A NEW BERNOULLI SUB-ODE METHOD FOR CONSTRUCTING TRAVELING WAVE SOLUTIONS FOR TWO NONLINEAR EQUATIONS WITH ANY ORDER

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In this paper, a new generalized Bernoulli sub-ODE method is proposed to construct exact solutions of nonlinear equations. The validity of the method is testified by finding new exact traveling wave solutions of the BBM equation with any order and general Gardner equation.

Keywords: sub-ODE method, traveling wave solution, BBM equation with any order, general Gardner equation, nonlinear equation.
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1. Introduction

The nonlinear phenomena exist in all the fields including either the scientific work or engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and so on. It is well known that many nonlinear evolution equations (NLEEs) are widely used to describe these complex phenomena. So, the powerful and efficient methods to find analytic solutions of nonlinear equations have drawn a lot of interest by a diverse group of scientists. Among the possible solutions to NLEEs, certain solutions for special form may depend only on a single combination of variables such as traveling wave variables. In the literature, there is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions. Some of these approaches are the homogeneous balance method [1,2], the hyperbolic tangent expansion method [3,4], the trial function method [5], the tanh-method [6-8], the non-linear transform method [9], the inverse scattering transform [10], the Backlund transform [11,12], the Hirotas bilinear method [13,14], the generalized Riccati equation [15,16], the Weierstrass elliptic function method [17], the theta function method [18-20], the sineCcosine method [21], the Jacobi elliptic function expansion [22,23], the complex hyperbolic function method [24-26], the truncated Painleve expansion [27], the F-expansion method [28], the rank analysis method [29], the exp-function expansion method [30] and so on.

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In this paper, we propose a Bernoulli sub-ODE method to construct exact traveling wave solutions for NLEEs. Firstly, we reduce the NLEEs to ODEs by a traveling wave variable transformation. Secondly, we suppose the solution can be expressed in a polynomial in a variable G, where G satisfies the Bernoulli equation. Thirdly, the degree of the polynomial can be determined by the homogeneous balance method, and the coefficients can be obtained by solving a set of algebraic equations.

The rest of the paper is organized as follows. In Section 2, we describe the Bernoulli sub-ODE method for finding traveling wave solutions of nonlinear evolution equations, and give the main steps of the method. In the subsequent sections, we will apply the method to find exact traveling wave solutions of the BBM equation with any order and general Gardner equation. In the last Section, some conclusions are presented.

2. **Description Of The Bernoulli Sub-ODE Method**

In this section we present the solutions of the following ODE:

$$G' + \lambda G = \mu G^2 \tag{1}$$

where $\lambda \neq 0$.

When $\mu \neq 0$, the Eq. (1) is the type of Bernoulli equation, and we can obtain the solution as

$$G = \frac{1}{\frac{\mu}{\lambda} + de^{\lambda \xi}}, \tag{2}$$

where d is an arbitrary constant.

When $\mu = 0$, the solution of Eq. (1) is denoted by

$$G = de^{-\lambda \xi}, \tag{3}$$

where d is an arbitrary constant.

Suppose that a nonlinear equation, say in two independent variables $x,\ t$, is given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{4}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. By using the solutions of Eq. (1), we can construct a serials of exact solutions of nonlinear equations.

Step 1. We suppose that

$$u(x,t) = u(\xi), \quad \xi = k(x - ct). \tag{5}$$

The traveling wave variable (5) permits us reducing (4) to an ODE for $u = u(\xi)$

$$P(u,\ u', u'',\ ...) = 0. \tag{6}$$

Step 2. Suppose that the solution of (6) can be expressed by a polynomial in G as follows:

$$u(\xi) = a_m G^m + a_{m-1} G^{m-1} +\ ..., \tag{7}$$

where $G = G(\xi)$ satisfies Eq. (1), and $a_m, a_{m-1}...$, and μ are constants to be determined later, $a_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (6).

