PHASE RETRIEVAL (DUAL) FRAMES: A NEW APPROACH

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The phase retrieval problem involves recovering a signal from the magnitude of its measurements, without knowledge of its phase. Motivated by recent investigations in the field of phase retrieval and frame theory in Hilbert spaces, we provide a characterization of the problem of phase retrieval frames in Euclidean space $\mathbb{R}^m$. The study focuses on the case where the signal and the measurements are both real-valued providing a comprehensive overview of phase retrieval frames. We derive specific forms of phase retrieval. Additionally, the general form of dual phase retrieval frames in $\mathbb{R}^m$ is analyzed and a complete description of phase retrieval dual frames is given. Furthermore, we confirm the efficiency of reconstruction using phase retrieval frames by performing numerical examples.

Keywords: Phase retrieval frames, dual frames, complement property.

MSC2020: Primary 42C15; Secondary 42C40.

1. Introduction

Frames are redundant sequence of vectors within a separable Hilbert space that provide many representations for every vector. This redundancy is what makes frames useful for applications. In addition, frames play a significant role not only in theory, but also in many type of applications such as noise reduction and suppression, signal processing [10], coding and communications [12], sampling [8, 9], time-frequency analysis, voice recognition, bio-imaging, system modelling [13] and so on. Actually, frame theory has been shown in practice to be an effective field of research with applications.

Phase retrieval is the problem of recovering a signal f in a Hilbert space $\mathcal{H}$, from a set of intensity measurements absolute of the frame coefficients. The concept of phase retrieval sequences in finite dimensional Hilbert spaces was first presented in [11] and then it is reformulated in terms of frame theory by Balan, Casazza and Edidin [4] in 2006.

With this background and view of phase retrieval, the motivation behind this work is to perform the reconstruction of each signal using the absolute value of the frame coefficients, essentially providing a global identification of phase retrieval frames in finite dimensional real Hilbert space based on their components. The concept of phase retrieval frames is defined by using frame vectors and so could not be formulated as an operator form. In view of this, the identifying of phase retrieval frames depends on the dimension of the underlying Hilbert space and the number of frame elements, and it is not possible to provide a unique formula for them. We will provide some concrete classifications for phase retrieval frames and their duals. The paper also compares the reconstruction of a random signal using the classical reconstruction formula with the phase retrieval reconstruction. This

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comparison could potentially provide insight into the efficiency and performance of different signal reconstruction methods.

The structure of this paper is arranged in the following manner. In Section 2, we will collect some necessary background from the basic concepts about phase retrieval frames in Hilbert spaces. Section 3 is devoted some characterizations of phase retrieval frames $\Phi = \{\varphi_i\}_{i=1}^n$ in $\mathbb{R}^m$ when $n = 2m-1$ based on the components of the frame elements. Furthermore, we discover useful and precise ways to figure out the phase retrieval frames in $\mathbb{R}^m$. Section 4, characterizes the phase retrieval of dual frames and we describe a detailed view on how to perform the phase retrieval of dual frames in $\mathbb{R}^m$. Section 5, presents some results about phase retrieval frames in $\mathbb{R}^m$ when $n > 2m-1$, also gives several examples to confirm our results. The last section provides, we analyzed a comparison of the reconstruction of a random signal by using the classical frame reconstruction formula with the phase retrieval reconstruction.

Throughout this paper, we assume that $\mathcal{H}^m$ is a m-dimensional real Hilbert space, $\mathcal{H}$ is a separable Hilbert space and I a countable index set. For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ we denote the collection of all bounded linear operators between $\mathcal{H}_1$ and $\mathcal{H}_2$ by $B(\mathcal{H}_1, \mathcal{H}_2)$ and we apply $B(\mathcal{H})$ for $B(\mathcal{H}, \mathcal{H})$. Also, we denote the range and kernel of $T \in B(\mathcal{H})$ by $\mathcal{R}(T)$, $\mathcal{N}(T)$, respectively, and $\{e_i\}_{i=1}^m$ denotes the standard orthonormal basis in $\mathbb{R}^m$.

2. **Preliminaries**

In this section, we present some concepts which will be used in the next sections.

2.1. **Frame theory**

A sequence $\Phi = \{\varphi_i\}_{i\in I}$ in $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $0 < A_\Phi \le B_\Phi$ such that

$$A_\Phi \|f\|^2 \le \sum_{i\in I} |\langle f, \varphi_i\rangle|^2 \le B_\Phi \|f\|^2, \qquad (f \in \mathcal{H}).$$

It is *tight* if $A_\Phi = B_\Phi$, and it is a *Bessel sequence* if at least the upper frame condition holds. If $\{\varphi_i\}_{i\in I}$ is a Bessel sequence, the *synthesis operator* is the operator $T_\Phi : \ell^2(I) \longrightarrow \mathcal{H}$ defined by $T_\Phi\{c_i\}_{i\in I} := \sum_{i\in I} c_i\varphi_i$. It is well known that T_Φ is well-defined and bounded. Its adjoint $T_\Phi^* : \mathcal{H} \longrightarrow \ell^2(I)$ of T_Φ which is called the *analysis operator*, is given by $T_\Phi^* f = \{\langle f, \varphi_i\rangle\}_{i\in I}$. Finally, the *frame operator* is defined by

$$S_\Phi : \mathcal{H} \longrightarrow \mathcal{H}, \qquad S_\Phi f := T_\Phi T_\Phi^* f = \sum_{i\in I} \langle f, \varphi_i\rangle \varphi_i.$$

It is bounded, bijective as well as self-adjoint; these properties immediately lead to the important frame decomposition

$$f = S_\Phi S_\Phi^{-1} f = \sum_{i\in I} \langle f, S_\Phi^{-1}\varphi_i\rangle \varphi_i, \qquad (f \in \mathcal{H}). \tag{1}$$

A *Riesz basis* for $\mathcal{H}$ is a sequence of the form $\{Ue_i\}_{i\in I}$ where $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded bijective operator. Every Riesz basis has a unique bi-orthogonal sequence which is also a Riesz basis [6].

If $\Phi = \{\varphi_i\}_{i\in I}$ is a frame on $\mathcal{H}$ with the frame operator S_Φ, the sequence $\{S_\Phi^{-1}\varphi_i\}_{i\in I}$ is also a frame which is called the *canonical dual frame*. Every Bessel sequence $\{\psi_i\}_{i\in I}$ satisfying

$$f = \sum_{i\in I} \langle f, \psi_i\rangle \varphi_i, \qquad (f \in \mathcal{H}), \tag{2}$$

is called a *dual frame* of $\{\varphi_i\}_{i\in I}$.

