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HYPERCYCLIC TUPLE C_0-SEMIGROUPS OF OPERATORS

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In this paper, we introduce the notions of hypercyclic and locally topologically hypercyclic multi-parameter C_0-semigroups of operators on separable Banach spaces. Then we show that every finite dimensional Banach space admits a hypercyclic multi-parameter C_0-semigroup. Next, the hypercyclicity of tensor product of one-parameter C_0-semigroups as a multi-parameter C_0-semigroup will be discussed. Finally, locally topologically transitive two-parameter C_0-semigroups are studied.

1. Introduction and preliminaries

A continuous linear operator T on a Banach space X is called hypercyclic if it has a hypercyclic vector $x \in X$, i.e. there is a vector $x \in X$ such that $Orb(T, x) := \{T^n x : n \in \mathbb{N} \cup 0\}$ is dense in X.

Ansari [1] and Bernal- Gonzalez [2] showed that every infinite-dimensional separable Banach space admits a hypercyclic operator. This result was also extended to the non-normable Frèchet case by Bonet and Peris [3]. For more details about hypercyclic operators see the surveys [4, 11, 12]. It is well known that there is no hypercyclic operator on a finite dimensional Banach space. Hypercyclic tuples of operators was introduced by N. S. Feldman [10]. A finite sequence $T = (T_1, T_2, ..., T_n)$ of commuting continuous linear operators on a locally convex space X is called hypercyclic if there is a vector $x \in X$ whose orbit under T, i.e. $Orb(T, x) := \{T_1^{k_1} T_2^{k_2} ... T_n^{k_n} x : k_i \in \mathbb{N}, i = 1, 2, ...n\}$ is dense in X. In [10] Feldman showed that hypercyclic n-tuples can arise in finite dimensions when $n > 1$; something that does not happen for single operators.

In the continuous case, a one-parameter family $T = \{T(t)\}_{t\geq 0}$ of continuous linear operators on X, is a strongly continuous semigroup (or C_0-semigroup) of operators if $T(0) = I$, $T(t)T(s) = T(t + s)$, for all $t, s \geq 0$, and $\lim_{t\to 0} T_t x = x$ for all $x \in X$. For more information on C_0-semigroups refer to the books [9, 19]. A C_0-semigroup $T = \{T(t)\}_{t\geq 0}$ is said to be hypercyclic if $Orb(T, x) := \{T(t)x : t \geq 0\}$ is dense in X for some $x \in X$. Desch, Schappacher and Webb in [7] initiated the investigation of hypercyclic semigroups. So far, several specific examples of hypercyclic strongly continuous semigroups have been studied (see for example [6, 7, 8, 17, 20]). Also, tensor product of hypercyclic semigroups was studied in [22]. It is well-known that there is no hypercyclic C_0-semigroups of operators on finite dimensional Banach spaces.

Multi-parameter C_0-semigroups of operators were investigated in [15, 16], [14]. Any homomorphism W from the semigroup $(\mathbb{R}^d_+, +)$ into $B(X)$, the space of all bounded linear operators on the Banach space X, with $W(0) = I$ is called a d-parameter semigroup of operators on X where $\mathbb{R}^d_+ = \{(t_1, t_2, ..., t_n) : t_i \geq 0, i = 1, 2, ..., n\}$. Also, if the mapping $t \mapsto$

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$W(t)x$ is continuous at $0 \in \mathbb{R}_+^d$, for any $x \in X$, then W is said to be strongly continuous or d-parameter C_0-semigroup. Ergodic properties of d-parameter semigroups has been studied in [21].

The concept of J-class C_0-semigroups of operators (or topologically transitive C_0-semigroups) also studied by Nasseri in [18]. Recall that a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a normed space X is called J-class if there exists $0 \neq x \in X$ such that $J_T(x) = X$, where

$$\begin{aligned} J_T(x) : \; = \; & \{y \in X : \text{ there exist a strictly increasing sequence} \\ & (t_n)_{n\in\mathbb{N}} \subseteq [0,\infty) \text{ with } t_n \to 0 \text{ and a sequence} \\ & (x_n)_{n\in\mathbb{N}} \text{ in } X \text{ such that } x_n \to x \text{ and } T(t_n)(x_n) \to y\} \end{aligned}$$