Step 3. Substituting (7) into (6) and using (1), collecting all terms with the same order of G together, the left-hand side of (6) is converted into another polynomial in G. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_m, a_{m-1}..., k, c, \lambda$ and μ.

Step 4. Solving the algebraic equations system in Step 3, and by using the solutions of Eq. (1), we can construct the traveling wave solutions of the nonlinear evolution equation (6).

In the following sections, we will apply the method described above to some examples.

3. Application Of The Bernoulli Sub-ODE Method For BBM Equation With Any Order

In this section we will consider the BBM equation with any order [31]:

$$u_t + au_x + bu^n u_x - ru_{xxt} = 0, \quad n > 0, \tag{8}$$

where $a,\ b$ and r are known constants.

In order to obtain the traveling wave solutions of Eq. (8), we suppose that

$$u(x,t) = u(\xi),\ \xi = x - ct, \tag{9}$$

where c is a constant that to be determined later.

By using the wave variable (9), (8) is converted into an ODE

$$-cu' + au' + bu^n u' + cru''' = 0. \tag{10}$$

Suppose that the solution of (10) can be expressed by a polynomial in G as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i G^i, \tag{11}$$

where a_i are constants. Balancing the order of $u^n u'$ and u''' in Eq. (10), we have $mn + m + 1 = m + 3 \ \Rightarrow \ m = \frac{2}{n}$. So we make a variable $u = v^{\frac{2}{n}}$, then (10) is converted into

$$2(a-c+bv)n^2v^2v' + 2cr(2-n)(2-2n)(v')^3 + bcrnvv'v'' + 2n^2crv^2v''' = 0. \tag{12}$$

Suppose that the solution of (12) can be expressed by a polynomial in G as follows:

$$v(\xi) = \sum_{i=0}^{l} b_i G^i, \tag{13}$$

where b_i are constants, $G = G(\xi)$ satisfies Eq. (1). Balancing the order of v^3v' and $(v')^3$ in Eq. (12), we have $4l + 1 = 3l + 3 \ \Rightarrow \ l = 2$. So Eq. (13) can be rewritten as

$$v(\xi) = b_2G^2 + b_1G + b_0,\ b_2 \neq 0, \tag{14}$$

where $b_2,\ b_1,\ b_0$ are constants to be determined later. Then with (1) we can obtain

$$v'(\xi) = 2b_2\mu G^3 + (b_1\mu - 2b_2\lambda)G^2 - b_1\lambda\mu$$
$$v''(\xi) = 6b_2\mu^2G^4 + (2b_1\mu^2 - 10b_2\mu\lambda)G^3 + (-3b_1\mu\lambda + 4b_2\lambda^2)G^2 + b_1\lambda^2G$$
$$v'''(\xi) = 24b_2\mu^3G^5 + (6b_1\mu^3 - 54b_2\mu^2\lambda)G^4 + (-12b_1\mu^2\lambda + 38b_2\mu\lambda^2)G^3 + (7b_1\mu\lambda^2 - 8b_2\lambda^3)G^2 - b_1\lambda^3G.$$

Substituting (14) into (12) and collecting all the terms with the same power of G together and equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

G^9 : $12bcrnb_2^3\mu^3 + 4n^2bb_2^4\mu + 80n^2crb_2^3\mu^3 - 96crnb_2^3\mu^3 + 64crb_2^3\mu^3 = 0$

G^8 : $-192crb_2^3\mu^2\lambda - 32bcrnb_2^3\mu^2\lambda + 22bcrnb_1b_2^2\mu^3 + 96crb_1b_2^2\mu^3 - 4n^2bb_2^4\lambda - 204n^2crb_2^3\mu^2\lambda - 144crnb_1b_2^2\mu^3 + 14n^2bb_2^3b_1\mu + 288crnb_2^3\mu^2lambda + 156n^2crb_1b_2^2\mu^3 = 0$

