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GROUPOID TRUNCATIONS

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O teorema celebra datorata lui Mundici [1] stabileste o stransa legatura intre MV-algebre si anumite intervale in grupuri laticial ordonate. Trecerea de la operatia de grup la cea de MV-algebra se face printr-un procedeu pe care l-am numit, in aceasta lucrare "trunchiere". Acest articol este o incercare de a trata, abstract, la nivel de grupoizi acest mod de a trunchia o operatie algebrica binara.

A famous theorem of Mundici gives a close connection between MV-algebras and some intervals in lattice ordered groups. The procedure of inducing the MV-algebra operation from the group operation was called, in this paper "truncation". We try, in this paper, to give a general abstract treatment, at the groupoids level, of the truncation of a binary algebraic operation.

Keywords: groupoid, truncation, MV-algebra.

1. Introduction

Let S be a (non empty) set and let S^S be the monoid (with respect to the composition) of all functions from S to S.
If φ is an idempotent ($\varphi \circ \varphi = \varphi$) of S^S then it defines a retraction of S onto $S_\varphi = \varphi(S)$ (φ is a left inverse of the canonical inclusion of S_φ into S).
It is easy to see that every non empty subset A of S is the image of (at least one) idempotent of S^S; for example, take $a \in A$ and define $\varphi(x) = x$ for $x \in A$ and $\varphi(x) = a$ for $x \in S - A$.
Consider the equivalence relation "~" defined by φ (x~y iff $\varphi(x) = \varphi(y)$), let M_φ be the quotient set and let π be the canonical projection.

Lemma. The restriction θ of π to S_φ is bijective.

Proof. We have $\varphi(x)=x$ iff $x \in S_\varphi$ and the injectivity of θ follows. Then, as for every $x \in S$, we have that $x \sim \varphi(x)$ the surjectivity of θ also follows.

Remark 1. $\theta(\varphi(x)) = \pi(x)$, for every $x \in S$.

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Let now $(S,+)$ be a groupoid ("+" is a composition law on S, not necessary associative or commutative) and let φ be an idempotent of S^S (not necessary a morphism of groupoids). We ask for conditions granting that the equivalence "~" defined by φ is a congruence (meaning that if $x \sim y$ then $x + z \sim y+z$ and $z+x \sim z+y$ for every $z \in S$).

Theorem 1. The equivalence defined by an idempotent $\varphi \in S^S$ is a congruence iff :

$$\varphi(x+y) = \varphi(\varphi(x) + \varphi(y)) \text{ for every } x,y \in S. \qquad (*)$$

Proof. Suppose, first, that "~" is a congruence. From $x \sim \varphi(x)$ we obtain that $\varphi(\varphi(x) + y) = \varphi(x+y)$; in the same manner we get that $\varphi(\varphi(x) + \varphi(y)) = \varphi(\varphi(x) + y)$. The relation (*) is proved.
If, now, we suppose the relation (*) to hold and taking $x \sim y$ and $z \in S$ we get: $\varphi(x+z) = \varphi(\varphi(x) + \varphi(z)) = \varphi(\varphi(y) + \varphi(z)) = \varphi(y+z)$ and so $x+z \sim y+z$ etc.

Remark 2. The condition (*) is trivially satisfied for $x, y \in S_\varphi$.

Remark 3. By modifying, in an obvious way, the condition (*) one obtains a similar result for partial groupoids.

Examples.

1. Suppose that S is an idempotent groupoid (i.e $x+x=x$ for every x). If $\varphi : S \to S$ satisfies the condition (*) then it is idempotent (in S^S). In fact we have that
$\varphi(x) = \varphi(x+x) = \varphi(\varphi(x) + \varphi(x)) = \varphi(\varphi(x))$.
2. Suppose that $(S, \leq)$ is a chain and let $(S, \wedge)$ the associated inf-semilattice. Then, every idempotent, increasing function φ satisfies the condition (*). In fact, if $x \leq y$ then $\varphi(x) \leq \varphi(y)$ so $\varphi(x \wedge y) = \varphi(x) = \varphi(\varphi(x) \wedge \varphi(y))$.

Definition. An idempotent φ satisfying (*) is called a **truncation**.

Proposition 1. If φ, ϕ are commuting truncations then $\phi \circ \varphi$ is a truncation.