In this paper, hypercyclicity and locally topologically transitivity of d-parameter C_0-semigroups of operators are investigated. In Section 2, some elementary hypercyclicity properties of d-parameter C_0-semigroups are studied. Finite dimensional Banach spaces and hypercyclic d-parameter C_0-semigroups on them will be discussed in Section 3. Indeed it will be shown that there are many hypercyclic d-parameter C_0-semigroups on $\mathbb{R}^n$ and $\mathbb{C}^n$ despite of they are finite dimensional. In Section 4, hypercyclicity of tensor product of one-parameter semigroups as d-parameter C_0-semigroups will be discussed. Finally, in the last section, locally topologically transitivity of d-parameter C_0-semigroups are investigated.
In the rest of the paper, X is a separable Banach space.

2. Hypercyclic d-parameter C_0-semigroup

Recall that any homomorphism $W : (\mathbb{R}_+^d, +) \to B(X)$ with $W(0) = I$ is called a d-parameter semigroup on X, which is denoted by $(W, \mathbb{R}_+^d, X)$. The family $\{W(t)\}_{t\in\mathbb{R}_+^d}$ is called strongly continuous (or C_0-semigroup) if the mapping $t \mapsto W(t)x : \mathbb{R}_+^d \to X$ is continuous for every $x \in X$. If $\{e_i : i = 1, 2, .., d\}$ is the standard basis of $\mathbb{R}^d$, then $u_i(t) := W(te_i)x$, $s \geq 0$, $x \in X$, is a C_0-one-parameter semigroup for which $u_i u_j = u_j u_i$ and $W(t_1, t_2, ..., t_n) = \prod_{i=1}^n u_i(t_i)$.
It is interesting to note that every one-parameter C_0-group $\{T(t)\}_{t\in\mathbb{R}}$ can be considered as a two-parameter C_0-semigroup of the form $W(s,t) := T(s-t)$, $s, t \geq 0$. In this case, $u_1(t) = T(t)$, and $u_2(t) = T(-t)$, $t \geq 0$.

Definition 2.1. *A d-parameter C_0-semigroup $(W, \mathbb{R}_+^d, X)$ is called*
i) tuple-hypercyclic (or simply, hypercyclic) if there exists a $x \in X$ such that $Orb(W, x) := \{W(t)x : t \in \mathbb{R}_+^d\}$ is dense in X. In this case, x is called the tuple-hypercyclic vector of W.
ii) tuple-transitive (or transitive) if for every pair of non-empty open subsets U, V of X, there is $t \in \mathbb{R}_+^d$ such that $W(t)(U) \cap V \neq \emptyset$.
We denote by $HC(W)$ the set of all hypercyclic vectors of W.

One can see that a d-parameter C_0-semigroup W on X is hypercyclic if and only if it is transitive. Also if $u_i(t) = W(te_i)$ is a hypercyclic one-parameter C_0-semigroup, for some $i = 1, 2, ..., d$, then W is hypercyclic. However, we will show that the converse is not true in general.

Remark 2.1. *Let X be a real-Banach space, $\widetilde{X}$ be the complexification of X, $\{T(t)\}_{t\geq 0}$ be a one-parameter C_0-semigroup on X and $\{\widetilde{T}(t)\}_{t\geq 0}$ be the complexification of $\{T(t)\}_{t\geq 0}$. If $\{\widetilde{T}(t)\}_{t\geq 0}$ is hypercyclic, then so is $\{T(t)\}_{t\geq 0}$ though its converse is not true in general. Suppose that W is a d-parameter C_0-semigroup on X and $\widetilde{W}$ is its complexification d-parameter C_0-semigroup on $\widetilde{X}$. One can see that hypercyclicity of $\widetilde{W}$ implies that W is also hypercyclic. Conversely, if W is d-parameter hypercyclic C_0-semigroup on X, then $(W_1, \mathbb{R}_+^{2d}, \widetilde{X})$ defined by*

$$W_1(s,t)(x+iy) = W(s)x + iW(t)y, \;\; s, t \in \mathbb{R}_+^d, \;\; x, y \in X$$

is a $2d$-parameter C_0-semigroup on $\widetilde{X}$.
In general case, if W_1 and W_2 are two hypercyclic d_1-parameter and d_2-parameter C_0-semigroup on Banach spaces X and Y, respectively, then $W(t_1,t_2) \in B(X \oplus Y)$, $t_1 \in \mathbb{R}_+^{d_1}$, $t_2 \in \mathbb{R}_+^{d_2}$, defined by $W(t_1,t_2)(x \oplus y) = W_1(t_1)x \oplus W_2(t_2)y$, is a $d_1 + d_2$-parameter hypercyclic C_0-semigroup on $X \oplus Y$. This together with the fact that $X \oplus Y \cong X \times Y$ implies that $W_1(t_1) \times W_2(t_2)$ is a tuple-hypercyclic on $X \times Y$.