G^7 : $12n^2bb_0b_2^3\mu + 172n^2crb_2^3\mu\lambda^2 - 4n^2cb_2^3\mu - 72crnb_1^2b_2\mu^3 + 28bcrnb_2^3\mu\lambda^2 - 288crb_1b_2^2\mu^2\lambda - 14n^2bb_2^3b_1\lambda + 12bcrnb_1^2b_2\mu^3 + 96n^2crb_1^2b_2\mu^3 - 58bcrnb_1b_2^2\mu^2\lambda + 432crnb_1b_2^2\mu^2\lambda + 12bcrnb_0b_2^2\mu^3 + 96n^2crb_0b_2^2\mu^3 + 4n^2ab_2^3\mu - 288crnb_2^3\mu\lambda^2 - 384n^2crb_1b_2^2\mu^2\lambda + 48crb_1^2b_2\mu^3 + 192crb_2^3\mu\lambda^2 + 18n^2bb_1^2b_2^2\mu = 0$

G^6 : $4n^2cb_2^3\lambda - 4n^2ab_2^3\lambda - 64crb_2^3\lambda^3 + 8crb_1^3\mu^3 + 30n^2bb_0b_1b_2^2\mu + 10n^2bb_1^3b_2\mu - 12n^2bb_0b_2^3\lambda + 10n^2ab_2^2b_1\mu - 10n^2cb_2^2b_1\mu - 18n^2bb_1^2b_2^2\lambda + 288crb_1b_2^2\mu\lambda^2 - 144crb_1^2b_2\mu^2\lambda - 31bcrnb_1^2b_2\mu^2\lambda - 8bcrnb_2^3\lambda^3 + 2bcrnb_1^3\mu^3 + 50bcrnb_1b_2^2\mu\lambda^2 + 10bcrnb_0b_1\mu^3b_2 - 216n^2crb_0b_2^2\mu^2\lambda - 32bcrnb_0b_2^2\mu^2\lambda - 228n^2crb_1^2b_2\mu^2\lambda - 48n^2crb_2^3\lambda^3 + 16n^2crb_1^3\mu^3 + 310n^2crb_1b_2^2\mu\lambda^2 + 120n^2crb_0b_1\mu^3b_2 + 216crnb_1^2b_2\mu^2\lambda - 432crnb_1b_2^2\mu\lambda^2 - 12crnb_1^3\mu^3 + 96crnb_2^3\lambda^3 = 0$

G^5 : $2n^2bb_1^4\mu + 24n^2bb_0b_1^2b_2\mu - 30n^2bb_0b_1b_2^2\lambda + 12n^2bb_0^2b_2^2\mu - 10n^2bb_1^3b_2\lambda + 8n^2ab_1^2b_2\mu - 8n^2cb_1^2b_2\mu - 10n^2ab_2^2b_1\lambda + 10n^2cb_2^2b_1\lambda + 8n^2ab_0b_2^2\mu - 8n^2cb_0b_2^2\mu - 24crb_1^3\mu^2\lambda - 96crb_1b_2^2\lambda^3 + 144crb_1^2b_2\mu\lambda^2 + 26bcrnb_1^2b_2\mu\lambda^2 - 26bcrnb_0b_1\mu^2b_2\lambda + 48n^2crb_0^2b_2\mu^3 + 152n^2crb_0b_2^2\mu\lambda^2 - 36n^2crb_1^3\mu^2\lambda + 28bcrnb_0b_2^2\mu\lambda^2 - 5bcrnb_1^3\mu^2\lambda - 14bcrnb_1b_2^2\lambda^3 + 2bcrnb_0b_1^2\mu^3 - 264n^2crb_0b_1\mu^2b_2\lambda - 82n^2crb_1b_2^2\lambda^3 + 24n^2crb_0b_1^2\mu^3 + 176n^2crb_1^2b_2\mu\lambda^2 + 144crnb_1b_2^2\lambda^3 - 216crnb_1^2b_2\mu\lambda^2 + 36crnb_1^3\mu^2\lambda = 0$