Theorem 2.1. [1] *Let $\Phi = \{\varphi_i\}_{i\in I}$ be a frame. Then every dual frame of Φ is of the form of $\Phi^d = \{S_\Phi^{-1}\varphi_i + u_i\}_{i\in I}$ where $\{u_i\}_{i\in I}$ is a Bessel sequence such that*

$$\sum_{i\in I}\langle f, \varphi_i\rangle u_i = 0, \qquad (f \in \mathcal{H}). \tag{3}$$

The *excess* of Φ, which is denoted by $E(\Phi)$, represents the maximum quantity n of elements Φ that can be removed from Φ while still preserving a frame. When two frames are dual to each other, they possess the same excess [3]. A frame Φ with the excess can be expressed as $\{\varphi_i\}_{i\in I\setminus\{i_1,\dots,i_n\}} \cup \{\varphi_{i_1}, \dots, \varphi_{i_n}\}$, where $\Phi = \{\varphi_i\}_{i\in I\setminus\{i_1,\dots,i_n\}}$ is a Riesz basis for $\mathcal{H}$ and $\{\varphi_{i_1}, \dots, \varphi_{i_n}\}$ are redundant elements of Φ.

2.2. Phase retrieval frames

A sequence $\Phi = \{\varphi_i\}_{i=1}^n \in \mathcal{H}^m$ is called *phase retrieval* if for any $x, y \in \mathcal{H}^m$ with

$$|\langle x, \varphi_i\rangle| = |\langle y, \varphi_i\rangle|, \qquad (i = 1, 2, \dots, n),$$

there exists $\theta \in \mathbb{R}$ such that $|\theta| = 1$ and $x = \theta y$, that is $x = \pm y$ [4]. The process of phase retrieval in $\mathbb{R}^m$ is characterized by a principal result, known as the *complement property*. This indicates that for all subsets $I \subset \{1, 2, .., n\}$ either $span\{\varphi_i\}_{i\in I} = \mathcal{H}^m$ or $span\{\varphi_i\}_{i\in I^c} = \mathcal{H}^m$. Given a family $\Phi = \{\varphi_i\}_{i=1}^n$ of vectors in $\mathcal{H}^m$, the *spark* of Φ is defined as the cardinality of the smallest linearly dependent subset of Φ. When $spark(\Phi) = m + 1$, every subset of size m is linearly independent and in that case, Φ is said to be *full spark*.

Theorem 2.2. [2] *A frame $\{\varphi_i\}_{i=1}^n$ in $\mathbb{R}^m$ yields phase retrieval if and only if it has the complement property. In particular, a full spark frame with $2m - 1$ vectors yields phase retrieval. Moreover, if $\{\varphi_i\}_{i=1}^n$ yields phase retrieval in $\mathbb{R}^m$, then $n \geq 2m - 1$ and no set of $2m - 2$ vectors yields phase retrieval.*

2.3. Signal reconstruction without phase

Allow us to denote by $\mathbb{H}^m = \mathcal{H}^m/\sim$ considered by recognizing two vectors which are divers in a phase factor, i.e., $x \sim y$ whenever there exists a scaler θ with $|\theta| = 1$ so that $y = \theta x$. Obviously in a real Hilbert space we have $\mathbb{H}^m = \mathcal{H}^m/\{1, -1\}$. The mapping

$$\alpha_\Phi : \mathbb{H}^m \longrightarrow \mathbb{R}^n, \qquad \alpha_\Phi[x] = \{|\langle x, \varphi_i\rangle|\}_{i=1}^n \tag{4}$$

can be defined on $\mathbb{H}^m$ where $[x] = \{y \in \mathcal{H}^m : y \sim x\}$. The injectivity of the non-linear mapping α_Φ leads to the phase retrieval property of Φ and vice versa. This means that we can reconstruct any signal in $\mathcal{H}^m$ by using the modulus of its frame coefficients.

3. Phase retrieval frames $\Phi = \{\varphi_i\}_{i=1}^n$ in $\mathbb{R}^m$ when $n = 2m - 1$

Let $\Phi = \{\varphi_i\}_{i=1}^n$ be a frame in $\mathbb{R}^m$. In [4], it is shown that $2m-1$ vectors are sufficient for Φ dose phase retrieval. In this section, we introduce a characterization of phase retrieval frames $\Phi = \{\varphi_i\}_{i=1}^{2m-1}$ based on components of frame elements. Obviously, phase retrieval property is preserved under invertible operators. On the other hand, every frame with the finite excess contains a Riesz basis [3]. Accordingly, without loss of the generality we can consider Φ as $\{e_i\}_{i=1}^m \cup \{\varphi_i\}_{i=1}^{m-1}$ where $\{e_i\}_{i=1}^m$ is an orthonormal basis of $\mathbb{R}^m$ and we refer to $\{\varphi_i\}_{i=1}^{m-1}$ as the redundancy elements. We know that if a frame contains exactly $2m - 1$ vectors, then it does phase retrieval if and only if it is full spark [2], therefore in this section, we indeed identify all the full spark frames in $\mathbb{R}^m$.

We begin with the following lemma to describe 1-excess phase retrieval frames in $\mathbb{R}^2$.

Lemma 3.1. *Let $\Phi = \{\varphi_i\}_{i=1}^3$ be a frame such that $\{\varphi_1, \varphi_2\}$ is a Riesz basis for $\mathbb{R}^2$ and $\varphi_i = (x_i, y_i)$, $1 \leqslant i \leqslant 3$. Then Φ is phase retrieval if and only if*

$$(y_2x_3 - x_2y_3)(x_1y_3 - y_1x_3) \neq 0.$$

Proof. The matrix
$$A = \frac{1}{x_1y_2 - y_1x_2}\begin{bmatrix} y_2 & -x_2 \\ -y_1 & x_1 \end{bmatrix}$$
is well-defined and invertible since $\{\varphi_1, \varphi_2\}$ is linearly independent. In fact, A is the change of basis matrix from $\{\varphi_1, \varphi_2\}$ to $\{e_1, e_2\}$. On the other hand, phase retrieval property is preserved under invertible operators, hence Φ is phase retrieval if and only if $A\Phi = \{e_1, e_2.A\varphi_3\}$ is phase retrieval. Now, Theorem 2.2 easily follows that $A\Phi$ is phase retrieval if and only if both components of $A\varphi_3 = (y_2x_3 - x_2y_3, x_1y_3 - y_1x_3)$ are non-zero. □

As a consequence of Lemma 3.1,
$$\left\{\left\{Ae_1, Ae_2, \alpha Ae_1 + \beta Ae_2\right\} : \alpha\beta \neq 0 \qquad A \textit{ is a } 2\times 2 \textit{ invertible matrix}\right\},$$
is the set of all phase retrieval frames in $\mathbb{R}^2$. Similarly a concrete characterization of phase retrieval frames in $\mathbb{R}^3$ is given in the following.