As a consequence of this remark, one may construct a hypercyclic d-parameter C_0-semigroup on $L^p_{m_a}(\mathbb{R}_+, \mathbb{C}^d)$, the space of all functions $f = (f_1, ..., f_d) : \mathbb{R}_+ \to \mathbb{C}^d$ with $f_i \in L^p_{m_{a_i}}(\mathbb{R}_+)$, where $a = (a_1, ...a_d)$ and $m_{a_i}(x) = e^{a_i|x|}$, $i = 1, 2, .., d$. Trivially the equality $L^p_{m_a}(\mathbb{R}_+, \mathbb{C}^d) = \bigoplus_{i=1}^d L^p_{m_{a_i}}(\mathbb{R}_+)$ holds. Now it is well-known that the family $\{U_i(t)\}_{t\geq 0}$ defined on $L^p_{m_{a_i}}(\mathbb{R}_+)$ by $U_i(t)f(x) = f(x+t)$ is a hypercyclic one-parameter C_0-semigroup, $i = 1, ..., d$. Thus, $W(t_1, ..., t_d)$ defined on $L^p_{m_a}(\mathbb{R}_+, \mathbb{C}^d)$ by $W(t_1, ..., t_d)f(x) = (f_1(x + t_1), ..., f_d(x + t_d)))$ is a hypercyclic d-parameter C_0-semigroup.

As another important class of tuple C_0-semigroups, we may consider hypercyclicity of the d-parameter translation semigroups on $L^p_\nu(\mathbb{R}^d_+)$.
Let $1 \leq p < \infty$ and $\nu : \mathbb{R}^2_+ \to \mathbb{R}$ be a locally integrable function. We consider the space of 2-variable weighted p-integrable functions defined by

$$L^p_\nu(\mathbb{R}^2_+) := \{f : \mathbb{R}^2_+ \to \mathbb{C} : f \text{ is measurable and } \|f\| < \infty\},$$

where $\|f\| := (\int\int_{\mathbb{R}^2_+} |f(x,y)|^p \nu(x,y)dxdy)^{\frac{1}{p}}$. For any $f \in L^p_\nu(\mathbb{R}^2_+)$, define $W(s,t)f(x,y) = f(x+s, y+t)$. The proof of the following lemma and proposition is similar to the proofs described in Example 7.4 and Example 7.10 [13].

Lemma 2.1. *The family $\{W(s,t)\}_{s,t\geq 0}$ is a two-parameter C_0-semigroup on $L^p_\nu(\mathbb{R}^2_+)$ if and only if there exist $M > 0$, $w_1, w_2 \in \mathbb{R}$ such that*

$$\nu(x,y) \leq M e^{sw_1 + tw_2} \nu(x+s, y+t), \quad (s,t \geq 0). \tag{2.1}$$

In the following proposition, we assume that the weight ν satisfies (2.1) for any $x, y > 0$. Note that this is equivalent to

$$\nu(x,y) \leq M e^{w_1(u-s)+w_2(v-s)} \nu(u,v) \quad (u \geq x \geq 0, \quad v \geq y \geq 0).$$

Proposition 2.1. *For the translation semigroup on the space $X = L^p_\nu(\mathbb{R}^2_+)$, the following assertions are equivalent:*
(i) the translation semigroup is hypercyclic;
(ii) $\liminf_{\|(x,y)\|\to\infty} v(x,y) = 0$.

The following is a hypercyclic criterion for d-parameter C_0-semigroups.