G^4 : $-2n^2bb_1^4\lambda + 2n^2ab_1^3\mu - 2n^2cb_1^3\mu - 24n^2bb_0b_1^2b_2\lambda + 12n^2ab_0b_1b_2\mu - 12n^2cb_0b_1b_2\mu + 18n^2bb_0^2b_2b_1\mu - 12n^2bb_0^2b_2^2\lambda + 6n^2bb_0b_1^3\mu + 8n^2cb_1^2b_2\lambda - 8n^2ab_1^2b_2\lambda + 8n^2cb_0b_2^2\lambda - 8n^2ab_0b_2^2\lambda + 24crb_1^3\mu\lambda^2 - 48crb_1^2b_2\lambda^3 - 8bcrnb_0b_2^2\lambda^3 - 5bcrnb_0b_1^2\mu^2\lambda + 22bcrnb_0b_1\mu\lambda^2b_2 - 48n^2crb_0b_1^2\mu^2\lambda + 180n^2crb_0b_1\mu\lambda^2b_2 - 32n^2crb_0b_2^2\lambda^3 + 12n^2crb_0^2\mu^3b_1 - 108n^2crb[0]^2b_2\mu^2\lambda + 26n^2crb_1^3\mu\lambda^2 + 4bcrnb_1^3\mu\lambda^2$

$-7bcrnb_1^2b_2\lambda^3 - 44n^2crb_1^2b_2\lambda^3 - 36crnb_1^3\mu\lambda^2 + 72crnb_1^2b_2\lambda^3 = 0$

$G^3:$ $12n^2cb_0b_1b_2\lambda - bcrnb_1^3\lambda^3 + 76n^2crb_0^2b_2\lambda^2\mu - 4n^2cb_0b_1^2\mu - 12n^2ab_0b_1b_2\lambda$
$-6n^2bb_0b_1^3\lambda - 6bcrnb_0b_1\lambda^3b_2 - 36n^2crb_0b_1\lambda^3b_2 - 24n^2crb_0^2\mu^2b_1\lambda + 12crnb_1^3\lambda^3$
$-2n^2ab_1^3\lambda - 6n^2crb_1^3\lambda^3 + 2n^2cb_1^3\lambda - 18n^2bb_0^2b_2b_1\lambda + 4bcrnb_0b_1^2\mu\lambda^2 + 4n^2ab_0^2b_2\mu$
$+4n^2bb_0^3b_2\mu + 6n^2bb_0^2b_1^2\mu + 4n^2ab_0b_1^2\mu + 28n^2crb_0b_1^2\mu\lambda^2 - 8crb_1^3lambda^3$
$-4n^2cb_0^2b_2\mu = 0$

$G^2:$ $-2n^2cb_0^2b_1\mu + 2n^2bb_0^3b_1\mu - 4n^2ab_1^2b_0\lambda - 4n^2ab_0^2b_2\lambda - 16n^2crb_0^2b_2\lambda^3 = 0$

$G^1:$ $-2n^2ab_0^2b_1\lambda - 2n^2bb_0^3b_1\lambda + 2n^2cb_0^2b_1\lambda - 2n^2crb_0^2b_1\lambda = 0.$
Solving the algebraic equations above, yields:

$$c = \frac{-2n^2a}{brn\lambda^2 - 12rn\lambda^2 + 6n^2r\lambda^2 - 2n^2 + 8r\lambda^2},\ b_0 = 0,$$
$$b_1 = \frac{-2(3bn + 20n^2 - 24n + 16)ar\mu\lambda}{b(brn\lambda^2 - 12rn\lambda^2 + 6n^2r\lambda^2 - 2n^2 + 8r\lambda^2)},$$
$$b_2 = \frac{2ar\mu^2(3bn + 20n^2 - 24n + 16)}{b(brn\lambda^2 - 12rn\lambda^2 + 6n^2r\lambda^2 - 2n^2 + 8r\lambda^2)}. \tag{15}$$

