A COMPOSITE SPLITTING ALGORITHM FOR SOLVING FIXED POINT AND VARIATIONAL INCLUSION PROBLEMS IN HILBERT SPACES

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In this paper, we present a composite splitting method for solving fixed point and variational inclusion problems in a real Hilbert space. The introduced method consists of forward-backward method, Tseng-type method and self-adaptive method. Under some additional conditions, we prove that the sequence generated by the splitting method strongly converges to a solution of fixed point and variational inclusion problems.

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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $g : \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ be a set-valued operator and $h : \mathcal{H} \to \mathcal{H}$ be a single-valued operator. In this paper, we investigate the variational inclusion problem which aims to find a point $u^\dagger \in \mathcal{H}$ such that

$$0 \in (g + h)(u^\dagger). \tag{1}$$

The reason why we are interested in the variational inclusion problem (1) is that is closely related to the following well-known minimization problem

$$\min_{x \in \mathcal{H}} \{G(x) + H(x)\}, \tag{2}$$

where $G, H : \mathcal{H} \to \mathbf{R} \cup \{+\infty\}$ are two proper, lower semicontinuous and convex functions such that G is subdifferentiable and H is differentiable. As a matter of fact, if set $g = \partial G$ and $h = \nabla H$, then solving (2) is equivalent to solving (1).

Now, it is well known that variational inclusion problems offer a model framework for discussing many interesting problems, such as fixed point problems ([19–21, 27]), optimization problems ([12, 17, 18, 24]), split problems ([10, 11, 29, 31, 32, 34, 39]), equilibrium problems ([30, 48]) and variational inequalities ([3, 4, 28, 33, 35–38, 41, 42, 46]). Numerous iterative algorithms are studied and designed for finding a solution of the variational inclusion (1), see [1, 2, 7, 8, 16, 43, 45]. Especially, an interesting algorithm for solving (1) is the forward-backward algorithm ([13–15]) which generates a sequence $\{x_n\}$ as follows: $x_0 \in \mathcal{H}$,

$$x_{n+1} = (I + \varpi_n g)^{-1}(I - \varpi_n h)(x_n), \ n \geq 0, \tag{3}$$

where ϖ_n is a positive constant for all $n \geq 0$.

In the sequel, we use $(g + h)^{-1}(0)$ to stand for the solution set of the variational inclusion (1). If g is maximal monotone and h is strongly monotone or inverse strongly monotone, then the sequence $\{x_n\}$ generated by (3) converges weakly to some point in

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$(g+h)^{-1}(0)$. Recently, Cholamjiak, Hieu and Cho [6] combined forward-backward method and Tseng's method ([23]) for solving (1) where the operator h is plain monotone.

In this paper, we consider a fixed point problem of finding a point $u^\dagger \in \mathcal{H}$ such that

$$f(u^\dagger) = u^\dagger, \tag{4}$$

where $f : \mathcal{H} \to \mathcal{H}$ is a pseudocontractive operator. Use $\mathrm{Fix}(f)$ to denote the solution set of (4). We are interest in pseudocontractive operators due to this kind of operators is closely related to monotone or/and accretive operators ([5, 9]). There are many iterative algorithms for solving fixed point problem (4), see [22, 25, 44].

Motivated and inspired by the work in the literature, in this paper we consider a common problem of finding a point $u^\dagger \in \mathcal{H}$ such that

$$u^\dagger \in (g+h)^{-1}(0) \cap \mathrm{Fix}(f). \tag{5}$$

We present a composite splitting method for solving fixed point of pseudocontractive operator and variational inclusion problems in a real Hilbert space $\mathcal{H}$. The introduced method consists of forward-backward method, Tseng-type method and self-adaptive method. Under some additional conditions, we prove that the sequence generated by the splitting method strongly converges to a point in $(g+h)^{-1}(0) \cap \mathrm{Fix}(f)$.

2. **Preliminaries**

Let $\mathcal{H}$ be a real Hilbert space. Use " $\rightharpoonup$ " and " $\to$ " to denote weak convergence and strong convergence, respectively. Let $\{x_n\} \subset \mathcal{H}$ be a sequence. Write $\omega_w(x_n) = \{x^\dagger : \exists \{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup x^\dagger (i \to \infty)\}$.

Recall that an operator $f : \mathcal{H} \to \mathcal{H}$ is said to be pseudocontractive if

$$\langle f(u) - f(u^\dagger), u - u^\dagger \rangle \le \|u - u^\dagger\|^2, \ \forall u, u^\dagger \in \mathcal{H}. \tag{6}$$

Note that the inequality (2) is equivalent to

$$\begin{aligned} &\|f(u) - f(u^\dagger)\|^2 \le \|u - u^\dagger\|^2 + \|(I-f)u - (I-f)u^\dagger\|^2, \\ \Leftrightarrow &\langle u - u^\dagger, (I-f)u - (I-f)u^\dagger \rangle \ge 0, \end{aligned}$$

for all $u, u^\dagger \in \mathcal{H}$.

