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EXISTENCE OF SOLUTIONS FOR RANDOM FUNCTIONAL-DIFFERENTIAL INCLUSIONS

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În această lucrare studiem existenţa soluţiilor pentru incluziuni diferenţiale funcţionale aleatoare definite de multifuncţii cu valori convexe în spaţii Banach separabile. Utilizăm teoreme de existenţă a soluţiilor în cazul determinist precum şi rezultate din analiza multivocă privind existenţa selecţiilor măsurabile.

This paper is devoted to the study of functional-differential inclusions with memory defined on a separable Banach space and depending in a measurable way on a random parameter. Two existence theorems are obtained through the use of analogous deterministic results and techniques from the theory of measurable multifunctions.

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1. Introduction

There are two typical methods in proving the existence of random solutions of differential inclusions; in the first one, the measurability of solutions with respect to a random parameter is proved step by step ([7], [8]), in the second one, random fixed point theorems are used ([10]). For other results on random differential inclusions we refer to [3].

In the case of random functional-differential inclusions, conditions for the existence of random viable solutions were obtained by Rybinski in [12]. The method proposed in this paper is based on a random fixed point principle for multivalued mappings which has appeared in the proofs of main theorems in Engl's paper ([4]). In [12] it is shown how the problem of the existence of a random solution may be reduced to the related deterministic problem; it is an indirect approach in which measurable selections are chosen "beyond" the differential problem.

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The aim of the present paper is to establish the existence of solutions for random functional-differential inclusions with memory governed by convex valued orientor fields which take values in a separable Banach space. In distinction to Rybinski's approach, our work is based on a direct "measurable selections approach" which seems to be most natural and powerful in every concrete situation. In this way our result may be interpreted, on one hand, as an extension of the results in [7], [8] and [10] to the case of functional-differential inclusions and, on the other hand, as an extension to the random case of the corresponding deterministic result in [5] and [11].

The paper is organized as follows: definitions, notations and basic results are given in the next section and the main results are presented in Section 3.

2. Notations and preliminary results

Throughout this paper X is a separable Banach space whose norm is denoted by $\|.\|$ and $\mathcal{P}(X)$ will stand for the set of all subsets of X. If $A \subset X$, by $cl(A)$ and $\overline{co}A$ we mean the closure and the closed convex hull of A, respectively. If $x \in X$, the distance from the point x to the set A will be denoted by $d(x, A)$. For any $A, B \in \mathcal{P}(X)$, the Hausdorff distance between A and B is defined as

$$d_H(A, B) := \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}.$$

For $a \in \overline{A} \subseteq X$, the contingent cone (or Bouligand cone) to A at a is defined by

$$K_a^+ A = \left\{ x \in X : \underline{\lim}_{\lambda\to 0+} \frac{d(a+\lambda x, A)}{\lambda} = 0 \right\}$$

It is easy to see that this cone is closed, but in general it is not convex. However when A is convex, $K_a^+ A$ is convex too and coincides with another very useful cone introduced by Clarke ([1]).

If X^* is the topological dual of X and $A \subset X$, by $\sigma_A(.)$ we denote the support function of A, i.e. $\sigma_A(x^*) = \sup_{a\in A}\langle x^*, a\rangle$. If I is a real interval, let $C(I, X)$ be the Banach space consisting of all continuous functions $x(.) : I \to X$ with the norm $\|x(.)\|_\infty = \sup\{\|x(t)\| : t \in I\}$. Similarly, $AC(I, X)$ will denote the space of absolutely continuous functions from I to X. By $L^1(I, X)$ we mean the Banach space of measurable functions $y(.) : I \to X$ which are Lebesgue integrable, endowed with norm $\|y(.)\|_1 = \int_I \|y(t)\| dt$.

Let (Ω, Σ, μ) be a σ-finite measure space (not necessarily complete) and $L^1(\Omega, X)$ be the space of integrable functions $f(.): \Omega \to X$ equipped with the norm $\|f(.)\|_1 = \int_\Omega \|f(\omega\| d\mu(\omega)$. For any topological space S, the script $B(S)$ will stand for the σ-field of Borel subsets of S.

Recall that a function $f(.,.): \Omega \times X \to X$ is said to be Caratheodory if $\omega \to f(\omega, x)$ is measurable for any $x \in X$ and $x \to f(\omega, x)$ is continuous for any $\omega \in \Omega$. In what follows we will need the following result.

Theorem 2.1. ([6]) Let (Ω, Σ, μ) be a σ-finite measure space, Y a locally compact separable metric space and Z a metric space. Then $f: \Omega \times Y \to Z$ is a Caratheodory function if and only if $\omega \to g(\omega)(.) := f(\omega,.)$ is measurable as a mapping from Ω to the space $C(Y, Z)$ endowed with the compact-open topology.