Proposition 3.1. *Let $\Phi = \{\varphi_i\}_{i=1}^5$ be a sequence in $\mathbb{R}^3$ such that $\beta = \{\varphi_i\}_{i=1}^3$ is a Riesz basis. Then Φ does phase retrieval if and only if all components of $[\varphi_4]_\beta, [\varphi_5]_\beta$ and $[\varphi_4]_\beta \times [\varphi_5]_\beta$ are non-zero, where $[\varphi]_\beta$ is the coordinate vector of φ with respect to the basis β.*

Proof. The set β is a Riesz basis, so there exists an invertible operator U on $\mathbb{R}^3$ such that $U\varphi_i = e_i$, $i = 1, 2, 3$. Using the fact that the phase retrieval property is preserved under invertible operators follows that Φ does phase retrieval if and only if $\Psi := \{e_1, e_2, e_3, U\varphi_4, U\varphi_5\}$ does phase retrieval. Note that if $[\varphi_4]_\beta = (\alpha, \beta, \gamma)$ and $U\beta = \{e_1, e_2, e_3\}$ then $[U\varphi_4]_{U\beta} = (\alpha, \beta, \gamma)$. Now if Ψ does phase retrieval, then

(i) $\text{span}\{e_1, e_2, U\varphi_4\} = \mathbb{R}^3$,
(ii) $\text{span}\{e_1, U\varphi_4, U\varphi_5\} = \mathbb{R}^3$.

Theorem 2.2, (i) easily follows that the first component of $U\varphi_4$ is non-zero. Moreover, (ii) implies that $U\varphi_4$ and $U\varphi_5$ are linearly independent and the first component of $U\varphi_4 \times U\varphi_5$ is non-zero, otherwise $U\varphi_4 \times U\varphi_5 \in \textit{span}\{e_1, U\varphi_4, U\varphi_5\}^\perp = \{0\}$, which is a contradiction. The proof for other components are similar.

Conversely, let all components of $[\varphi_4]_\beta, [\varphi_5]_\beta$ and $[\varphi_4]_\beta \times [\varphi_5]_\beta$ are non-zero. It is enough to show that Φ has the complement property. For this, it is enough to prove that the set $\{\varphi_4, \varphi_5, e_i\}$ is linearly independent. To this end, assume that
$$c_1\varphi_4 + c_2\varphi_5 + c_3e_i = 0, \tag{5}$$
for some $c_i \in \mathbb{R}$, $i = 1, 2, 3$. Then $\langle \varphi_4 \times \varphi_5, c_1\varphi_4 + c_2\varphi_5 + c_3e_i\rangle = c_3\langle \varphi_4 \times \varphi_5, e_i\rangle = 0$. The fact that all components of $[\varphi_4]_\beta \times [\varphi_5]_\beta$ are non-zero, implies that $c_3 = 0$. Moreover, φ_4 and φ_5 are linearly independent and so we have $c_1 = c_2 = 0$. Other cases to obtain the complement property is easily follows from the fact that all components of $[\varphi_4]_\beta, [\varphi_5]_\beta$ are non-zero. □

Corollary 3.1. *Let $\Phi = \{e_i\}_{i=1}^3 \cup \{\varphi_i\}_{i=1}^2$ be a frame in $\mathbb{R}^3$. Then Φ does phase retrieval if and only if all components of φ_1, φ_2 and $\varphi_1 \times \varphi_2$ are non-zero.*

Proposition 3.1, immediately leads to the following.

Corollary 3.2. *Let $a, b, c \in \mathbb{R} \setminus \{0\}$. Then $\{e_i\}_{i=1}^3 \cup \{(a, b, c), (x, y, z)\}$ for all $(x, y, z) \in \mathbb{R}^3$ does phase retrieval except on the axes and the planes $x = \frac{a}{b}y, x = \frac{a}{c}z$ and $y = \frac{b}{c}z$.*

We now continue to describe the general form of phase retrieval frames in $\mathbb{R}^4$.

Theorem 3.1. *A frame* $\Phi = \{e_i\}_{i=1}^4 \cup \{\varphi_i\}_{i=1}^3$ *in* $\mathbb{R}^4$ *does phase retrieval if and only if the following conditions are satisfied*

(i) $\langle \varphi_i, e_j\rangle \neq 0$ *for all* $i = 1, 2, 3$ *and* $j = 1, 2, 3, 4$.

(ii) $\{\varphi_i\}_{i=1}^3$ *is linearly independent and* $\langle \psi, e_j\rangle \neq 0$, *for all* $j = 1, 2, 3, 4$ *where* $\psi \in (\text{span}\{\varphi_i\}_{i=1}^3)^\perp$.

(iii) $\dfrac{\langle e_j, \psi_1^\sigma\rangle}{\langle e_l, \psi_1^\sigma\rangle} \neq \dfrac{\langle e_j, \psi_2^\sigma\rangle}{\langle e_l, \psi_2^\sigma\rangle}$, *for* $j, l \in \{1, 2, 3, 4\}$ *when* $\text{span}\{\varphi_i\}_{i\in\sigma}^\perp = \text{span}\{\psi_1^\sigma, \psi_2^\sigma\}$ *for every* $\sigma \subseteq \{1, 2, 3\}$ *with* $|\sigma| = 2$.