Proof. It is obvious that $\phi \circ \varphi$ is idempotent. Then we have:
$\phi \circ \varphi(x+y) = \phi(\varphi(\varphi(x) + \varphi(y))) = \varphi(\phi(\varphi(x) + \varphi(y))) = \varphi(\phi(\phi \circ \varphi(x) + \phi \circ \varphi(y))) = \phi(\varphi(\phi \circ \varphi(x) + \phi \circ \varphi(y))) = \phi \circ \varphi(\phi \circ \varphi(x) + \phi \circ \varphi(y))$.

Remark 4. If φ is a truncation then, with the notation above, M_φ has a canonical structure of a groupoid and π is a morphism of groupoids. If S is a semigroup so is M_φ.

Definition. Let φ be a truncation. On S_φ define $x \oplus y = \varphi(x+y)$. Remark that , if $x+y \in S_\varphi$ then $x \oplus y = x+y$.

Theorem 2. With the notation above, θ is an isomorphism of groupoids. If $(S, +)$ is a semigroup then $(S_\varphi, \oplus)$ is a semigroup.

Proof. We have, by using Remark 1:
$\theta(x \oplus y) = \theta(\varphi(x+y) = \pi(x+y) = \pi(x) + \pi(y)$, so θ is a morphism of groupoids etc.

Corollary. If we consider φ as taking values in S_φ then φ is a morphism of groupoids.

Remark 5. In general S_φ is not a subgroupoid of S.

Remark 6. If S is a monoid with unit 0 then S_φ is a monoid with unit $\varphi(0)$.

Remark 7. Not every non empty subset of a groupoid can be turned into a groupoid using a truncation. For example, take the set of rational numbers Q with addition as composition law and take the subset Z of the integers. Any truncation of image Z will induce the usual addition on Z and, by the previous corollary, a surjective morphism from $(Q,+)$ to $(Z,+)$ which is not possible.

Proposition 2. Let $(S, +, \le)$ be an ordered groupoid and φ an increasing truncation. Then $(S_\varphi, \oplus, \le)$ is an ordered groupoid (we use the same notation for the induced order as for the initial one).

Proof. Let $x, y, z \in S_\varphi$, $x \le y$; then,

$x \oplus z = \varphi(x+z) \le \varphi(y+z) = y \oplus z$,etc.

Remark 8. One can define a partial operation on S_φ (and in fact on every subset of S) by restricting the operation "+" to those $x, y \in S_\varphi$ satisfying the condition $x + y \in S_\varphi$ (in this case, of course, $x \oplus y = x + y$).

Examples.

1. For a groupoid S take $\varphi = id_S$. Then, obviously, $S_\varphi = S$ as groupoids.

2. For a groupoid S let φ be the constant map $\varphi(x) = a$ for every $x \in S$. Then φ is a truncation and $S_\varphi = \{a\}$ and the groupoid operation is trivial.

3. Let $(S,\wedge)$ be an inf-semilattice and let $a \in S$. Define a function $\varphi : S \to S$ by $\varphi(x) = x \wedge a$ for every $x \in S$. It is clear that φ is an idempotent of S^S. From $\varphi(x \wedge y) = (x \wedge y) \wedge a = (x \wedge a) \wedge (y \wedge a) = \varphi(\varphi(x) \wedge \varphi(y))$ we get that φ is a truncation. In this case the semigroup structure on $S_\varphi = (, a]$ is the same as the induced inf-semilattice structure (here, $(,a] = \{x; x \in S, x \leq a\}$).

4. Let A be a commutative ring with unit and let Λ be the monoid of the ideals of A, the operation being the addition of ideals. Define $\varphi : \Lambda \to \Lambda$, $\varphi(\alpha) = \sqrt{\alpha}$. It is known that φ is an idempotent and from the formula $\sqrt{\alpha + \beta} = \sqrt{\sqrt{\alpha} + \sqrt{\beta}}$ we obtain that φ is a truncation. The set Λ_φ is the set of radical ideals of A and the monoid structure is obtained as above (by taking the radical of the sum of two radical ideals). The unit of Λ_φ is the nilradical of the ring (the ideal of the nilpotent elements of A).