By a Kamke function we mean a function $w(.,.): [t_0, T] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the Caratheodory conditions (i.e. $t \to w(t,.)$ measurable and $x \to w(., x)$ continuous), $w(t, x) \le \varphi(t)$ a.e.$[t_0, T]$ with $\varphi(.) \in L^1(I, \mathbb{R}_+)$, $w(t,0) = 0$ a.e.$[t_0, T]$ and $u(t) \equiv 0$ is the only solution of the problem

$$u(t) \le \int_{t_0}^{t} w(s, u(s))ds, \quad u(t_0) = 0 .$$

Definition 2.2. Let $F(.): \Omega \to \mathcal{P}(X)$ with nonempty, closed values. $F(.)$ is said to be (weakly) measurable if any of the following equivalent conditions holds:

i) for any open subset $U \subseteq X$, $\{\omega \in \Omega : F(\omega) \cap U \neq \varnothing\} \in \Sigma$;

ii) for all $x \in X$, $\omega \to d(x, F(\omega))$ is measurable.

If, in addition, μ is complete, then the statements i) and ii) above are equivalent to any of the following ones

iii) Graph$(F(.)) := \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \otimes B(X)$ (graph measurability);

iv) for any closed subset $C \subseteq X$, $\{\omega \in \Omega : F(\omega) \cap C \neq \varnothing\}$ (strong measurability).

For a measurable multifunction $F(.):\Omega \to \mathcal{P}(X)$ we denote by S_F the set of Bochner integrable selections of $F(.)$,

$$S_F = \{f(.) \in L^1(\Omega, X) : f(\omega) \in F(\omega) \text{ a.e. on } \Omega\}.$$

As one can easily see, the above set is nonempty and closed if and only if $\omega \to \inf_{z \in F(\omega)} \|z\| \in L^1(\Omega, \mathbb{R}_+)$. Obviously, if $F(.)$ is integrably bounded in the sense that there exists $r(.) \in L^1(\Omega, \mathbb{R}_+)$ such that

$$\|z\| \le r(\omega), \ \forall z \in F(\omega), \forall \omega \in \Omega,$$

then $S_F \neq \emptyset$. For other properties of measurable multifunctions we refer to [2]

Definition 2.3. Let Y be a topological space, Z a metric space and $F(.):Y \to \mathcal{P}(Z)$ be a multifunction with nonempty, closed values. $F(.)$ is said to be Hausdorff continuous at $y_0 \in Y$ if for any $\varepsilon > 0$ there exists $U \subset X$ open, $y \in U$ such that

$$d_H(F(y_0), F(y)) < \varepsilon, \ \forall y \in U.$$

We say that $F(.)$ is Hausdorff continuous on Y if it is so at every $y_0 \in Y$.

Definition 2.4. Let $\mathcal{P}_b(X)$ be the family of bounded subsets of X. The Kuratowski measure of noncompactness $\alpha : \mathcal{P}_b(X) \to \mathbb{R}_+$ is defined by

$$\alpha(B) = \inf\{r > 0 : B \text{ admits a finite cover by sets of diameter } \le r\},$$

while the Hausdorff (ball) measure of noncompactness $\beta : \mathcal{P}_b(X) \to \mathbb{R}_+$ is defined by

$$\beta(B) = \inf\{r > 0 : B \text{ admits a finite cover by balls of radius } r\}.$$

It is easy to see that these measures are related by

$$\beta(B) \le \alpha(B) \le 2\beta(B) \ \ \forall B \in \mathcal{P}_b(X),$$

hence they are equivalent.

Let $I = [t_0, T]$ be a real interval and $0 < \Delta < T - t_0$. Consider the following functional-differential inclusion with nonconvex valued orientor field $F(.,.)$

$$x'(t) \in F(t, x_t(.)) \text{ a.e. } (I), \tag{1}$$

$$x(.)\big|_{[t_0-\Delta, t_0]} \in M, \tag{2}$$

where $F(.,.): I \times C([t_0 - \Delta, t_0], X) \to \mathcal{P}(X)$ is a given set-valued map, M is a nonempty compact subset of $C([t_0 - \Delta, t_0], X)$ and for all $t \in I$, $x_t : [t_0 - \Delta, t_0] \to X$ is a continuous function defined by $x_t(s) = x(t + s - t_0)$. Hence $x_t(.)$ describes the history of the state from time $t - \Delta$ up to the present time t.

A solution of the problem (1)-(2) is a continuous function $x(.) : [t_0 - \Delta, T] \to X$ such that $x(.)\big|_I \in AC(I, X)$, $x(.)\big|_{[t_0-\Delta, t_0]} \in M$ and the inclusion (1) holds a.e. on I.