Theorem 2.1 (Hypercyclic criterion)**.** *Let X be a separable Banach space and $(W, \mathbb{R}^d_+, X)$ be a two-parameter semigroup on X. Let $Y, Z \subseteq X$ be dense subsets of X and $(t_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^d_+$ where $t_k = (t^1_k, t^2_k, ..., t^d_k)$, and $S_s : Z \to E, s \geq 0$, be a family of linear mapping such that for some $j = 1, 2, ..., d$*

i) $\lim_{k\to\infty} W(t_k)y = 0$ *for all* $y \in Y$*;*
ii) $\lim_{k\to\infty} S_{t^j_k} z = 0$ *for all* $z \in Z$*;*
iii) $W(t)S_{t^j_k} z = z$ *for all* $z \in Z$, $t \in \mathbb{R}^d_+$ *and* $k \in \mathbb{N}$,

then $(W, \mathbb{R}^d_+, X)$ is transitive and in particular hypercyclic.

Proof. Let U, V be a pair of non-empty open subsets of X. Fix $y \in U \cap Y$ and $z \in V \cap Z$. By i) and iii), $\lim_{k\to\infty} W(t_k)y = 0$ and $W(t)S_{t^j_k} z = z$. Put $y_k = y + S_{t^j_k} z$. Trivially $\lim_{k\to\infty} W(t_k)y_k = z$. This implies that $W(t_k)(U) \cap V \neq \emptyset$ for sufficiently large $k \in \mathbb{N}$. ☐

3. Hypercyclic d-parameter C_0-semigroups on finite dimensional Banach spaces

It is well-known that there is no hypercyclic one-parameter C_0-semigroup on a finite dimensional Banach space. In this section, we show that there are many tuple-hypercyclic C_0-semigroups on $\mathbb{R}^n$ and $\mathbb{C}^n$. In a n-dimensional complex (or real) Banach space, a one-parameter C_0-semigroup is essentially uniformly continuous and so is of the form $\{e^{tA}\}_{t\geq 0}$, for some $A \in M_n(\mathbb{C})$ (respectively, $A \in M_n(\mathbb{R})$). Thus a d-parameter C_0-semigroup on this space is of the form

$$W(t_1, t_2, ..., t_d) = e^{t_1 A_1 + t_2 A_2 + ... + t_d A_d}$$

where $A_j \in M_n(\mathbb{R})$ and $A_i A_j = A_j A_i$, for $i, j = 1, ..., d$.

In the following proposition, we construct tuple-hypercyclic C_0-semigroups on $\mathbb{R}^n$, using the translation semigroups.

Proposition 3.1. *For any $n \in \mathbb{N}$, there exists a hypercyclic $2n$-parameter C_0-semigroup on $\mathbb{R}^n$.*

Proof. Let $\{e_i : i = 1, 2, ..., n\}$ be the standard basis for $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, define $u_i(t)x = x + te_i$ and $v_i(t)x = x - te_i$, $i = 1, ..., n$. Trivially $\{u_i(t)\}_{t\geq 0}$ and $\{u_i(t)\}_{t\geq 0}$, $i = 1, 2, ..., n$, are commuting C_0-semigroup of operators on $\mathbb{R}^n$ and so

$$W(t_1, t_2, ..., t_n, s_1, s_2, ..., s_n)x := \prod_{i=1}^{n} u_i(t_i) v_i(s_i)x, \quad (x \in \mathbb{R}^n)$$

is a $2n$-parameter C_0-semigroup. Also, one can see that in this case, $Orb(W, (1, 1, ..., 1)) = \mathbb{R}^n$. Thus W is a hypercyclic $2n$-parameter C_0-semigroup on $\mathbb{R}^n$ with the hypercyclic vector $(1, 1, ..., 1) \in \mathbb{R}^n$. □

As a consequence of Proposition 3.1, and Remark 2.1, $\mathbb{C}^n$ admits a $4n$-parameter hypercyclic C_0-semigroup.

In the following proposition, we construct a tuple-hypercyclic C_0-semigroup on $\mathbb{R}^n$ with less parameters using rotations and homogeneities.