Proof. Suppose that Φ is a phase retrieval frame, then $\text{span}\{e_i\}_{i\in\sigma} \cup \{\varphi_i\}_{i=1}^3 = \mathbb{R}^4$ for $\sigma \subseteq \{1, 2, 3, 4\}$ with $|\sigma| = 3$, by using the complement property. Thus we conclude that $\langle \varphi_i, e_j\rangle \neq 0$ for some $1 \le i \le 3, 1 \le j \le 4$. Also, $\text{span}\{\varphi_1, \varphi_2, \varphi_3, e_i\} = \mathbb{R}^4$. This shows that $\{\varphi_1, \varphi_2, \varphi_3\}$ is linearly independent. Furthermore, if $\psi \in (\text{span}\{\varphi_i\}_{i=1}^3)^\perp$ and $\langle \psi, e_j\rangle = 0$ for some $1 \le j \le n$, then we get $\psi \in \text{span}\{e_j, \varphi_1, \varphi_2, \varphi_3\}^\perp = \{0\}$, which is a contradiction, hence $\langle \psi, e_j\rangle \neq 0$ for all $1 \le j \le 4$. Finally, to obtain (iii), without loss of the generality, assume that $\sigma = \{1, 2\}$ and choose two vectors $\psi_1^\sigma, \psi_2^\sigma$ orthogonal to φ_1 and φ_2. Now, if $\begin{vmatrix} \langle e_j, \psi_1^\sigma\rangle & \langle e_j, \psi_2^\sigma\rangle \\ \langle e_l, \psi_1^\sigma\rangle & \langle e_l, \psi_2^\sigma\rangle \end{vmatrix} = 0$, then the linear system

$$\begin{cases} \langle e_j, \psi_1^\sigma\rangle\alpha_1 + \langle e_j, \psi_2^\sigma\rangle\alpha_2 = 0, \\ \langle e_l, \psi_1^\sigma\rangle\alpha_1 + \langle e_l, \psi_2^\sigma\rangle\alpha_2 = 0, \end{cases}$$

has a non-trivial solution $(\alpha_1, \alpha_2) \neq (0, 0)$. Put $\xi = \alpha_1 e_j + \alpha_2 e_l$ and assume that $m, n \notin \{j, l\}$. Trivially $\xi \neq 0$ and $\xi \in \{e_m, e_n, \psi_1^\sigma, \psi_2^\sigma\}^\perp$ which is a contradiction. Conversely, let conditions (i), (ii) and (iii) be satisfied, it is enough to show that Φ has the complement property. Obviously by (i), we have $\text{span}\{e_1, e_2, e_3, \varphi_i\} = \mathbb{R}^4$ for $i = 1, 2, 3$. Furthermore, we have to show that $\{e_1, e_2, \varphi_3, \varphi_4\}$ is linearly independent. Indeed, if

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3\varphi_3 + \alpha_4\varphi_4 = 0, \tag{6}$$

for real scalars $\alpha_i, 1 \le i \le 4$ and $\text{span}\{\psi_1^\sigma, \psi_2^\sigma\} = \text{span}\{\varphi_3, \varphi_4\}^\perp$, then

$$\langle e_1, \psi_1^\sigma\rangle\alpha_1 + \langle e_1, \psi_2^\sigma\rangle\alpha_2 = 0,$$
$$\langle e_2, \psi_1^\sigma\rangle\alpha_2 + \langle e_2, \psi_2^\sigma\rangle\alpha_2 = 0.$$

Using (iii) it follows that $\alpha_1 = 0 = \alpha_2$ also $\alpha_3 = \alpha_4 = 0$, by (ii). Thus it is enough to prove that the set $\{\varphi_1, \varphi_2, \varphi_3, e_i\}$ is linearly independent for each $1 \le i \le 4$. Let

$$\alpha_1\varphi_1 + \alpha_2\varphi_2 + \alpha_3\varphi_3 + \alpha_4 e_i = 0, \qquad (\alpha_i \in \mathbb{R}). \tag{7}$$

and choose a non-zero $\psi \in (\text{span}\{\varphi_i\}_{i=1}^3)^\perp$, then $0 = \langle \psi, \alpha_1\varphi_1 + \alpha_2\varphi_2 + \alpha_3\varphi_3 + \alpha_4 e_i\rangle = \alpha_4\langle \psi, e_i\rangle$. Using (ii) implies that $\alpha_4 = 0$. Moreover φ_1, φ_2 and φ_3 are linearly independent and so, $\alpha_1 = \alpha_2 = \alpha_3 = 0$ by (7). Other cases to get the complement property are similar. □

As a consequence, we obtain the following interesting results for phase retrieval frame in $\mathbb{R}^4$.

Corollary 3.3. *Let* $\{u_i\}_{i=1}^4$ *be an orthogonal basis in* $\mathbb{R}^4$. *Then* $\Phi = \{e_i\}_{i=1}^4 \cup \{u_i\}_{i=1}^3$ *does phase retrieval if and only if we have*

(i) $\langle e_j, u_i\rangle \neq 0$ *for all* $i = 1, 2, 3$ *and* $j = 1, 2, 3, 4$.

(ii) $\dfrac{\langle e_j, u_i\rangle}{\langle e_l, u_i\rangle} \neq \dfrac{\langle e_j, u_4\rangle}{\langle e_l, u_4\rangle}$ *for all* $j, l \in \{1, 2, 3, 4\}$ *and* $1 \le i \le 3$.

By an inductive approach, we are now ready to describe phase retrieval frames in $\mathbb{R}^m$.

Proposition 3.2. *Let $\Phi = \{e_i\}_{i=1}^m \cup \{\varphi_i\}_{i=1}^{m-1}$ be a frame in $\mathbb{R}^m$. Then Φ does phase retrieval if and only if:*

(i) $\langle \varphi_i, e_j\rangle \neq 0$ *for all* $1 \leq i \leq m$ *and* $1 \leq j \leq m-1$.

(ii) $\{\varphi_1, \varphi_2, ..., \varphi_{m-1}\}$ *is linearly independent and* $\langle \psi, e_j\rangle \neq 0$ *for* $1 \leq j \leq m$, *where* $(\mathrm{span}\{\varphi_i\}_{i=1}^{m-1})^\perp$ *is generated by* ψ.

(iii) $\left|(\langle e_{j_l}, \psi_r\rangle)_{\substack{1\leq l\leq m-k \\ 1\leq r\leq m-k}}\right| \neq 0$, *where* $\mathrm{span}\{\varphi_{i_1}, ..., \varphi_{i_k}\}^\perp = \mathrm{span}\{\psi_1, ..., \psi_{\mathrm{m-k}}\}$ *for all* $\{i_1, ..., i_k\} \subseteq \{1, ..., m-1\}$ *and* $\{j_1, ..., j_{m-k}\} \subseteq \{1, ..., m\}$.

4. Phase retrieval dual frames

In this section, we address the problem that, given a phase retrieval frame Φ in $\mathbb{R}^m$ how we can characterize their phase retrieval dual frames. For a frame Φ, we denote by D_Φ the set of all its dual frames. Also we use PD_Φ for the subset of all phase retrieval dual frames. We first state the following lemma which is a very useful tool for obtaining the main results of this section.