Hypothesis 2.5. i) $F(.,.): I \times C([t_0 - \Delta, t_0], X) \to \mathcal{P}(X)$ has nonempty compact values and is graph measurable;

ii) for every $t \in I$, $y \to F(t, y)$ is lower semi-continuous (l.s.c.);

iii) there exist $a(.), b(.) \in L^1(I, \mathbb{R}_+)$ such that for almost all $t \in I$ and $x(.) \in C([t_0 - \Delta, t_0], X)$, $\|z\| \le a(t)\|x(.)\|_\infty + b(t)$ for all $z \in F(t, x)$;

iv) there exists a Kamke function $w(.,.) : I \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $B \subset C([t_0 - \Delta, t_0], X)$ bounded and nonempty we have

$$2\beta(F(t, B)) \le w(t, \beta(B)) \text{ a.e. } (I),$$

where $\beta(.)$ is the Hausdorff measure of noncompactness from Definition 2.4.

An important tool in proving our main results is the following existence theorem which is due to Papageorgiou.

Theorem 2.6. ([10]) Let $M \subset C([t_0 - \Delta, t_0], X)$ be a given compact family of continuous functions and assume that $F(.,.): I \times C([t_0 - \Delta, t_0], X) \to \mathcal{P}(X)$ satisfies Hypothesis 2.5. Then the problem (1)-(2) admits a solution.

The problem that we will consider in the next theorem is the following

$$x'(t) \in F(t, x_t(.)) \tag{3}$$

$$x(t) \in U(t), \quad \forall t \in I \tag{4}$$

$$x(.)\big|_{[t_0-\Delta, t_0]} = y(.), \tag{5}$$

where $F(.,.) : I \times C([t_0 - \Delta, t_0], X) \to \mathcal{P}(X)$, $U(.) : I \to \mathcal{P}(X)$ are two given set-valued maps and $y(.) \in C([t_0 - \Delta, t_0], X)$ satisfies $y(t_0) \in U(t_0)$.
Set $\Lambda = \{(t, \phi(.)) \in I \times C([t_0 - \Delta, t_0], X) : \phi(t_0) \in U(t_0)\}$ and note that $(t_0, y(.)) \in \Lambda$.
For any $(t, x) \in \text{Graph } U(.)$, we define

$$T_{(t,x)}\text{Graph}(U(.)) = \{y \in X : (1, y) \in K^{+}_{(t,x)}\text{Graph}(U(.))\}.$$

Hypothesis 2.7. i) $F(.,.) : I \times C([t_0 - \Delta, t_0], X) \to \mathcal{P}(X)$ has nonempty compact convex values and is jointly measurable;

ii) for every $t \in I$, $y \to F(t, y)$ is upper semi-continuous (u.s.c.) from $C([t_0 - \Delta, t_0], X)$ endowed with the norm $\|.\|_\infty$ to the space X with the weak topology;

iii) there exist $a(.) \in L^1(I, \mathbb{R}_+)$ and a subset $J \subset I$ with $\mu(I \setminus J) = 0$ such that for all $(t, \phi(.)) \in \Lambda \cap (J \times C([t_0 - \Delta, t_0], X))$ one has

$$\|z\| \le a(t)(1 + \|\phi(t_0)\|), \quad \forall z \in F(t, \phi(.))$$

$$F(t, \phi(.)) \cap T_{(t,\phi(t_0))}\text{Graph}U(.) \ne \emptyset.$$

iv) $U(.) : I \to \mathcal{P}(X)$ is upper semi continuous (u.s.c), with nonempty compact values.

In proving the existence of random viable solutions in the next section, we need to have the analogous deterministic result obtained by Gavioli and Malaguti in [5].

Theorem 2.8. ([5]) Assume that $F(.,.) : \Lambda \to \mathcal{P}(X)$ and $U(.) : I \to \mathcal{P}(X)$ satisfy Hypothesis 2.6. Then the problem (3)-(5) admits a solution.

The following result will be also useful.

Theorem 2.9. ([9]) Let $F(.) : \Omega \to \mathcal{P}(X)$ be an integrably bounded set-valued map with weakly compact (w-compact) and convex values. Then S_F is nonempty, convex, w-compact with respect to the norm $\|.\|_1$ of $L^1(\Omega, X)$.

3. Main results

Consider the following Cauchy problem concerning random functional-differential inclusions of the form

$$\frac{d}{dt}x(\omega,t)\in F(\omega,t,x_t(\omega,.)) \text{ a.e.}(I), \tag{6}$$

$$x(\omega,.)\big|_{[t_0-\Delta,t_0]}\in G(\omega), \tag{7}$$

where
$F(.,.,.):\Omega\times I\times C([t_0-\Delta,t_0],X)\to\mathcal{P}(X)$ and $G(.,.):\Omega\to\mathcal{P}(C([t_0-\Delta,t_0],X))$ are two given set-valued maps and for all $t\in I$ and $\omega\in\Omega$, $x_t(\omega,.):[t_0-\Delta,t_0]\to X$ is a continuous function defined by $x_t(\omega,s)=x(\omega,t+s-t_0)$.