Proposition 3.2. *Let $n \in \mathbb{N}$ be given. If n is even, then there exists a hypercyclic n-parameter C_0-semigroup, and if n is odd, then there exists a hypercyclic $n+1$-parameter C_0-semigroup on $\mathbb{R}^n$.*

Proof. First let n be even and $m = \frac{n}{2}$. For $k = 1, 2, ..., m$ and $s, t \geq 0$, define

$$W_k(s,t) := e^{2s} I_n \begin{pmatrix} 1 & 0 & 0 & \dots & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & \dots & 0 & 0 \\ \vdots & \dots & \ddots & & \dots & & & \vdots \\ 0 & \dots & 0 & \cos t & \sin t & 0 & \dots & 0 \\ 0 & \dots & 0 & -\sin t & \cos t & 0 & \dots & 0 \\ 0 & \dots & & 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \dots & \dots & & \ddots & \vdots \\ 0 & 0 & & \dots & \dots & 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

where I_n is the identity matrix on $\mathbb{R}^n$ and the matrix $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ begin from $2k-1^{th}$ columns and row. One can easily see that $W_k(s,t)$ is a two-parameter C_0-semigroup on $\mathbb{R}^n$. Now, define $W : \mathbb{R}_+^n \to M_n(\mathbb{R})$ as follows

$$W(s_1, t_1, s_2, t_2, ..., s_m, t_m) := \prod_{k=1}^{m} W_k(s_k, t_k). \tag{3.2}$$

It is not hard to see that W is a n-parameter C_0-semigroup on $\mathbb{R}^n$. Also, letting $v := (1,0,1,0,...,1,0)^T$, one can show that $Orb(w,v) = \mathbb{R}^n$. Thus, W is a hypercyclic n-parameter C_0-semigroup on $\mathbb{R}^n$.
For an odd number $n = 2m+1$, let W_k be defined as (3.1), $k = 1,2,...,m$. Now define $W_{m+1} : \mathbb{R}^2_+ \to M_n(\mathbb{R})$ by

$$W_{m+1}(s,t)x := x + (s-t)e_n.$$

If we define $W : \mathbb{R}^{n+1}_+ \to M_n(\mathbb{R})$ by

$$W(s_1,t_1,s_2,t_2,...,s_m,t_m,s_{m+1},t_{m+1}) := \prod_{k=1}^{m+1} W_k(s_k,t_k), \tag{3.3}$$

then W is a hypercyclic $(n+1)$-parameter C_0-semigroup on $\mathbb{R}^n$. □

Applying Proposition 3.2 and Remark 2.1, if n is even (respectively, odd), then there exists a $2n$-parameter (respectively, $2n+2$-parameter) C_0-semigroup on $\mathbb{C}^n$.

Remark 3.1. *(1) It is well-known that if $\{T_t\}_{t\geq 0}$ is a hypercyclic C_0-Semigroup of operators then each T_t $(t>0)$ is hypercyclic as a single operator (see [5]). This is not true for the d-parameter case. Indeed in Section 3, it will be proved that finite dimensional Banach spaces admit a tuple-hypercyclic C_0-semigroup although there is no hypercyclic operator on these spaces.*

(2) As another consequence of existences of a tuple-hypercyclic C_0-semigroup on finite dimensional Banach spaces, it can be concluded that if W is a tuple-hypercyclic C_0-semigroup on a finite dimensional Banach space X, then $W^*(t)$, $t \in \mathbb{R}^d_+$ could has an eigenvalue.

4. Tensor product of tuple-hypercyclic C_0-semigroups

Recall that for Banach spaces X and Y, we denote their (algebraic) tensor product by $X \otimes Y$. Furthermore, let α be a tensor norm (or uniform cross-norm) on $X \otimes Y$. Then α is, in particular, a reasonable cross-norm on $X \otimes Y$, which implies that for any $x \in X$ and $y \in Y$ we have

$$\alpha(x \otimes y) = \|x\|_X \cdot \|y\|_Y.$$

It is well known that

$$\pi(z) = \inf\Big\{\sum_{i=1}^{n} \|x_i\|_X \cdot \|y_i\|_Y : z = \sum_{i=1}^{n} x_i \otimes y_i\Big\}, \qquad (z \in X \otimes Y)$$

defines a tensor norm on $X \otimes Y$, which is called the projective tensor norm. Actually, this norm is the greatest reasonable cross-norm on $X \otimes Y$. For any norm α on $X \otimes Y$, we denote by $X \tilde{\otimes}_\alpha Y$ the completion of the normed space $(X \otimes Y, \alpha)$.

For bounded operators $T : X \to X$, $S : Y \to Y$ and any uniform cross-norm α, the product $T \otimes S$ is a bounded operator on $(X \otimes Y, \alpha)$ by definition of uniform cross-norm. The unique extension of $T \otimes S$ to $X \tilde{\otimes}_\alpha Y$ is for simplicity, also denoted by $T \otimes S$.