Let $h : \mathcal{H} \to \mathcal{H}$ be a mapping. h is said to be

(i) monotone if $\langle h(x) - h(y), x - y \rangle \ge 0, \ \forall x, y \in \mathcal{H}$.

(ii) L-Lipschitz continuous if $\|h(x) - h(y)\| \le L\|x - y\|, \ \forall x, y \in \mathcal{H}$, where $L > 0$ is a constant. If $L < 1$, then h is said to be L-contractive.

Recall that a bounded linear operator $B : \mathcal{H} \to \mathcal{H}$ is said to be σ-strongly positive if there exists a constant $\sigma > 0$ such that $\langle B(x), x \rangle \ge \sigma \|x\|^2, \ \forall x \in \mathcal{H}$.

Let $g \colon \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ be a mapping. g is said to be monotone if and only if $\forall p, q \in \mathcal{H}$, $\langle p - q, u - v \rangle \ge 0$ where $u \in g(p)$ and $v \in g(q)$. A monotone operator g is said to be maximal monotone if and only if its graph is not strictly contained in the graph of any other monotone operator. Let $g \colon \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator. Define its resolvent $\mathrm{Res}^g_\varpi : \mathcal{H} \to \mathcal{H}$ by $\mathrm{Res}^g_\varpi := (I + \varpi g)^{-1}$ where $\varpi > 0$ is any constant. It is well-known that Res^g_ϖ is a single-valued operator and $x \in (g+h)^{-1}(0) \Leftrightarrow x = \mathrm{Res}^g_\varpi (I - \varpi h)(x)$.

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Recall the orthogonal projection $\mathrm{proj}_C : \mathcal{H} \to C$, denoted by $\mathrm{proj}_C(x) := \arg\min_{y \in C} \|x - y\|$ satisfies the following inequality ([40])

$$x \in \mathcal{H}, \ \langle x - \mathrm{proj}_C(x), y - \mathrm{proj}_C(x) \rangle \le 0, \ \forall y \in C. \tag{7}$$

In any Hilbert space $\mathcal{H}$, $\forall u, u^\dagger \in \mathcal{H}$ and $\forall \varsigma \in \mathcal{R}$, we have

$$\|\varsigma u + (1-\varsigma) u^\dagger\|^2 = \varsigma \|u\|^2 + (1-\varsigma)\|u^\dagger\|^2 - \varsigma(1-\varsigma)\|u - u^\dagger\|^2. \tag{8}$$

Lemma 2.1 ([30, 47]). *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $f : C \to C$ be an L-Lipschitz pseudocontractive operator. Then,*

(i) f is demiclosed, i.e., $u_n \rightharpoonup \tilde{z}$ and $f(u_n) \to z^\dagger \Rightarrow f(\tilde{z}) = z^\dagger$.

(ii) $\forall \tilde{u} \in C$ and $u^\dagger \in \mathrm{Fix}(f)$, we have

$$\|f[(1-\tau)\tilde{u} + \tau f(\tilde{u})] - u^\dagger\|^2 \leq \|\tilde{u} - u^\dagger\|^2 + (1-\tau)\|f[(1-\tau)\tilde{u} + \tau f(\tilde{u})] - \tilde{u}\|^2,$$

where $0 < \tau < \frac{1}{\sqrt{1+L^2}+1}$.

Lemma 2.2 ([26]). *Suppose that $\{a_n\} \subset (0, \infty)$, $\{b_n\} \subset (0, 1)$ and $\{c_n\}$ are three real number sequences. Suppose that $a_{n+1} \leq (1 - b_n)a_n + a_n c_n, \forall n \geq 0$, $\sum_{n=1}^{\infty} b_n = \infty$ and $\limsup_{n\to\infty} c_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.*

3. **Main results**

In this section, we introduce an iterative algorithm and prove that it converges strongly to an element in $\mathrm{Fix}(f) \cap (g+h)^{-1}$. Let $\mathcal{H}$ be a real Hilbert space. Let $f : \mathcal{H} \to \mathcal{H}$ be an L_1-Lipschitz pseudocontractive operator and $g : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator. Let $h : \mathcal{H} \to \mathcal{H}$ be an L_2-Lipschitz monotone operator and $\varphi : \mathcal{H} \to \mathcal{H}$ be a κ-contractive operator. Let $B : \mathcal{H} \to \mathcal{H}$ be a σ-strong positive bounded linear operator. Suppose that $\Omega := \mathrm{Fix}(f) \cap (g+h)^{-1} \neq \emptyset$.