5. Let S be the set of all subgroups of the additive group of the reals R together with the addition of subgroups. It is well-known that a subgroup of R is either discrete (of the form aZ with $a \in R$) or dense in R. Consider the function $\varphi : S \to S$ given by $\varphi(H) = \overline{H}$. Then φ satisfies the condition (*). Indeed, if, say, $H,K \in S$ are discrete then (*) obvious holds. If H is dense then so is $H+K$ and one obtains the condition (*). It is worth noting that, in general, $\overline{H + K} \neq \overline{H} + \overline{K}$ (take, for example, $H = Z$ and $K = \pi Z$).

6. Let G be a lattice ordered group (l-group for short) and let G^+ be the set of elements of G which are ≥ 0. G^+ is a lattice ordered monoid with respect to the induced structures. Take $0 < u$ and consider the function $\varphi : G^+ \to G^+$

defined by $\varphi(x) = x \wedge u$. It is clear that φ is idempotent. Let us prove that φ is a truncation with respect to the group operation "+" (not necessary commutative). We prove, first, that if , for $x, y \in G^+, x \wedge u = y \wedge u$ then $(x+z) \wedge u = (y+z) \wedge u$ for every $z \in G^+$. In fact, if $t \leq x+z, u$ we get that $t - z \leq x, u$ and so $t - z \leq y, u$ and $t \leq y + z, u$ etc. Now, using the trivial identities $x \wedge u = (x \wedge u) \wedge u$, $y \wedge u = (y \wedge u) \wedge u$ we obtain the condition (*) $(x+y) \wedge u = (x \wedge u + y \wedge u) \wedge u$. It is clear that, in this case, we have that $S_\varphi = [0, u] = \{x; 0 \leq x \leq u\}$ and, using the theorem above, we get a monoid structure on $[0, u]$ by putting $x \oplus y = (x+y) \wedge u$ for every $x, y \in [0, u]$. Of course, if G is commutative then so is the monoid $([0, u], \oplus)$.

7. Consider the ring $R[X]$ of polynomials with real coefficients and let $R_n[X]$ be the subset of the polynomials of degree less or equal than *n*. Consider the function φ which associates to a given polynomial the polynomial obtained by neglecting the terms of degree greater than *n*. It is clear that φ is idempotent and trivially a truncation with respect to addition. More interesting, maybe, is that φ is also a truncation with respect to multiplication. This result is of some interest, for example, in the calculus of limited expansions of functions.

8. Consider the additive group (*Q*, +) of the rationals and take $S = Z$ the subgroup of the integers. Then no idempotent of image *Z* will be a truncation. Indeed, if such a truncation exists, then the operation $\oplus$ on *Z* will be the usual addition and the truncation will induce a surjective homomorphism of *Q* onto *Z* which is not possible.

Proposition 3. Suppose that φ, ϕ are commuting truncations. Then $S_{\phi \circ \varphi} = (S_\varphi)_\phi$ (in an obvious sense).

Proof. From Proposition 1 we know that $\phi \circ \varphi$ is a truncation. Let $\oplus$ denote the operation on $S_{\phi \circ \varphi}$ and $\oplus'$ the operation on S_φ. We get that $x \oplus y = \phi \circ \varphi(x+y) = \phi(x \oplus' y)$ and this is what is needed. It is useful to note that, if $\varphi(x) = x$, then $\phi(\varphi(x)) = \phi(x)$ and, by commutativity, $\varphi(\phi(x)) = \phi(x)$; this means that $\phi(S_\varphi) \subseteq S_\varphi$ so that (the restriction of) ϕ could be thought as a truncation on S_φ.

Corollary. In the conditions of the proposition above, one has that $(S_\varphi)_\phi = (S_\phi)_\varphi$.

Proposition 4. Let $\pi : S \to T$ be a morphism of groupoids and let $\alpha : T \to S$ be a section of π in the category of sets ($\pi \circ \alpha = id_T$). Define $\varphi = \alpha \circ \pi$. Then φ is a truncation with $S_\varphi = \alpha(T)$.

Proof. First we show that $\varphi : S \to S$ is an idempotent; we have that $\varphi \circ \varphi = (\alpha \circ \pi) \circ (\alpha \circ \pi) = \alpha \circ (\pi \circ \alpha) \circ \pi = \alpha \circ \pi = \varphi$. We have to check condition (*). We need to show that $\alpha \circ \pi(x + y) = \alpha \circ \pi(\alpha \circ \pi(x) + \alpha \circ \pi(y))$. It is enough to prove that $\pi(x + y) = \pi(\alpha \circ \pi(x) + \alpha \circ \pi(y))$.
But, π being a morphism of groupoids, this reduces to
$\pi(x) + \pi(y) = \pi((\alpha \circ \pi)(x)) + \pi((\alpha \circ \pi(y))$ which is true due to the fact that α is a section of π.