Provided that $\mu \neq 0$, combining with (2) and (3) we can obtain the traveling wave solutions of the BBM equation (8) as follows:

$$v(\xi) = \frac{-2(3bn + 20n^2 - 24n + 16)ar\mu\lambda}{b(brn\lambda^2 - 12rn\lambda^2 + 6n^2r\lambda^2 - 2n^2 + 8r\lambda^2)}\left(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}}\right)$$
$$+\frac{2ar\mu^2(3bn + 20n^2 - 24n + 16)}{b(brn\lambda^2 - 12rn\lambda^2 + 6n^2r\lambda^2 - 2n^2 + 8r\lambda^2)}\left(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}}\right)^2. \tag{16}$$

Then

$$u(\xi) = \Big[\frac{-2(3bn + 20n^2 - 24n + 16)ar\mu\lambda}{b(brn\lambda^2 - 12rn\lambda^2 + 6n^2r\lambda^2 - 2n^2 + 8r\lambda^2)}\left(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}}\right)$$
$$+\frac{2ar\mu^2(3bn + 20n^2 - 24n + 16)}{b(brn\lambda^2 - 12rn\lambda^2 + 6n^2r\lambda^2 - 2n^2 + 8r\lambda^2)}\left(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}}\right)^2\Big]^{\frac{2}{n}}, \tag{17}$$

where $\xi = x - [\frac{-2n^2a}{brn\lambda^2 - 12rn\lambda^2 + 6n^2r\lambda^2 - 2n^2 + 8r\lambda^2}]t$, and d is an arbitrary constant.

Remark 3.1. If we take $\mu = 0$, then we obtain trivial solution $u \equiv 0$.

Remark 3.2. In [31], the author has reported the following exact solutions of BBM equation (8).

$$u(x,t) = \{ \begin{array}{l} \frac{aA^2(1+n)(2+n)r}{2b(n^2-A^2r)} sech^2[\frac{1}{2}A(x+\frac{An^2}{-n^2+A^2r}t+C_0)]^{\frac{1}{n}}, if \quad a\neq 0,\ A^2r-n^2\neq 0 \\ \pm\frac{B(1+n)(2+n)\sqrt{r}}{2bn} sech^2[\frac{1}{2}(\pm\sqrt{\frac{n^2}{r}}x+Bt+C_0)]^{\frac{1}{n}}, \quad otherwise \end{array}$$

Also some other exact solutions have been reported in [32-33]. Our result (17) is different from the results in [31-33], and have not been reported in the literature to our best knowledge.

4. Application Of The Bernoulli Sub-ODE Method For General Gardner Equation

We consider the general Gardner equation [32]:

$$u_t + (p + qu^n + ru^{2n})u_x + u_{xxx} = 0, \quad n \geq 0,\ r < 0. \tag{18}$$

When $n = 1, q \neq 0, r \neq 0$, Eq. (18) becomes the KdV-mKdV equation

$$u_t + (p + qu + ru^2)u_x + u_{xxx} = 0.$$

When $n = 1, q \neq 0, r = 0$, Eq. (18) becomes the KdV equation

$$u_t + (p + qu)u_x + u_{xxx} = 0.$$

When $n = 1, q = 0, r \neq 0$, Eq. (18) becomes the mKdV equation

$$u_t + (p + ru^2)u_x + u_{xxx} = 0.$$

In the following, we shall construct exact traveling wave solutions of Eq. (18). In order to obtain the traveling wave solutions of Eq. (18), we suppose that

$$u(x,t) = u(\xi),\ \xi = k(x - \omega t), \tag{19}$$

where k, ω are constants that to be determined later.