Let $\{T(t)\}_{t\geq 0}$ and $\{S(s)\}_{s\geq 0}$ be two one-parameter C_0-semigroups on Banach spaces X and Y, respectively. One can prove that the tensor product $T(t) \otimes S(s)$ is a two-parameter C_0-semigroup on $(X \otimes Y, \alpha)$ for any uniform cross-norm α. The following theorem shows that the tensor product of two hypercyclic one-parameter C_0-semigroups is a tuple-hypercyclic C_0-semigroup.

Theorem 4.1. *Let $\{T(t)\}_{t\geq 0}$ and $\{S(s)\}_{s\geq 0}$ be two hypercyclic one-parameter C_0-semigroups on Banach spaces X and Y, respectively. Then the two-parameter C_0-semigroup $T(t) \otimes S(s)$ is tuple-hypercyclic on $X \tilde{\otimes}_\alpha Y$ for any reasonable cross norm α.*

Proof. Let α be a reasonable cross norm. Consider the norm $\|(x,y)\| := \sup\{\|x\|_X, \|y\|_Y\})$ on $X \times Y$. The canonical bilinear map

$$\psi : X \times Y, \longrightarrow (X \otimes Y, \alpha)$$
$$(x,y) \mapsto x \otimes y$$

is continuous and $\|\psi\| \leq 1$. So for any $n \geq 1$, the mapping

$$\psi_n : X^n \times Y^n \longrightarrow X \otimes Y$$
$$(x_1, \cdots, x_n, y_1, \cdots y_n) \mapsto \sum_{k=1}^{n} \psi(x_k, y_k)$$

is continuous with the norm

$$\|(x_1, \cdots, x_n, y_1, \cdots y_n)\| := \max\{\|x_1\|_X, \|y_1\|_Y, ..., \|x_n\|_X, \|y_n\|_Y\}$$

on $X^n \times Y^n$. We shall show that $T(t) \otimes S(s)$ is tuple-transitive. Let U and V be non-empty open subsets of $X \tilde{\otimes}_\alpha Y$. As $X \otimes Y = span(\psi(X \times Y))$ is dense in $X \tilde{\otimes}_\alpha Y$, we may find elements $u = \sum_{k=1}^{n} x_k \otimes y_k \in U$ and $v = \sum_{k=1}^{n} p_k \otimes q_k \in V$. Without loss of generality, we may assume $m = n$ (by extending one of the sums by zero elements if necessary). Continuity of ψ_n implies that $\psi_n^{-1}(U)$ and $\psi_n^{-1}(V)$ are non-empty open subsets of $X^n \times Y^n$. From hypercyclicity of $\{T(t)\}_{t\geq 0}$ and $\{S(s)\}_{s\geq 0}$, we deduce that the diagonal semigroups $T^n(t) = T(t) \times ... \times T(t)$ and $S^n(s) = S(s) \times ... \times S(s)$ are hypercyclic on X^n and Y^n, respectively (see Corollary1.3, [22]). By Remark 2.1, this implies that $T(t)^n \times S(s)^n$ is hypercyclic. So there exist $t, s > 0$ such that

$$T^n(t) \times S^n(s)(\psi_n^{-1}(U)) \cap \psi_n^{-1}(V) \neq \varnothing.$$

But

$$\psi_n(T^n(t) \times S^n(s)(\psi_n^{-1}(U)) \cap \psi_n^{-1}(V)) \subseteq T(t) \otimes S(s)(U) \cap V.$$

Therefore $\{T(t) \otimes S(s)\}_{s,t\geq 0}$ is tuple-transitive two-parameter C_0-semigroup. □

5. J-class two-parameter C_0-semigroup

In this section, we study the locally topologically transitive two-parameter C_0-semigroups (or simply J-class two-parameter C_0-semigroups).

Definition 5.1. *A two-parameter C_0-semigroup $\{W(s,t)\}_{s,t\geq 0}$ on a normed space X is called J-class if there exists $0 \neq x \in X$, such that $J_W(x) = X$, where*

$$\begin{aligned} J_W(x) : \;=\; & \{y \in X : \text{ there exist strictly increasing sequences} \\ & (s_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}} \subseteq [0,\infty) \text{ with } s_n, t_n \to \infty \text{ and a sequence} \\ & (x_n)_{n\in\mathbb{N}} \text{ in } X \text{ such that } x_n \to x \text{ and } W(s_n,t_n)(x_n) \to y\}. \end{aligned}$$

We put $A_W := \{x \in X : J_W(x) = X\}$.