Let $\{\varsigma_n\}_{n=0}^{\infty} \subset (0, 1]$, $\{\tau_n\}_{n=0}^{\infty} \subset (0, 1)$, $\{\lambda_n\}_{n=0}^{\infty} \subset (0, 1)$ and $\{\gamma_n\}_{n=0}^{\infty} \subset (0, 1)$ be four real number sequences. Let $\delta \in (0, 1)$ and $\alpha \in (0, \sigma/\kappa)$ be two constants. In what follows, suppose that $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=0}^{\infty} \gamma_n = \infty$, $\{\varsigma_n\}_{n=0}^{\infty} \subset [\underline{\varsigma}, \bar{\varsigma}] \subset (0, 1]$ and $0 < \vartheta < \lambda_n < \tau_n < \frac{1}{\sqrt{1+L_1^2}+1} (\forall n \geq 0)$.

Next, we present an iterative algorithm for finding an element in Ω.

Algorithm 3.1. *Let $x_0 \in \mathcal{H}$ be an initial value and $\varpi_0 > 0$ be a fixed constant.*

Step 1. Let x_n be known. Compute

$$\begin{cases} z_n = \mathrm{Res}^g_{\varpi_n}(x_n - \varpi_n h(x_n)), \\ y_n = z_n - \varpi_n(h(z_n) - h(x_n)), \\ u_n = (1-\varsigma_n)x_n + \varsigma_n y_n. \end{cases} \tag{9}$$

Step 2. Compute

$$\begin{cases} v_n = (1-\tau_n)u_n + \tau_n f(u_n), \\ w_n = (1-\lambda_n)u_n + \lambda_n f(v_n). \end{cases} \tag{10}$$

Step 3. Compute

$$x_{n+1} = \alpha\gamma_n\varphi(x_n) + (I - \gamma_n B)w_n. \tag{11}$$

Step 4. Update

$$\varpi_{n+1} = \begin{cases} \min\left\{\varpi_n, \frac{\delta\|x_n - z_n\|}{\|h(x_n) - h(z_n)\|}\right\}, & \text{if } x_n \neq z_n, \\ \varpi_n, & \text{else}. \end{cases} \tag{12}$$

Replace n by $n+1$ and return to Step 1.

Remark 3.1. *We have the following statements: (i) $\mathrm{Res}^g_{\varpi_n}(x_n - \varpi_n h(x_n)) = x_n \Rightarrow x_n \in (g+h)^{-1}(0)$. (ii) $\varpi_n \geq \varpi_{n+1} \geq \min\{\varpi_0, \frac{\delta}{L_2}\}$ and $\lim_{n\to\infty} \varpi_n = \varpi > 0$. (iii) By the assumptions, we deduce that $\mathrm{proj}_\Omega \circ (I - B + \alpha\varphi)$ is contractive. Thus, $\mathrm{proj}_\Omega \circ (I - B + \alpha\varphi)$ has a unique fixed point in Ω, denoted by $z^\dagger$. Therefore, $\langle \alpha\varphi(z^\dagger) - B(z^\dagger), x - z^\dagger\rangle \leq 0$, $\forall x \in \Omega$.*

Proposition 3.1. *The sequence $\{x_n\}$ generated by Algorithm 3.1 is bounded.*

Proof. From (9), we have

$$\begin{aligned}\|y_n - z^\dagger\|^2 &= \|z_n - z^\dagger - \varpi_n(h(z_n) - h(x_n))\|^2 \\ &= \|z_n - z^\dagger\|^2 - 2\varpi_n\langle h(z_n) - h(x_n), z_n - z^\dagger\rangle + \varpi_n^2\|h(x_n) - h(z_n)\|^2.\end{aligned} \tag{13}$$

Observe that $\|z_n - z^\dagger\|^2 = \|x_n - z^\dagger\|^2 + 2\langle z_n - x_n, z_n - z^\dagger\rangle - \|x_n - z_n\|^2$. By (13), we get

$$\begin{aligned}\|y_n - z^\dagger\|^2 &= \|x_n - z^\dagger\|^2 - 2\varpi_n\langle h(z_n) - h(x_n), z_n - z^\dagger\rangle - \|x_n - z_n\|^2 \\ &\quad + 2\langle z_n - x_n, z_n - z^\dagger\rangle + \varpi_n^2\|h(x_n) - h(z_n)\|^2 \\ &= \|x_n - z^\dagger\|^2 + \varpi_n^2\|h(x_n) - h(z_n)\|^2 - \|x_n - z_n\|^2 \\ &\quad + 2\langle z_n - x_n - \varpi_n(h(z_n) - h(x_n)), z_n - z^\dagger\rangle.\end{aligned} \tag{14}$$

Since $z_n = (I + \varpi_n g)^{-1}(x_n - \varpi_n h(x_n))$,

$$x_n - \varpi_n h(x_n) \in z_n + \varpi_n g(z_n). \tag{15}$$

It yields

$$x_n - z_n - \varpi_n(h(x_n) - h(z_n)) \in \varpi_n(g(z_n) + h(z_n)). \tag{16}$$