Lemma 4.1. *Let $\Phi = \{\varphi_i\}_{i=1}^{2m-1}$ be a frame in $\mathbb{R}^m$. Then*

$$D_\Phi = \left\{\left\{S_\Phi^{-1}\varphi_i + u_i\right\}_{i=1}^m \cup \left\{S_\Phi^{-1}\varphi_{m+j} + v_j\right\}_{j=1}^{m-1}; \{v_j\}_{j=1}^{m-1} \in \mathbb{R}^m, u_i = -\sum_{j=1}^{m-1}\langle S_\Phi^{-1}\varphi_i, \varphi_{m+j}\rangle v_j\right\},$$

where $\Phi_0 = \{\varphi_i\}_{i=1}^m$ is a Riesz basis.

Proof. Applying Theorem 2.1 shows that every dual frame Φ^d of Φ has the form

$$\Phi^d = \left\{S_\Phi^{-1}\varphi_i + u_i\right\}_{i=1}^m \cup \left\{S_\Phi^{-1}\varphi_{m+j} + v_j\right\}_{j=1}^{m-1},$$

where $\{u_i\}_{i=1}^m \cup \{v_j\}_{j=1}^{m-1}$ is a Bessel sequence and for every $f \in \mathcal{H}$ we have

$$\sum_{i=1}^m \langle f, u_i\rangle\varphi_i + \sum_{j=1}^{m-1}\langle f, v_j\rangle\varphi_{m+j} = 0.$$

Hence,

$$\begin{aligned}\sum_{i=1}^m\langle f, u_i + \sum_{j=1}^{m-1}\langle S_{\Phi_0}^{-1}\varphi_i, \varphi_{m+j}\rangle v_j\rangle\varphi_i &= \sum_{i=1}^m\langle f, u_i\rangle\varphi_i + \sum_{i=1}^m\sum_{j=1}^{m-1}\langle f, v_j\rangle\langle\varphi_{m+j}, S_{\Phi_0}^{-1}\varphi_i\rangle\varphi_i \\ &= \sum_{i=1}^m\langle f, u_i\rangle\varphi_i + \sum_{j=1}^{m-1}\langle f, v_j\rangle\varphi_{m+j} = 0.\end{aligned}$$

So, $u_i = -\sum_{j=1}^{m-1}\langle S_{\Phi_0}^{-1}\varphi_i, \varphi_{m+j}\rangle v_j$, for all $j = 1, 2, \cdots, m$. □

Motivated by Lemma 4.1, we are now ready to characterize phase retrieval dual frames. The following important and applied result, will later play a key role in our main results. For simplicity, we consider Φ of the form $\Phi = \{e_i\}_{i=1}^m \cup \{\varphi_i\}_{i=1}^{m-1}$.

Theorem 4.1. *Let $\Phi = \{e_i\}_{i=1}^m \cup \{\varphi_i\}_{i=1}^{m-1}$ be a frame in $\mathbb{R}^m$. If $\{v_j\}_{j=1}^{m-1} \subseteq \mathbb{R}^m$ such that $\{e_i - \sum_{j=1}^{m-1}\langle e_i, \varphi_j\rangle S_\Phi v_j\}_{j=1}^m$ is a Riesz basis and the vector $\{\eta_i\}_{i=1}^{m-1} = \{A(\varphi_j + S_\Phi v_j)\}_{j=1}^{m-1}$ satisfies Proposition 3.2 where A is the matrix of change basis $\{e_i + u_i\}_{i=1}^m$ to $\{e_i\}_{i=1}^m$. Then*

$$\Phi^d = \left\{S_\Phi^{-1}e_i - \sum_{j=1}^{m-1}\langle e_i, \varphi_j\rangle v_j\right\} \cup \left\{S_\Phi^{-1}\varphi_j + v_j\right\}_{j=1}^{m-1} \in PD_\Phi.$$

Proof. Consider that $\Phi = \{e_i\}_{i=1}^m \cup \{\varphi_i\}_{i=1}^{m-1}$ is a frame in $\mathbb{R}^m$ and $\Phi^d \in D_\Phi$. Applying Lemma 4.1 for following holds $\Phi^d = \{S_\Phi^{-1}e_i + u_i\}_{i=1}^m \cup \{S_\Phi^{-1}\varphi_j + v_j\}_{j=1}^{m-1}$, where $\{v_j\}_{j=1}^{m-1}$ are arbitrary and $u_i = -\sum_{j=1}^{m-1}\langle e_i, \varphi_j\rangle v_j$ for all $1 \le i, j \le m$. On the other hand, the matrix of change basis $\{e_i + u_i\}_{i=1}^m$ to $\{e_i\}_{i=1}^m$, denoted by A, is invertible by the assumption. In fact $A^{-1} = I_{m\times m} + (\langle u_j, e_i\rangle)_{1\le i,j\le m}$. Thus $AS_\Phi\Phi^d = \{e_i\}_{i=1}^m \cup \{A(\varphi_j + S_\Phi v_j)\}_{j=1}^{m-1}$, using Proposition 3.2 implies that $\Phi^d \in PD_\Phi$ if $\{\eta_i\}_{i=1}^{m-1}$ satisfies (3). This completes the proof. □

The next theorem provides a natural and intrinsic characterization of general form of phase retrieval dual frames on $\mathbb{R}^2$. Note that, by Lemma 4.1, there is an one to one correspondence between dual frames $\Phi = \{e_1, e_2, ae_1 + be_2\}$ as following

$$D_\Phi = \Big\{\Phi_u^d := \{S_\Phi^{-1}e_1 - au, S_\Phi^{-1}e_2 - bu, S_\Phi^{-1}(a,b) + u\}; u \in \mathbb{R}^2\Big\}.$$

Proposition 4.1. *Let $\Phi = \{e_1, e_2, ae_1 + be_2\}$ be a phase retrieval frame in $\mathbb{R}^2$ and $u = (x, y)$. Then the following are equivalent*

(i) *Φ_u^d does phase retrieval.*

(ii)

$$\begin{cases} ax + by \neq \dfrac{1}{1+a^2+b^2}, \\ (ab^2)x^2 + (b^3 + b - a^2b)xy + (-1-a^2)x + aby + (-a-ab^2)y^2 \neq a^2, \\ (-a^2b - b)x^2 + (-ab^2 + a^3 + a)xy + a^2by^2 + (-1-b^2)y - abx \neq b. \end{cases} \tag{8}$$

(iii) *$(x', y') := S_\Phi u$ satisfies*

$$\begin{cases} 1 - ax' \neq by', \\ b + (1+a^2) \neq abx', \\ a + (1+b^2)x' \neq aby'. \end{cases} \tag{9}$$