By using (19), (18) is converted into an ODE

$$-k\omega u' + k(p + qu^n + ru^{2n})u' + k^3u''' = 0 \tag{20}$$

Suppose that the solution of (20) can be expressed by a polynomial in G as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i G^i \tag{21}$$

where a_i are constants, $G = G(\xi)$ satisfies Eq. (1). Balancing the order of $u^{2n}u'$ and u''' in Eq. (20), we have $2mn + m + 1 = m + 3 \ \Rightarrow\ m = \frac{1}{n}$. So we make a variable $u = v^{\frac{1}{n}}$, then (20) is converted into

$$-k(\omega-p-qv-rv^2)n^2v^2v'+k^3(1-n)(1-2n)(v')^3+3k^3n(1-n)vv'v''+k^3n^2v^2v''' = 0. \tag{22}$$

Suppose that the solution of (22) can be expressed by a polynomial in G as follows:

$$v(\xi) = \sum_{i=0}^{l} b_i G^i, \tag{23}$$

where b_i are constants. Balancing the order of $v^4 v'$ and $(v')^3$ in Eq. (22), we have $4l + l + 1 = 3l + 3 \ \Rightarrow \ l = 1$. So Eq. (23) can be rewritten as

$$v(\xi) = b_1 G + b_0, \ b_1 \neq 0, \tag{24}$$

where $b_1, \ b_0$ are constants to be determined later.

Substituting (24) into (22) and collecting all the terms with the same power of G together and equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

$$G^1: \ kn^2b_1\omega b_0^2\lambda - kn^2b_1pb_0^2\lambda - kn^2b_1rb_0^4\lambda - kn^2b_1qb_0^3\lambda - k^3n^2b_0^2b_1\lambda^3 = 0$$

$$\begin{aligned} G^2: \ & -4kn^2b_1^2rb_0^3\lambda + kn^2b_1pb_0^2\mu + kn^2b_1rb_0^4\mu - 3k^3nb_1^2b_0\lambda^3 - 2kn^2b_1^2pb_0\lambda \\ & +2kn^2b_1^2\omega b_0\lambda - kn^2b_1\omega b_0^2\mu + kn^2b_1qb_0^3\mu + k^3n^2b_1^2b_0\lambda^3 - 3kn^2b_1^2qb_0^2\lambda \\ & +7k^3n^2b_0^2b_1\mu\lambda^2 = 0 \end{aligned}$$

$$\begin{aligned} G^3: \ & -k^3b_1^3\lambda^3 - 3kn^2b_1^3qb_0\lambda + 3kn^2b_1^2qb_0^2\mu + kn^2b_1^3\omega\lambda + 4kn^2b_1^2rb_0^3\mu - 6kn^2b_1^3rb_0^2\lambda \\ & -kn^2b_1^3p\lambda - 2kn^2b_1^2\omega b_0\mu + 2kn^2b_1^2pb_0\mu + 2k^3n^2b_1^2b_0\mu\lambda^2 + 12k^3nb_1^2b_0\mu\lambda^2 \\ & -12k^3n^2b_0^2b_1\mu^2\lambda = 0 \end{aligned}$$

$$\begin{aligned} G^4: \ & -kn^2b_1^3\omega\mu + 6k^3n^2b_0^2b_1\mu^3 + kn^2b_1^3p\mu - kn^2b_1^4q\lambda + 6kn^2b_1^3rb_0^2\mu + 3kn^2b_1^3qb_0\mu \\ & -4kn^2b_1^4rb_0\lambda - 9k^3n^2b_1^2b_0\mu^2\lambda + 3k^3b_1^3n\mu\lambda^2 + k^3b_1^3n^2\mu\lambda^2 - 15k^3nb_1^2b_0\mu^2\lambda \\ & +3k^3b_1^3\mu\lambda^2 = 0 \end{aligned}$$

$$\begin{aligned} G^5: \ & -kn^2b_1^5r\lambda + kn^2b_1^4q\mu + 6k^3n^2b_1^2b_0\mu^3 - 3k^3b_1^3n^2\mu^2\lambda - 6k^3b_1^3n\mu^2\lambda + 6k^3nb_1^2b_0\mu^3 \\ & +4kn^2b_1^4rb_0\mu - 3k^3b_1^3\mu^2\lambda = 0 \end{aligned}$$

$$G^6: \ kn^2b_1^5r\mu + k^3b_1^3\mu^3 + 2k^3b_1^3n^2\mu^3 + 3k^3b_1^3n\mu^3 = 0.$$