Since $z^\dagger \in (g+h)^{-1}$, we have $0 \in \varpi_n(g+h)z^\dagger$. By the monotonicity of $\varpi_n(g+h)$ and (16), we obtain

$$\langle x_n - z_n - \varpi_n(h(x_n) - h(z_n)), z_n - z^\dagger\rangle \geq 0. \tag{17}$$

According to (12), we have $\|h(x_n) - h(z_n)\| \leq \frac{\delta\|x_n - z_n\|}{\varpi_{n+1}}$. Combining (14) with (17), we acquire

$$\|y_n - z^\dagger\|^2 \leq \|x_n - z^\dagger\|^2 - \big(1 - \delta^2\frac{\varpi_n^2}{\varpi_{n+1}^2}\big)\|x_n - z_n\|^2. \tag{18}$$

In the light of (7) and (9), we achieve

$$\begin{aligned}\|u_n - z^\dagger\|^2 &= \|(1 - \varsigma_n)(x_n - z^\dagger) + \varsigma_n(y_n - z^\dagger)\|^2 \\ &= (1 - \varsigma_n)\|x_n - z^\dagger\|^2 + \varsigma_n\|y_n - z^\dagger\|^2 - (1 - \varsigma_n)\varsigma_n\|x_n - y_n\|^2.\end{aligned}$$

This together with (18) implies that

$$\begin{aligned}\|u_n - z^\dagger\|^2 &\leq \|x_n - z^\dagger\|^2 - \varsigma_n\big(1 - \delta^2\frac{\varpi_n^2}{\varpi_{n+1}^2}\big)\|x_n - z_n\|^2 - (1 - \varsigma_n)\varsigma_n\|x_n - y_n\|^2 \\ &\leq \|x_n - z^\dagger\|^2.\end{aligned} \tag{19}$$

From (7), we gain

$$\begin{aligned}\|w_n - z^\dagger\|^2 &= \|(1 - \lambda_n)(u_n - z^\dagger) + \lambda_n(f(v_n) - z^\dagger)\|^2 \\ &= (1 - \lambda_n)\|u_n - z^\dagger\|^2 - \lambda_n(1 - \lambda_n)\|f(v_n) - u_n\|^2 + \lambda_n\|f(v_n) - z^\dagger\|^2.\end{aligned} \tag{20}$$

Applying Lemma 2.1, we deduce

$$\begin{aligned}\|f(v_n) - z^\dagger\|^2 &= \|f[(1 - \tau_n)u_n + \tau_n f(u_n)] - z^\dagger\|^2 \\ &\leq \|u_n - z^\dagger\|^2 + (1 - \tau_n)\|f(v_n) - u_n\|^2.\end{aligned} \tag{21}$$

Substituting (21) into (20), we receive

$$\begin{aligned}\|w_n - z^\dagger\|^2 &\leq (1 - \lambda_n)\|u_n - z^\dagger\|^2 - \lambda_n(1 - \lambda_n)\|f(v_n) - u_n\|^2 \\ &\quad + \lambda_n\|u_n - z^\dagger\|^2 + \lambda_n(1 - \tau_n)\|f(v_n) - u_n\|^2 \\ &= \|u_n - z^\dagger\|^2 - \lambda_n(\tau_n - \lambda_n)\|f(v_n) - u_n\|^2 \\ &\leq \|u_n - z^\dagger\|^2.\end{aligned} \tag{22}$$

Taking into account (11), (19) and (22), we derive

$$\begin{aligned}
\|x_{n+1}-z^\dagger\| &= \|\alpha\gamma_n\varphi(x_n)+(I-\gamma_n B)w_n-z^\dagger\| \\
&= \|\alpha\gamma_n(\varphi(x_n)-\varphi(z^\dagger))+(I-\gamma_n B)(w_n-z^\dagger)+\gamma_n(\alpha\varphi(z^\dagger)-B(z^\dagger))\| \\
&\le \alpha\gamma_n\|\varphi(x_n)-\varphi(z^\dagger)\|+|I-\gamma_n B|\|w_n-z^\dagger\|+\gamma_n\|\alpha\varphi(z^\dagger)-B(z^\dagger)\| \\
&\le \alpha\gamma_n\kappa\|x_n-z^\dagger\|+(1-\sigma\gamma_n)\|w_n-z^\dagger\|+\gamma_n\|\alpha\varphi(z^\dagger)-B(z^\dagger)\| \\
&\le [1-(\sigma-\alpha\kappa)\gamma_n]\|x_n-z^\dagger\|+\gamma_n\|\alpha\varphi(z^\dagger)-B(z^\dagger)\|.
\end{aligned} \tag{23}$$