Proof. $(i) \iff (ii)$ Since Φ is phase retrieval frame we get $ab \neq 0$ by Lemma 3.1. An easy computations shows that

$$S_\Phi^{-1} = \frac{1}{a^2+b^2+1}\begin{bmatrix} 1+b^2 & -ab \\ -ab & 1+a^2 \end{bmatrix}.$$

Then $\{S_\Phi^{-1}e_1 - a(x,y), S_\Phi^{-1}e_2 - b(x,y)\}$ is linearly independent if and only if

$$\begin{vmatrix} \frac{1+b^2}{\frac{1}{a^2+b^2+1}} - ax & \frac{-ab}{a^2+b^2+1} - ay \\ \frac{-ab}{a^2+b^2+1} - bx & \frac{1+a^2}{\frac{1}{a^2+b^2+1}} - by \end{vmatrix} \neq 0.$$

Or equivalently, $ax + by \neq \dfrac{1}{1+a^2+b^2}$. Furthermore, a dual frame of Φ is given by

$$\Phi_u^d = \{S_\Phi^{-1}e_1 - au, S_\Phi^{-1}e_2 - bu, S_\Phi^{-1}\varphi + u\},$$

where $u = (x, y) \in \mathbb{R}^2$. So

$$\begin{aligned} S_\Phi\Phi_u^d = \ & \Big\{\Big(1 - ax - a^3x - a^2by, -a^2bx - ay - ab^2y\Big), \\ & \Big(-bx - ba^2x - b^2ay, 1 - ab^2x - by - b^3y\Big), \\ & \Big(a + x(1+a^2) + aby, b + abx + y(1+b^2)\Big)\Big\}. \end{aligned}$$

The matrix

$$A = \frac{1}{a^3x + ax + ab^2x + b^3y + a^2by + by - 1}\begin{bmatrix} 1 - ab^2x - b(1+b^2)y & b(1+a^2)x + ab^2y \\ a^2bx + a(1+b^2)y & 1 - a(1+a^2)x - a^2by \end{bmatrix},$$

is the change of basis matrix from

$$\Big\{\Big(1-ax-a^3x-a^2by,-a^2bx-ay-ab^2y\Big),\Big(-bx-ba^2x-b^2ay,1-ab^2x-by-b^3y\Big)\Big\}$$

to $\{e_1,e_2\}$. Thus

$$A^{-1}S_\Phi\Phi_u^d=\Big\{e_1,e_2,\frac{1}{(a^3+a+ab^2)x+(b^3+a^2b+b)y-1}\times \Big((ab^2)x^2+(b^3+b-a^2b)xy-(1+a^2)x+aby-(a+ab^2)y^2-a^2, -(a^2b+b)x^2+(-ab^2+a^3+a)xy+a^2by^2-(1+b^2)y-abx-b\Big)\Big\}.$$

Therefore, Φ_u^d is phase retrieval dual frame on $\mathbb{R}^2$ if and only if (8) is satisfied by the assertion after Lemma 3.1.

$(i)\iff(iii)$ Phase retrieval property is preserved under invertible operators. Hence Φ_u^d does phase retrieval if and only if $\Phi^\dagger=\Big\{e_1-aS_\Phi u,e_2-bS_\Phi u,(a,b)+S_\Phi u\Big\}$ does phase retrieval. Due to Lemma 3.1, $\Phi^\dagger=\Big\{(1-ax',ay'),(-bx',1-by'),(a-x',b-y')\Big\}$ does phase retrieval if and only if (9) holds. □

5. Phase retrieval frames $\Phi=\{\varphi_i\}_{i=1}^n$ in $\mathbb{R}^m$ when $n>2m-1$

Let $\Phi=\{\varphi_i\}_{i=1}^n$ be a frame in $\mathbb{R}^m$. Due to Theorem 2.2, $n\geq 2m-1$. In Section 3, we found a concrete form of phase retrieval frames $\Phi=\{\varphi_i\}_{i=1}^{2m-1}$ based on some conditions for the components of φ_i' s. In this section, we continue to describe phase retrieval frames $\Phi=\{\varphi_i\}_{i=1}^n$ when $n>2m-1$. The next lemma describes this fact for $m=2$.

Lemma 5.1. *Let $\Phi=\{\varphi_i\}_{i=1}^n$ be a frame for $\mathbb{R}^2$ such that $\{\varphi_1,\varphi_2\}$ is a Riesz basis for $\mathbb{R}^2$ and $0\neq\varphi_i=(x_i,y_i)$, $1\leq i\leq n$. Then Φ does phase retrieval if and only if*

$$(x_iy_2-x_2y_i)(x_1y_i-x_iy_1)\neq 0, \tag{10}$$

for some $3\leq i\leq n$.

Proof. Assume that (10) holds for some i, then by Lemma 3.1, $\{\varphi_1,\varphi_2,\varphi_i\}$ does phase retrieval and so Φ is also phase retrieval.

Conversely, in contrary assume that

$$(x_iy_2-x_2y_i)(x_1y_i-x_iy_1)=0,\qquad (3\leq i\leq n) \tag{11}$$

Let $\{\varphi_i\}_{i=1}^n$ does phase retrieval, then $\{A\varphi_i\}_{i=1}^n$ is also phase retrieval, where

$$A=\frac{1}{x_1y_2-y_1x_2}\begin{bmatrix} y_2 & -x_2\\ -y_1 & x_1\end{bmatrix}.$$

Actually A is the change of basis matrix from $\{\varphi_1,\varphi_2\}$ to $\{e_1,e_2\}$.

Take $\sigma=\{1\leq i\leq n;\ x_iy_2-x_2y_i=0\}$. Using (11) we obtain

$$\sigma^c=\{1\leq i\leq n;\ x_1y_i-x_iy_1=0\},$$

since $\varphi_i\neq 0$, for $1\leq i\leq n$. On the other hand neither $\text{span}\{A\varphi_i\}_{i\in\sigma}=\mathbb{R}^2$ nor $\text{span}\{A\varphi_i\}_{i\in\sigma^c}=\mathbb{R}^2$. Hence, Φ does not phase retrieval. This is contrary to the assumption. □

As a demonstration of Theorem 2.2, we immediately obtain the following characterization for phase retrieval frames on $\mathbb{R}^3$.

Proposition 5.1. *Let $\Phi=\{e_i\}_{i=1}^3\cup\{\varphi_i\}_{i=1}^3$ be a frame in $\mathbb{R}^3$. Then Φ does phase retrieval if and only if the following holds*

(i) *For all $1\leq l\leq 3$ there exist distinct i,j,k such that $\langle\varphi_i,e_l\rangle\neq 0$ or $\langle\varphi_j\times\varphi_k,e_l\rangle\neq 0$.*

(ii) *The vectors* $\{\varphi_i\}_{i=1}^3$ *are linearly independent or for all* $1 \le l \le 3$ *there exist distinct* $1 \le j,k \le 3$ *such that* $\langle \varphi_j \times \varphi_k, e_l\rangle \neq 0$.