The following is a topological characterization of J-class two-parameter C_0-semigroups. Our proof is a modification of the proof of a similar result in one-parameter case in [18].

Theorem 5.1. *A Two-parameter C_0-semigroup $\{W(s,t)\}_{s,t\geq 0}$ on a Banach space X is tuple J-class C_0-semigroup with a non-zero vector x if and only if for every neighborhood U of x and open set $V \subset X$ there exist $s, t > 0$ such that $W(s,t)U \cap V \neq \phi$.*

Proof. Assume that for every neighborhoods U of x and open set $V \subset X$ there exist $s, t > 0$ such that $W(s,t)U \cap V \neq \phi$. We show that $J_W(x) = X$. Let $y \in X$ be given and assume first that $y \notin \{W(s,t)x : s, t \geq 0\}$. For each $n \in \mathbb{N}$, there exists $s_n, t_n \geq 0$ with $W(s_n,t_n)(B_{\frac{1}{n}}(x)) \cap B_{\frac{1}{n}}(y) \neq \phi$, which in particular means that there exists $x_n \in X$ with $||x_n - x|| < \frac{1}{n}$ and $||W(s_n,t_n)x_n - y|| < \frac{1}{n}$. This implies that $\lim_{n\to\infty} W(s_n,t_n)x_n = y$. We are going to show that $(t_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ have a strictly increasing subsequences tending to infinity. Suppose in contrary that $(t_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ are bounded. Then there are subsequences of $(t_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ converging to some t and s respectively. Without lose of generality, we assume that $t_n \to t$ and $s_n \to t$. It follows from the continuity of the map $g : \mathbb{R}_+^2 \times X \to X, \;\; g(s,t,x) := W(s,t)x$, that $\lim_{n\to\infty} W(s_n,t_n)x_n = W(s,t)x = y$, which

is impossible, since $y \notin \{W(s,t) : s,t \geq 0\}$.
Now let $y \in \{W(s,t)x : s,t \geq 0\}$. Continuity of g implies that, the image of g under t,s for fixed $x \in X$ is a countable union of compact sets, in particular $g_x(\mathbb{R}^2_+) = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} g_x([0,n] \times [0,m])$ and therefore is of the first category. Hence

$$B_{\frac{1}{n}}(y) \setminus \{W(s,t)x : s,t \geq 0\} \neq \emptyset$$

holds for each $n \in \mathbb{N}$. Letting $y_n \in B_{\frac{1}{n}}(y) \setminus \{W(s,t)x : s,t \geq 0\}$ and using the first part of the proof, there exists $s_n, t_n > n$, such that $||W(t_n, s_n)x_n - y_n|| < \frac{1}{n}$. Now we choose $y_{n+1} \in B_{\frac{1}{n+1}}(y) \setminus \{W(s,t)x : s,t \geq 0\}$. Then again we choose $t_{n+1} > \max\{t_n, n+1\}$, $s_{n+1} > \max\{s_n, n+1\}$ and x_{n+1} with $||x_{n+1} - x|| < \frac{1}{n+1}$, such that

$$||W(s_{n+1}, t_{n+1})x_{n+1} - y_{n+1}|| < \frac{1}{n+1}.$$

It follows that $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are strictly increasing sequences of positive real numbers tending to infinity and

$$\begin{aligned} ||W(s_n,t_n)x_n - y|| &\leq ||W(s_n,t_n)x_n - y_n|| + ||y_n - y|| \\ &\leq \frac{1}{n} + \frac{1}{n} \\ &= \frac{2}{n} \end{aligned}$$

This implies $\lim_{n\to\infty} W(s_n,t_n)x_n = y$ and it shows the first direction. The other direction is obvious. □

Trivially transitivity of $\{W(s,t)\}_{s,t\geq 0}$ implies its locally topologically transitivity for all $x \in X$. Hence for any $x \in HC(W)$, $J_W(x) = X$. As another consequence of Theorem 5.1, we get the following proposition.