By induction, we have $\|x_n-z^\dagger\|\le\max\{\|\alpha\varphi(z^\dagger)-B(z^\dagger)\|/(\sigma-\alpha\kappa),\|x_0-z^\dagger\|\}$ and the sequence $\{x_n\}$ is bounded. □

Proposition 3.2. *$\omega_w(x_n)\subset \mathrm{Fix}(f)$.*

Proof. On account of (11), we achieve

$$\begin{aligned}
\|x_{n+1}-z^\dagger\|^2 &= \|\alpha\gamma_n(\varphi(x_n)-\varphi(z^\dagger))+(I-\gamma_n B)(w_n-z^\dagger)+\gamma_n(\alpha\varphi(z^\dagger)-B(z^\dagger))\|^2 \\
&\le \|(I-\gamma_n B)(w_n-z^\dagger)\|^2+2\alpha\gamma_n\langle\varphi(x_n)-\varphi(z^\dagger),x_{n+1}-z^\dagger\rangle \\
&\quad +2\gamma_n\langle\alpha\varphi(z^\dagger)-B(z^\dagger),x_{n+1}-z^\dagger\rangle \\
&\le (1-\sigma\gamma_n)^2\|w_n-z^\dagger\|^2+2\alpha\kappa\gamma_n\|x_n-z^\dagger\|\|x_{n+1}-z^\dagger\| \\
&\quad +2\gamma_n\langle\alpha\varphi(z^\dagger)-B(z^\dagger),x_{n+1}-z^\dagger\rangle \\
&\le (1-\sigma\gamma_n)^2\|w_n-z^\dagger\|^2+\alpha\kappa\gamma_n\|x_n-z^\dagger\|^2+\alpha\kappa\gamma_n\|x_{n+1}-z^\dagger\|^2 \\
&\quad +2\gamma_n\langle\alpha\varphi(z^\dagger)-B(z^\dagger),x_{n+1}-z^\dagger\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1}-z^\dagger\|^2 \le{}& \frac{(1-\sigma\gamma_n)^2}{1-\alpha\kappa\gamma_n}\|w_n-z^\dagger\|^2+\frac{\alpha\kappa\gamma_n}{1-\alpha\kappa\gamma_n}\|x_n-z^\dagger\|^2 \\
&+\frac{2\gamma_n}{1-\alpha\kappa\gamma_n}\langle\alpha\varphi(z^\dagger)-B(z^\dagger),x_{n+1}-z^\dagger\rangle.
\end{aligned} \tag{24}$$

Substituting (19) and (22) into (24), we attain

$$\begin{aligned}
\|x_{n+1}-z^\dagger\|^2 \le{}& \frac{(1-\sigma\gamma_n)^2}{1-\alpha\kappa\gamma_n}\big(\|x_n-z^\dagger\|^2-\varsigma_n\big(1-\delta^2\frac{\varpi_n^2}{\varpi_{n+1}^2}\big)\|x_n-z_n\|^2 \\
&-(1-\varsigma_n)\varsigma_n\|x_n-y_n\|^2-\lambda_n(\tau_n-\lambda_n)\|f(v_n)-u_n\|^2\big) \\
&+\frac{\alpha\kappa\gamma_n}{1-\alpha\kappa\gamma_n}\|x_n-z^\dagger\|^2+\frac{2\gamma_n}{1-\alpha\kappa\gamma_n}\langle\alpha\varphi(z^\dagger)-B(z^\dagger),x_{n+1}-z^\dagger\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1}-z^\dagger\|^2 \le{}& [1-\frac{2(\sigma-\alpha\kappa)\gamma_n}{1-\alpha\kappa\gamma_n}]\|x_n-z^\dagger\|^2+\frac{2(\sigma-\alpha\kappa)\gamma_n}{1-\alpha\kappa\gamma_n}\times\big(-\frac{(1-\sigma\gamma_n)^2}{2(\sigma-\alpha\kappa)} \\
&\times\varsigma_n\big(1-\delta^2\frac{\varpi_n^2}{\varpi_{n+1}^2}\big)\frac{\|x_n-z_n\|^2}{\gamma_n}-\frac{(1-\sigma\gamma_n)^2}{2(\sigma-\alpha\kappa)}(1-\varsigma_n)\varsigma_n\frac{\|x_n-y_n\|^2}{\gamma_n} \\
&-\frac{(1-\sigma\gamma_n)^2}{2(\sigma-\alpha\kappa)}\lambda_n(\tau_n-\lambda_n)\frac{\|f(v_n)-u_n\|^2}{\gamma_n}+\frac{\sigma^2\gamma_n}{2(\sigma-\alpha\kappa)}M \\
&+\frac{1}{\sigma-\alpha\kappa}\langle\alpha\varphi(z^\dagger)-B(z^\dagger),x_{n+1}-z^\dagger\rangle\big),
\end{aligned} \tag{25}$$

where M is a constant such that $M\ge\sup_n\{\|x_n-z^\dagger\|^2+\|x_n-x^*\|\}$.