Proof. Assume that Φ dose phase retrieval and $1 \le l \le 3$. Using Theorem 2.2 for distinct $i,j,k \in \{1,2,3\}$ where $l \neq j,k$ we have $\mathrm{span}\{\mathrm{e_j},\mathrm{e_k},\varphi_\mathrm{i}\} = \mathbb{R}^3$ or $\mathrm{span}\{\mathrm{e_l},\varphi_\mathrm{j},\varphi_\mathrm{k}\} = \mathbb{R}^3$ for $i,j,k \in \{1,2,3\}$, or equivalently, $\langle \varphi_i, e_l\rangle \neq 0$ or $\langle \varphi_j \times \varphi_k, e_l\rangle \neq 0$. This proves (i). To show (ii), assume that $\{\varphi_i\}_{i=1}^3$ is not linearly independent then $\mathrm{span}\{\mathrm{e_l},\varphi_1,\varphi_2,\varphi_3\} = \mathbb{R}^3$ for every $l \in \{1,2,3\}$. By using Theorem 2.2 this easily follows that $\mathrm{span}\{\mathrm{e_l},\varphi_\mathrm{j},\varphi_\mathrm{k}\} = \mathbb{R}^3$ for some $1 \le j \neq k \le 3$. In particular $\langle \varphi_j \times \varphi_k, e_l\rangle \neq 0$. Conversely, let conditions (i) and (ii) be satisfied. It is enough to show that Φ has the complement property. By (i), $\mathrm{span}\{\mathrm{e_j},\mathrm{e_k},\varphi_\mathrm{i}\} = \mathbb{R}^3$ or $\mathrm{span}\{\mathrm{e_l},\varphi_\mathrm{j},\varphi_\mathrm{k}\} = \mathbb{R}^3$ is fulfilled. Other cases to obtain the complement property is easily obtained by condition (ii). □

The advantage of using Proposition 5.1 lies in the fact that, if a frame with five elements in $\mathbb{R}^3$ does not phase retrieval, then by adding an appropriate member, we may obtain a phase retrieval frame. The following example illustrates this point well.

Example 5.1. *Let* $\Phi = \{e_i\}_{i=1}^3 \cup \{\varphi_i\}_{i=1}^3$ *be a frame in* $\mathbb{R}^3$ *where* $\varphi_1 = (a,b,c), \varphi_2 = \alpha\varphi_1, \varphi_3 = (x,y,z)$ *and* $abc \neq 0$. *Then* Φ *dose phase retrieval if and only if*

(i) $xyz \neq 0$,

(ii) $x \neq \frac{a}{b}y, x \neq \frac{a}{c}z, y \neq \frac{b}{c}z$.

Proof. Assume that Φ dose phase retrieval frame in $\mathbb{R}^3$. Clearly, by the assumption $\langle \varphi_1 \times \varphi_2, e_l\rangle = 0$ for all $l = 1,2,3$. Using (i) of Proposition 5.1 we conclude that $xyz \neq 0$. Furthermore, according to (ii) of Proposition 5.1 and the fact that $\langle \varphi_1 \times \varphi_2, e_l\rangle = 0$ we have $\langle \varphi_1 \times \varphi_3, e_l\rangle \neq 0$ or $\langle \varphi_2 \times \varphi_3, e_l\rangle \neq 0$ for all $l = 1,2,3$. So, since φ_1 and φ_2 are linearly independent. It is sufficient to $\langle \varphi_1 \times \varphi_2, e_l\rangle \neq 0$ for all $l = 1,2,3$. That means $x \neq \frac{a}{b}y, x \neq \frac{a}{c}z, y \neq \frac{b}{c}z$. Conversely, if conditions (i) and (ii) are satisfied. It is enough to show that Φ has the complement property. Obviously, $\mathrm{span}\{\mathrm{e_j},\mathrm{e_k},\varphi_\mathrm{i}\} = \mathbb{R}^3$ by (i). Moreover, $\mathrm{span}\{\mathrm{e_j},\mathrm{e_k},\varphi_1,\varphi_3\} = \mathrm{span}\{\mathrm{e_j},\mathrm{e_k},\varphi_2,\varphi_3\} = \mathbb{R}^3$, by (ii) for every $1 \le j,k \le 3$. Finally, $\mathrm{span}\{\mathrm{e_j},\mathrm{e_k},\varphi_1,\varphi_2\} = \mathbb{R}^3$ because of the assumption $abc \neq 0$. □

6. Numerical Results: Application and Analysis

Let $\Phi = \{\varphi_i\}_{i=1}^n$ be a frame for $\mathcal{H}^m$. It is easy to show that $\Phi = \{\varphi_i\}_{i=1}^n$ is a phase retrievable frame for $\mathcal{H}^m$ if and only if α_Φ is given by (4) is injective. Thus we can reconstruct every vector in $\mathcal{H}^m$ from the magnitude of its frame coefficients as follows

$$\beta_\Phi \alpha_\Phi[x] = [x], \qquad (x \in \mathcal{H}^m), \tag{12}$$

where β_Φ is a left inverse of α_Φ. Finding a general form of β_Φ is highlighted in the next.

6.1. Reconstruction by 2D phase retrieval frames

Consider a phase retrieval frame $\Phi = \{e_1, e_2, (\alpha,\beta)\}$ in $\mathbb{R}^2$, then

$$\alpha_\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \qquad \alpha_\Phi[(x_1,x_2)] = \{|x_1|, |x_2|, |\alpha x_1 + \beta x_2|\}.$$

It is not difficult to see that $\beta_\Phi : \alpha_\Phi \longrightarrow \mathbb{H}$ is given by

$$\beta_\Phi(a,b,c) = \begin{cases} [(a,b)], & c = |\alpha a + \beta b|, \\ [(a,-b)], & c = |\alpha a - \beta b|, \end{cases}$$

where $a = |x_1|, b = |x_2|, c = |\alpha x_1 + \beta x_2|$. Applying this approach, for phase retrieval frame $\Phi = \{(1,0),(0,1),(1,1)\}$ in $\mathbb{R}^2$ we obtain

$$[(x_1,x_2)] = \beta_\Phi\Big(|x_1|, |x_2|, |x_1 + x_2|\Big), \tag{13}$$

where

$$\beta_\Phi(|x_1|,|x_2|,|x_1+x_2|)=\begin{cases}[(|x_1|,|x_2|)], & |x_1+x_2|=\Big||x_1|+|x_2|\Big|,\\ [(|x_1|,-|x_2|)], & |x_1+x_2|=\Big||x_1|-|x_2|\Big|.\end{cases}$$

Take $x=(4,1)$. To examine the effectiveness of (13) and (1) we remove the third sentence in both of them. This means that

$$\beta_\Phi\alpha_\Phi(|4|,|1|,0)=\pm[(4,-1)], \qquad \sum_{i=1}^{2}\langle x,S_\Phi^{-1}\varphi_i\rangle\varphi_i=(\frac{7}{3},-\frac{2}{3}),$$

are approximations for x. Obviously, (4) may be deals to adaptable results.