Solving the algebraic equations above, yields:

Case 1:

$$\omega = \frac{n^3pr + 5n^2pr + 8npr + 4pr - 2nq^2 - q^2}{r(n^3 + 5n^2 + 8n + 4)}, \ b_0 = 0,$$

$$b_1 = \frac{-\mu q(2n+1)}{\lambda(n+2)r}, \ k = \pm\frac{\sqrt{\frac{-(2n+1)}{nr+r}}qn}{\lambda(n+2)}. \tag{25}$$

Substituting (25) into (24), we have

$$v(\xi) = \frac{-\mu q(2n+1)}{\lambda(n+2)r}G.$$

By (18) and $u = v^{\frac{1}{n}}$, we can obtain the traveling wave solution of general Gardner equation as follows:

When $\mu \neq 0$

$$u_1(\xi) = [\frac{-\mu q(2n+1)}{\lambda(n+2)r}(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}})]^{\frac{1}{n}}, \tag{26}$$

where d is an arbitrary constant,

$$\xi = \pm\frac{\sqrt{\frac{-(2n+1)}{nr+r}}qn}{\lambda(n+2)}(x - \frac{n^3pr + 5n^2pr + 8npr + 4pr - 2nq^2 - q^2}{r(n^3 + 5n^2 + 8n + 4)}t).$$

When $\mu = 0$, we obtain trivial solution $u \equiv 0$.

Case 2:

$$\omega = \frac{n^3pr + 5n^2pr + 8npr + 4pr - 2nq^2 - q^2}{r(n^3 + 5n^2 + 8n + 4)},\ b_0 = \frac{-(2n+1)q}{(n+2)r},$$

$$b_1 = \frac{\mu q(2n+1)}{\lambda(n+2)r},\ k = \pm\frac{\sqrt{\frac{-(2n+1)}{nr+r}}qn}{\lambda(n+2)}. \tag{27}$$

Substituting (27) into (24), we have

$$v(\xi) = \frac{\mu q(2n+1)}{\lambda(n+2)r}G + \frac{-(2n+1)q}{(n+2)r}$$

Similarly, we can obtain another traveling wave solution of general Gardner equation as follows:

When $\mu \neq 0$

$$u_3(\xi) = [\frac{\mu q(2n+1)}{\lambda(n+2)r}(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}}) + \frac{-(2n+1)q}{(n+2)r}]^{\frac{1}{n}}, \tag{28}$$

where d and ξ are the same as Case 1.

When $\mu = 0$, we have $u \equiv 0$.

Remark 4.1. Compared with the exact solutions of general Gardner equation reported by the authors in [32,34,35], the results (26) and (28) are new solutions.

5. Conclusions

In this paper, we have seen that some new traveling wave solutions of BBM equation with any order and general Gardner equation are successfully found by using the Bernoulli sub-ODE method. Now we briefly summarize the method in the following.

The main points of the method are that assuming the solution of the ODE reduced by using the traveling wave variable as well as integrating can be expressed by an m-th degree polynomial in G, where $G = G(\xi)$ is the general solutions of a Bernoulli sub-ODE equation. The positive integer m can be determined by the general homogeneous balance method, and the coefficients of the polynomial can be obtained by solving a set of simultaneous algebraic equations .

Compared to the methods used before, one can see that this method is concise and effective. Also this method can be used to many other nonlinear problems.

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