Put $a_n = \|x_n - z^\dagger\|^2$, $b_n = \frac{2(\sigma-\alpha\kappa)\gamma_n}{1-\alpha\kappa\gamma_n}$ and

$$\begin{aligned}c_n = &-\frac{(1-\sigma\gamma_n)^2}{2(\sigma-\alpha\kappa)}\varsigma_n\big(1-\delta^2\frac{\varpi_n^2}{\varpi_{n+1}^2}\big)\frac{\|x_n-z_n\|^2}{\gamma_n} - \frac{(1-\sigma\gamma_n)^2}{2(\sigma-\alpha\kappa)}(1-\varsigma_n)\varsigma_n\frac{\|x_n-y_n\|^2}{\gamma_n}\\ &-\frac{(1-\sigma\gamma_n)^2}{2(\sigma-\alpha\kappa)}\lambda_n(\tau_n-\lambda_n)\frac{\|f(v_n)-u_n\|^2}{\gamma_n}+\frac{\sigma^2\gamma_n}{2(\sigma-\alpha\kappa)}M\\ &+\frac{1}{\sigma-\alpha\kappa}\langle\alpha\varphi(z^\dagger)-B(z^\dagger),x_{n+1}-z^\dagger\rangle,\end{aligned}\tag{26}$$

for all $n \geq 0$.

Owing to (25) and (26), we have

$$a_{n+1} \leq (1-b_n)a_n + b_n c_n,\ \forall n \geq 0.\tag{27}$$

Next, we show $-1 \leq \limsup_{n\to\infty} c_n < +\infty$. First, from (26), we have

$$\begin{aligned}c_n &\leq \frac{\sigma^2\gamma_n}{2(\sigma-\alpha\kappa)}M+\frac{1}{\sigma-\alpha\kappa}\langle\alpha\varphi(z^\dagger)-B(z^\dagger),x_{n+1}-z^\dagger\rangle\\ &\leq \frac{\sigma^2 M}{2(\sigma-\alpha\kappa)}+\frac{M}{\sigma-\alpha\kappa}\|\alpha\varphi(z^\dagger)-B(z^\dagger)\|,\end{aligned}$$

which implies that $\limsup_{n\to\infty} c_n < +\infty$. Now, we show $\limsup_{n\to\infty} c_n \geq -1$ by reduction to absurdity. Suppose that $\limsup_{n\to\infty} c_n < -1$. Then there exists N_0 such that $c_n < -1, \forall n \geq N_0$. On account of (27), we achieve $a_{n+1} \leq a_n - b_n$ when $n \geq N_0$. This implies that $a_{n+1} \leq a_{N_0} - \sum_{k=N_0}^{n} b_k$. It follows that $\limsup_{n\to\infty} a_n \leq a_{N_0} - \limsup_{n\to\infty}\sum_{k=N_0}^{n} b_k = -\infty$. This contradicts the hypothesis. So, $-1 \leq \limsup_{n\to\infty} c_n < +\infty$. Choose any $x^\dagger \in \omega_w(x_n)$. Then, there is a subsequence $\{n_i\} \subset \{n\}$ such that $x_{n_i} \rightharpoonup x^\dagger$ as $i \to \infty$ and

$$\begin{aligned}\limsup_{n\to\infty} c_n = \lim_{i\to\infty} c_{n_i} = \lim_{i\to\infty}\Big[&-\frac{(1-\sigma\gamma_{n_i})^2}{2(\sigma-\alpha\kappa)}\varsigma_{n_i}\big(1-\delta^2\frac{\varpi_{n_i}^2}{\varpi_{n_i+1}^2}\big)\frac{\|x_{n_i}-z_{n_i}\|^2}{\gamma_{n_i}}\\ &-\frac{(1-\sigma\gamma_{n_i})^2}{2(\sigma-\alpha\kappa)}(1-\varsigma_{n_i})\varsigma_{n_i}\frac{\|x_{n_i}-y_{n_i}\|^2}{\gamma_{n_i}}+\frac{1}{\sigma-\alpha\kappa}\langle\alpha\varphi(z^\dagger)-B(z^\dagger),x_{n_i+1}-z^\dagger\rangle\\ &-\frac{(1-\sigma\gamma_{n_i})^2}{2(\sigma-\alpha\kappa)}\lambda_{n_i}(\tau_{n_i}-\lambda_{n_i})\frac{\|f(v_{n_i})-u_{n_i}\|^2}{\gamma_{n_i}}+\frac{\sigma^2\gamma_{n_i}}{2(\sigma-\alpha\kappa)}M\Big].\end{aligned}\tag{28}$$