6.2. Reconstruction by 3D phase retrieval frames

Suppose that $\Phi=\{e_1,e_2,e_3,(\alpha_1,\alpha_2,\alpha_3),(\beta_1,\beta_2,\beta_3)\}$ does phase retrieval in $\mathbb{R}^3$, then $\alpha_\Phi:\mathcal{H}\longrightarrow\mathbb{R}^5$, $\quad \alpha_\Phi[(x_1,x_2,x_3)]=\{|x_1|,|x_2|,|x_3|,|\alpha_1x_1+\alpha_2x_2+\alpha_3x_3|,|\beta_1x_1+\beta_2x_2+\beta_3x_3|\}$. Straightforward calculation indicates

$$\beta_\Phi(a,b,c,d,e)=\begin{cases}[(a,b,c)], & d=|\alpha_1a+\alpha_2b+\alpha_3c|,\ e=|\beta_1a+\beta_2b+\beta_3c|,\\ [(a,-b,-c)], & d=|\alpha_1a-\alpha_2b-\alpha_3c|,\ e=|\beta_1a-\beta_2b-\beta_3c|,\\ [(-a,b,-c)], & d=|-\alpha_1a+\alpha_2b-\alpha_3c|,\ e=|-\beta_1a+\beta_2b-\beta_3c|,\\ [(-a,-b,c)], & d=|-\alpha_1a-\alpha_2b+\alpha_3c|,\ e=|-\beta_1a-\beta_2b+\beta_3c|,\end{cases} \tag{14}$$

where

$$a=|x_1|,b=|x_2|,c=|x_3|,d=|\alpha_1x_1+\alpha_2x_2+\alpha_3x_3|,e=|\beta_1x_1+\beta_2x_2+\beta_3x_3|.$$

We want to compare the classical reconstruction formula (1) and the phase retrieval reconstruction (12). To serve this purpose, consider the phase retrieval frame

$$\Phi=\{(1,0,0),(0,1,0),(0,0,1),(1,2,1),(1,1,2)\},$$

obtained by Proposition 3.1. With the canonical dual

$$\left\{(\frac{5}{6},-\frac{1}{4},-\frac{1}{4}),(-\frac{1}{4},\frac{3}{8},-\frac{1}{8}),(-\frac{1}{4},-\frac{1}{8},\frac{3}{8}),(\frac{1}{12},\frac{3}{8},-\frac{1}{8}),(\frac{1}{12},-\frac{1}{8},\frac{3}{8})\right\}.$$

We are going to reconstruct a vector $x=(x_1,x_2,x_3)$ in $\mathbb{R}^3$ by using the frame coefficients and their magnitude. Consider a random signal (a vector with the length 60) on the interval $[0,1]$ and divide it in to sub vectors with the length 3. Then each sub vector can be reconstructed by (1) and (12) after removing the fourth or the fifth frame coefficient in each process. Mean squared error (MSE) is defined as mean or average of the square of the difference between actual and estimated values. For example, if $\{\hat{y}_j\}_{j=1}^n$ is an estimate of $\{y_j\}_{j=1}^n$, then its MSE is calculated as

$$MSE=\frac{1}{N}\sum_{j=1}^{N}(\hat{y}_j-y_j)^2.$$

In signal processing, a lower MSE value indicates a closer match between the estimated signal and the true signal, implying better accuracy or fidelity of the estimation method.

Finally, we collect the average mean square error (MSE) of every sub vector and compute the MSE for both approximations on $[0,1]$. We summarize the results for some random signals in the Table 1. It is worth wide to mention that in phase retrieval reconstruction by (12) after assuming $|\langle x,\varphi_i\rangle|=0$, $i=4$ or $i=5$ for a sub vector x it may $(|x_1|,|x_2|,|x_3|,|\langle x,\varphi_4\rangle|,0)$ or $(|x_1|,|x_2|,|x_3|,0,|\langle x,\varphi_5\rangle|)$ be outside of $\mathcal{R}(\alpha_\Phi)$ and so we can not calculate β_Φ for them from (14). In practice, we achieve this, by finding an element in the $\mathcal{R}(\alpha_\Phi)$ in which has minimum distance to $\beta_\Phi(|x_1|, |x_2|, |x_3|, |\langle x, \varphi_4\rangle|, 0)$. Then we consider β_Φ on this element instead of $\beta_\Phi(|x_1|, |x_2|, |x_3|, |\langle x, \varphi_4\rangle|, 0)$. A similar argument can be applied for the element $\beta_\Phi(|x_1|, |x_2|, |x_3|, 0, |\langle x, \varphi_5\rangle|)$.

	MSE	
Signals	Classical reconstruction	Phase retrieval reconstruction
Signal 1	0.0043	**0.0036**
Signal 2	0.0084	**0.0056**
Signal 3	**0.0144**	0.0179
Signal 4	0.0098	**0.0060**
Signal 5	0.0142	**0.0030**

	MSE	
Signals	Classical reconstruction	Phase retrieval reconstruction
Signal 1	0.0104	**0.0103**
Signal 2	0.0059	**0.0037**
Signal 3	**0.0014**	0.0038
Signal 4	0.0052	**0.0012**
Signal 5	0.0163	**0.0071**

TABLE 1. The table up (down) displays the MSE of estimates for different random signals by the classical reconstruction formula (1) and phase retrieval reconstruction (12) when $\Phi = \{(1,0,0),(0,1,0),(0,0,1),(1,2,1),(1,1,2)\}$ under removing the fourth (fifth) coefficient.

Overall, our results confirm that using the phase retrieval method is more effective in reconstructing signals. This is especially evident when considering that the phase retrieval method can reconstruct the signal based solely on the magnitudes of the frame coefficients.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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