Remark 9. It follows that every congruence of a groupoid generates truncations but not in a canonical way; it depends on the section one chooses.

Examples.

1. Consider the additive group *R* of the real numbers, let *Z* be the subgroup of the integers and denote by *T* the quotient group R/Z. Let $\alpha(\hat{x}) = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the floor of x. Then α is well defined and is a section of the canonical projection $\pi : R \to T$. Using the previous proposition we obtain a truncation φ such that $\varphi(x)$ is the fractionary part of $x \in R$. In this case $R_\varphi = [0,1)$ and the induced structure is given by the operation, say $\bullet$, with $x \bullet y = x + y$ if $x + y < 1$ and $x \bullet y = x + y - 1$ if $x + y \geq 1$.

2. Let *S* be a semigroup and *I* a proper two-sided ideal of *S*. Consider the Rees congruence "~" defined by *I* : *x*~*y* iff either *x*=*y* or $x, y \in I$. The quotient semigroup *S/~* consists of the class *I* and the classes $\{x\}$ for every $x \in S - I$. It is easy to choose a section α of the canonical projection: take an element $a \in I$ and define $\alpha(\{x\}) = x$ for $x \notin I$ and $\alpha(I) = a$. We get that $S_\varphi = \{a\} \cup (S - I)$ and the operation $\oplus$, on S_φ will be: $x \oplus a = a \oplus x = a$ and $x \oplus y = a$ if $x + y \in I$ or $x \oplus y = x + y$ if $x + y \notin I$ where, of course, "+" denotes the operation of *S*. Remark that *a* is a zero of the semigroup S_φ and that, if S is a monoid so is S_φ . We also see that if S is cancellative then, in general, S_φ will be not.

We shall now express some of the ideas above in the language of categories. Consider the category Δ:

- the objects of Δ are the pairs (S,φ) where S is a groupoid and φ a truncation on S.

- an arrow from (S,φ) to (T,ϕ) is a morphism of groupoids $f:S\to T$ such that $f\circ\varphi=\phi\circ f$.

- the composition of arrows is the usual composition of functions.

(it is trivial to check that Δ is, in fact, a category).

Proposition 5. Let f be an arrow from (S,φ) to (T,ϕ) and let S_φ, T_ϕ be as above. Then f naturally induces a morphism of groupoids $\bar{f}:S_\varphi\to T_\phi$.

Proof. It is trivial that $f(S_\varphi)\subseteq T_\phi$ and we define $\bar{f}$ as being the restriction of f to S_φ. We have (with obvious notations):

$$\bar{f}(x\oplus y)= f(x\oplus y)= f(\varphi(x+y))=\phi(f(x+y))=\phi(f(x)+f(y))=$$
$$=\bar{f}(x)\oplus\bar{f}(y).$$

Proposition 6. If $(S,\varphi),(T,\phi)$ are objects of the category Δ then $(S\times T,\varphi\times\phi)$ is an object of Δ and, together with the natural projections, a direct product of $(S,\varphi),(T,\phi)$.

Proof. Easy checking.

It would be of some interest to treat the topological case also. So, let S be a topological groupoid (meaning a groupoid together a topology on the support set such that the operation is continuous viewed as a function of two variables).
It is then natural to consider continuous truncations.

Proposition 7. Let a given continuous truncation $\varphi:S\to S$ be given. Then the operation $\oplus$ on S_φ is continuous in the induced topology.

Proof. In fact we have that the operation $\oplus$ on S_φ is obtained as a composition of continuous functions: shortly define $x\oplus y=\varphi(x+y)$.

R E F E R E N C E S

[1]. *R. Cignoli, I.M.L. D'Ottaviano, D. Mundici*, Algebraic Foundations of Many-Valued Reasoning, Kluwer Academic, Dordrecht, 2000.

[2]. *A.H.Clifford, G.B.Preston*, The Algebraic theory of semigroups, American Mathematical Society, 1964.

[3]. *P. Flondor, M. Sularia*, On a class of residuated semilattice monoids, Fuzzy sets and systems, 138, 2003, pg. 149-176.

[4]. *Afrodita Iorgulescu*, Algebras of logic as BCK algebras, Ed. ASE, 2008.