Since the sequence $\{x_{n_i+1}\}$ is bounded, there exists a subsequence of $\{x_{n_i+1}\}$, without loss of generality, still denoted by $\{x_{n_i+1}\}$ such that $x_{n_i+1} \rightharpoonup \hat{x}(i\to\infty)$. Thus, $\lim_{i\to\infty}\langle\alpha\varphi(z^\dagger)-B(z^\dagger),x_{n_i+1}-z^\dagger\rangle = \langle\alpha\varphi(z^\dagger)-B(z^\dagger),\hat{x}-z^\dagger\rangle$ exists. It follows from (28) that

$$\begin{aligned}\lim_{i\to\infty}\Big[&-\frac{(1-\sigma\gamma_{n_i})^2}{2(\sigma-\alpha\kappa)}\varsigma_{n_i}\big(1-\delta^2\frac{\varpi_{n_i}^2}{\varpi_{n_i+1}^2}\big)\frac{\|x_{n_i}-z_{n_i}\|^2}{\gamma_{n_i}}-\frac{(1-\sigma\gamma_{n_i})^2}{2(\sigma-\alpha\kappa)}\\ &\times(1-\varsigma_{n_i})\varsigma_{n_i}\frac{\|x_{n_i}-y_{n_i}\|^2}{\gamma_{n_i}}-\frac{(1-\sigma\gamma_{n_i})^2}{2(\sigma-\alpha\kappa)}\lambda_{n_i}(\tau_{n_i}-\lambda_{n_i})\frac{\|f(v_{n_i})-u_{n_i}\|^2}{\gamma_{n_i}}\Big]\end{aligned}\tag{29}$$

exists. Since $\lim_{i\to\infty}\gamma_{n_i} = 0$, from (29), we deduce

$$\lim_{i\to\infty}\|x_{n_i}-z_{n_i}\| = 0,\tag{30}$$

$$\lim_{i\to\infty}\|x_{n_i}-y_{n_i}\| = 0,\tag{31}$$

and

$$\lim_{i\to\infty}\|f(v_{n_i})-u_{n_i}\| = 0.\tag{32}$$

Since h is L_2-Lipschitz continuous, by (30), we have

$$\lim_{i\to\infty}\|h(z_{n_i})-h(x_{n_i})\| = 0.\tag{33}$$

Owing to $u_{n_i} - x_{n_i} = \varsigma_{n_i}(y_{n_i} - x_{n_i})$ and (31), we derive

$$\lim_{i\to\infty} \|u_{n_i} - x_{n_i}\| = 0. \tag{34}$$

In accordance with (10), we receive

$$\begin{aligned}\|u_{n_i} - f(u_{n_i})\| &\leq \|u_{n_i} - f(v_{n_i})\| + \|f(v_{n_i}) - f(u_{n_i})\| \\ &\leq \|u_{n_i} - f(v_{n_i})\| + L_1\|v_{n_i} - u_{n_i}\| \\ &= \|u_{n_i} - f(v_{n_i})\| + L_1\tau_{n_i}\|u_{n_i} - f(u_{n_i})\|,\end{aligned}$$

which yields

$$\|u_{n_i} - f(u_{n_i})\| \leq \frac{1}{1 - L_1\tau_{n_i}}\|u_{n_i} - f(v_{n_i})\|.$$

This together with (32) implies that

$$\lim_{i\to\infty} \|u_{n_i} - f(u_{n_i})\| = 0. \tag{35}$$

By (34), we have $u_{n_i} \rightharpoonup x^\dagger$ because $x_{n_i} \rightharpoonup x^\dagger$. Applying Lemma 2.1 to (35), we deduce $x^\dagger \in \mathrm{Fix}(f)$. Therefore, $\omega_w(x_n) \subset \mathrm{Fix}(f)$. □

Proposition 3.3. $\omega_w(x_n) \subset (g + h)^{-1}(0)$.

Proof. Pick any $(u^\dagger, v^\dagger) \in \mathrm{Graph}(g + h)$. Then,

$$v^\dagger - h(u^\dagger) \in g(u^\dagger). \tag{36}$$

Thanks to (15), we have

$$\frac{x_{n_i} - z_{n_i}}{\varpi_{n_i}} - h(x_{n_i}) \in g(z_{n_i}). \tag{37}$$

Since g is monotone, from (36) and (37), we acquire

$$\langle v^\dagger - h(u^\dagger) - (\frac{x_{n_i} - z_{n_i}}{\varpi_{n_i}} - h(x_{n_i})), u^\dagger - z_{n_i}\rangle \geq 0.$$