Proposition 5.1. *Let $\{T(t)\}_{t\geq 0}$ and $\{S(s)\}_{s\geq 0}$ be two J-class C_0-semigroups on Banach spaces X and Y, respectively. Then the two-parameter C_0-semigroup $\{T(t) \times S(s)\}_{s,t\geq 0}$ is tuple-J-class on $X \times Y$.*

Proof. Suppose that $J_T(x) = X$ and $J_S(y) = Y$ for some $0 \neq x \in X$ and $0 \neq y \in Y$. We show that $J_{T\times S}(x,y) = X \times Y$.
Let U be a non-empty open subset of $x \times y$ and V be an arbitrary non-empty open subset of $X \times Y$. So there exists non-empty open subsets $U_1, U_2 \subseteq X$ and $V_1, V_2 \subseteq Y$, such that $(x,y) \in U_1 \times V_1 \subseteq U$ and $U_2 \times V_2 \subseteq V$.
The semigroups $\{T(t)\}_{t\geq 0}$ and $\{S(s)\}_{s\geq 0}$ are J-class so there exists $t,s > 0$, such that $T(t)U_1 \cap U_2 \neq \emptyset$ and $S(s)V_1 \cap V_2 \neq \emptyset$. But

$$T(t)U_1 \cap U_2 \times S(s)V_1 \cap V_2 \subset T(t) \times S(s)U \cap V,$$

which completes the proof. □

Proposition 5.2. *Let X be a Banach space and $\{W(s,t)\}_{s,t\geq 0}$ be a two-parameter C_0-semigroup. Suppose that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are two convergent sequences to x and y, respectively, with the property that $y_n \in J_W(x_n)$. Then $y \in J_W(x)$. In particular, $J_W(x)$ and A_W are closed subsets of X.*

Proof. Take y_{n_1} with $n_1 \geq 1$ large enough, such that $||y_{n_1} - y|| < 1$. Since $y_{n_1} \in J_W(x_{n_1})$, choose z_1 with $||z_1 - x_{n_1}|| < 1$ and $s_1, t_1 \geq 1$ such that $||W(s_1,t_1)z_1 - y_{n_1}|| < 1$. Next choose y_{n_2} with $n_2 > n_1$ and $||y_{n_2} - y|| < \frac{1}{2}$ and as above, since $y_{n_2} \in J_W(x_{n_2})$, choose z_2, $t_2 \geq \max\{2, t_1\}$ and $s_2 \geq \max\{2, s_1\}$ with $||z_2 - x_{n_2}|| < \frac{1}{2}$ and $||W(s_2,t_2)z_2 - y_{n_2}|| < \frac{1}{2}$.

Inductively, we get sequences $(n_m)_{m\in\mathbb{N}}$, $(t_m)_{m\in\mathbb{N}}$, $(s_m)_{m\in\mathbb{N}}$ and $(z_m)_{m\in\mathbb{N}}$ with the property that $t_m \to \infty$, $s_m \to \infty$ and

$$||z_m - x_{n_m}|| < \frac{1}{m} \qquad ||y - y_{n_m}|| < \frac{1}{m} \qquad ||W(s_m, t_m)z_m - y_{n_m}|| < \frac{1}{m}$$

hold for each $m \in \mathbb{N}$. Now, $||W(t_m, s_m)z_m - y|| \leq ||W(t_m, s_m)z_m - y_{n_m}|| + ||y_{n_m} - y|| \leq \frac{2}{m}$ and therefore $\lim_{m\to\infty} z_m = x$ and $\lim_{m\to\infty} W(t_m, s_m)z_m = y$. So we proved $y \in J_W(x)$. □

Theorem 5.2. *A two-parameter C_0-semigroup $\{W(s,t)\}_{s,t\geq 0}$ on a Banach space X is tuple-hypercyclic if and only if $J_W(x) = X$ for every $x \in X$.*

Proof. Let $\{W(s,t)\}_{s,t\geq 0}$ be tuple-hypercyclic. Trivially $HC(W) \subset A_W$. Since A_W is closed subset of X and $HC(W)$ is dense in X, we get $A_W = X$. Conversely, suppose that $J_W(x) = X$ for all $x \in X$. It is enough to show that W is tuple-transitive. Let U and V be non empty open sets in X and $x \in U$ be chosen. From $J_W(x) = X$ there exists $(t,s) \in \mathbb{R}_+^2$ such that $W(t,s)U \cap V \neq \phi$, which implies that W is tuple transitive. □

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