J. Sheu, H.K. Chen, J.H. Chen, L.M. Tam, W.C. Chen, K.T. Lin, et al.* Chaos in the Newton-Leipnik system with fractional order, Chaos Solitons Fractals, **36** (2008) 98-103.
[44] *V.D. Gejji , S. Bhalekar*, Chaos in fractional ordered Liu system, Comput. Math. Appl., **59** (2010) 1117-1127.
[45] *P. Gholamin, A.H.R. Sheikhani*, Dynamical anlysis of a new three-dimensional fractional chaotic system. Paramana Journal of Physics, **92**(6), Article: 91 (2019) 1–14.
[46] *I. Petras*, A note on the fractional-order Chua's system, Chaos Solitons Fractals, **38**(1) (2008) 140-147.
[47] *M. Roohi, M.P. Aghababa, A.R. Haghighi*, Switching adaptive controllers to control fractional-order complex systems with unknown structure and input nonlinearities, Complexity, **21**(2) (2014) 211-223.
[48] *I. Pan, S. Das, S. Das*, Multi-objective active control policy design for commensurate and incommensurate fractional order chaotic nancial systems, Appl. Math. Model., **39**(2) (2015) 500-514.
[49] *Z. Wang, Y. Sun, G. Qi, and B.J.v. Wyk*, The effects of fractional order on a 3-D quadratic autonomous system with four-wing attractor, Nonlinear Dyn., **62**(1-2) (2010) 139-150.
[50] *M. Srivastava, S.K. Agrawal, K. Vishal, S. Das*, Chaos control of fractional order Rabinovich-Fabrikant system and synchronization between chaotic and chaos controlled fractional order Rabinovich-Fabrikant system, Appl. Math. Model., **38**(13) (2014) 3361-3372.
[51] *S.H. Eshaghi, R. Khoshsiar Ghaziani, A. Ansari*, Hopfbifurcation, chaos control and synchronization of a chaotic fractional- order system with chaos entanglement function. Mathematics and Computers in Simulation,**172** (2020) 321–340.
[52] *M. Ourahou, W. Ayrir, B.EL. Hassouni, A. Haddi*, Review on smart grid control and reliability in presence of renewable energies: Challenges and prospects. Mathematics and Computers in Simulation, **167** (2020) 19–31.
[53] *P. Ghasemi, K. Khalili Damghani*, A robust simulation- optimization approach for pre-disaster multi- period location- allocation- inventory planning. Mathematics and Computers in simulation, **179** (2021) 65-95.
[54] *K. Rabah, S. Ladaci*, A Fractional Adaptive Sliding Mode Control Conguration for Synchronizing Disturbed FractionalOrder Chaotic Systems, Circuits, Circuits Systems Signal Process., (2019) 1-21.
[55] *L. Chen, C. Huang, H. Liu, Y. Xia*, Anti-Synchronization of a Class of Chaotic Systems with Application to Lorenz System: A Unied Analysis of the Integer Order and Fractional Order, Mathematics, **7**(6) (2019) 559.
[56] *D. Nadjette, Z. Okba, Z. Sihem*, Dynamics of Fractional-Order Chaotic and Hyper-Chaotic Systems, Nonlinear Stud., **26**(2) (2019) 401-413.
[57] *G. Li, C. Sun*, Adaptive neural network backstepping control of fractional-order Chua–Hartley chaotic system, Adv. Difference Equ., **2019** (2019) 148.
[58] *A. Atangana, S. Qureshi*, Modeling attractors of chaotic dynamical systems with fractal–fractional operators, Chaos Solitons Fractals **1**(123) (2019) 320-337.
[59] *R. Behinfaraz, S. Ghaemi, S. Khanmohammadi*, Adaptive synchronization of new fractional-order chaotic systems with fractional adaption laws based on risk analysis, Math. Methods Appl. Sci., **42**(6) (2019) 1772-1785.