It follows that

$$\begin{aligned}\langle v^\dagger, u^\dagger - z_{n_i}\rangle &\geq \langle h(u^\dagger) - h(x_{n_i}) + \frac{x_{n_i} - z_{n_i}}{\varpi_{n_i}}, u^\dagger - z_{n_i}\rangle \\ &= \langle h(u^\dagger) - h(z_{n_i}), u^\dagger - z_{n_i}\rangle + \langle h(z_{n_i}) - h(x_{n_i}), u^\dagger - z_{n_i}\rangle \\ &\quad + \frac{1}{\varpi_{n_i}}\langle x_{n_i} - z_{n_i}, u^\dagger - z_{n_i}\rangle.\end{aligned} \tag{38}$$

Since h is monotone, $\langle h(u^\dagger) - h(z_{n_i}), u^\dagger - z_{n_i}\rangle \geq 0$. By (38), we get

$$\langle v^\dagger, u^\dagger - z_{n_i}\rangle \geq \langle h(z_{n_i}) - h(x_{n_i}), u^\dagger - z_{n_i}\rangle + \frac{1}{\varpi_{n_i}}\langle x_{n_i} - z_{n_i}, u^\dagger - z_{n_i}\rangle. \tag{39}$$

Owing to $x_{n_i} \rightharpoonup x^\dagger$, from (30), we also have $z_{n_i} \rightharpoonup x^\dagger$. Taking into account (30), and (33), we conclude that $\langle v^\dagger, u^\dagger - x^\dagger\rangle \geq 0$ for all $(u^\dagger, v^\dagger) \in \mathrm{Graph}(g + h)$. Thus, $x^\dagger \in (g + h)^{-1}$. Therefore, $\omega_w(x_n) \in (g + h)^{-1}$. □

Finally, we demonstrate the convergence of the sequence $\{x_n\}$ generated by Algorithm 3.1.

Theorem 3.1. *The sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $z^\dagger$.*

Proof. First, by Propositions 3.2 and 3.3, we have $\omega_w(x_n) \in \Omega$. From (10) and (11), we obtain

$$\begin{aligned} x_{n_i+1} - x_{n_i} &= \gamma_{n_i}(\alpha\varphi(x_{n_i}) - B(w_{n_i})) + (w_{n_i} - x_{n_i}) \\ &= \gamma_{n_i}(\alpha\varphi(x_{n_i}) - B(w_{n_i})) + (1 - \lambda_{n_i})(u_{n_i} - x_{n_i}) \\ &\quad + \lambda_{n_i}(f(v_{n_i}) - u_{n_i} + u_{n_i} - x_{n_i}). \end{aligned}$$

This together with (32) and (34) implies that $\|x_{n_i+1} - x_{n_i}\| \to 0$. Hence, $x_{n_i+1} \rightharpoonup x^\dagger$ $(i \to \infty)$. According to (28), we derive

$$\begin{aligned} \limsup_{n\to\infty} c_n &\leq \lim_{i\to\infty} \big(\frac{\sigma^2\gamma_{n_i}}{2(\sigma - \alpha\kappa)}M + \frac{1}{\sigma - \alpha\kappa}\langle \alpha\varphi(z^\dagger) - B(z^\dagger), x_{n_i+1} - z^\dagger\rangle\big) \\ &= \frac{1}{\sigma - \alpha\kappa}\langle \alpha\varphi(z^\dagger) - B(z^\dagger), x^\dagger - z^\dagger\rangle \leq 0. \end{aligned} \tag{40}$$

From (25), we obtain

$$\|x_{n+1} - z^\dagger\|^2 \leq [1 - \frac{2(\sigma - \alpha\kappa)\gamma_n}{1 - \alpha\kappa\gamma_n}]\|x_n - z^\dagger\|^2 + \frac{2(\sigma - \alpha\kappa)\gamma_n}{1 - \alpha\kappa\gamma_n}c_n, \tag{41}$$

It is obviously that $\frac{2(\sigma-\alpha\kappa)\gamma_n}{1-\alpha\kappa\gamma_n} \to 0$ and $\sum_n \frac{2(\sigma-\alpha\kappa)\gamma_n}{1-\alpha\kappa\gamma_n} = +\infty$. By Lemma 2.2, (40) and (41), we conclude that $x_n \to z^\dagger$. This completes the proof. □

4. **Conclusions**

In this paper, we propose a composite splitting algorithm [Algorithm 3.1] for solving variational inclusion problem (1) and fixed point problem (4) in a real Hilbert space $\mathcal{H}$. The introduced algorithm [Algorithm 3.1] consists of forward-backward algorithm (3), Tseng-type algorithm (9) and self-adaptive rule (12). Under some additional conditions, we prove that the sequence $\{x_n\}$ generated by the splitting algorithm [Algorithm 3.1] strongly converges to a point $z^\dagger = \text{proj}_\Omega \circ (I - B + \alpha\varphi)z^\dagger$ which solves the variational inequality $\langle \alpha\varphi(z^\dagger) - B(z^\dagger), x - z^\dagger\rangle \leq 0$, $\forall x \in \text{Fix}(f) \cap (g + h)^{-1}(0)$.

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