Before stating and proving our main result we give the definition of a solution to the above problem.

Definition 3.1. A solution to the random functional-differential inclusions (6)-(7) is a stochastic process $x(.,.):\Omega\times[t_0-\Delta,T]\to X$ with continuous paths (i.e., for all $t\in[t_0-\Delta,T]$, $x(.,t)$ is measurable and for all $\omega\in\Omega$, $x(\omega,.)\in C([t_0-\Delta,T],X))$ such that $x(\omega,.)\big|_I\in AC(I,X)$ for every $\omega\in\Omega$ and inclusions (6)-(7) are verified for almost all $\omega\in\Omega$.

Hypothesis 3.2. i) $F(.,.,.):\Omega\times I\times C([t_0-\Delta,t_0],X)\to\mathcal{P}(X)$ has nonempty, compact convex values and is jointly measurable;

ii) for all $(\omega,t)\in\Omega\times I$, the set-valued map $F(\omega,t,.)$ is Hausdorff continuous;

iii) there exist $a(.,.)$, $b(.,.):\Omega\times I\to\mathbb{R}_+$ such that for all $\omega\in\Omega$, $a(\omega,.)$, $b(\omega,.)\in L^1(I,\mathbb{R}_+)$ and for all $z\in F(\omega,.t,x(.))$, $\|z\|\le a(\omega,t)\|x(.)\|_\infty$ $+b(\omega,t)$ a.e. (I) and $\forall\omega\in\Omega$, $x(.)\in C([t_0-\Delta,t_0],X)$;

iv) there exists a Kamke function $w(.,.,.):\Omega\times I\times\mathbb{R}_+\to\mathbb{R}_+$ such that $(\omega,t)\to w(\omega,t,.)$ is jointly measurable and for all $B\subset C([t_0-\Delta,t_0],X)$ bounded one has

$$2\beta(F(\omega,t,B))\le w(\omega,t,\beta(B)) \quad \text{a.e. } (I),\ \forall\omega\in\Omega$$

where $\beta(.)$ is the Hausdorff measure of noncompactness;

v) $G(.):\Omega \to \mathcal{P}(C([t_0-\Delta,t_0],X))$ has nonempty, compact convex values and is measurable.

Theorem 3.3. Assume that $F(.,.,.):\Omega\times I\times C([t_0-\Delta,t_0],X)\to \mathcal{P}(X)$ and $G(.):\Omega\to C([t_0-\Delta,t_0],X)$ fulfill the assumptions of Hypothesis 3.2. Then the Cauchy problem (6)-(7) admits a solution.

Proof. Consider the set-valued map $R(.):\Omega\to\mathcal{P}([t_0-\Delta,T],X))$ defined by

$$R(\omega)=\{x(.)\in C([t_0-\Delta,t_0],X): x'(t)\in F(\omega,t,x_t(.)) \text{ a.e. } (I),$$
$$x(.)\big|_{[t_0-\Delta,t_0]}\in G(\omega)\}.$$

We are going to prove that $R(.)$ has measurable graph. By assumption, $F(.,.,.)$ has nonempty, compact convex values, so letting

$R_1(\omega)=\{x(.)\in C([t_0-\Delta,T],X): x(t)\in x(t')+\int_t^{t'}F(\omega,r,x_r(.))dr$ for all $t,t'\in I\}$,

$R_2(\omega)=\{x(.)\in C([t_0-\Delta,T],X): x(.)\big|_{[t\ -\Delta,t_0]}\in G(\omega)\}$,

we have $R(\omega)=R_1(\omega)\cap R_2(\omega)$. From Theorem 2.6 we obtain that for all $\omega\in\Omega$, $R(\omega)\neq\varnothing$ and with a same reasoning as in the proof of Theorem 3.1 in [9], we claim that it is closed. By Theorem 2.9, for all $\omega\in\Omega$, $\int_t^{t'}F(\omega,r,x_r(.))dr$ is a w-compact and convex subset of X, hence $x(t')+\int_t^{t'}F(\omega,r,x_r(.))dr$ is closed and convex.. Thus we can write

$$R_1(\omega)=\{x(.)\in C([t_0-\Delta,T],X): d(x(t),x(t')+\int_t^{t'}F(\omega,r,x_r(.))dr)=0$$
$$\text{for all } t,t'\in I\}.$$

Set $\phi(\omega,t,t',x(.)):=d\left(x(t),x(t')+\int_t^{t'}F(\omega,r,x_r(.))dr\right)$. For all $x^*\in X^*$ one has

$$\sigma_{x(t')+\int_t^{t'}F(\omega,r,x_r(.))dr}(x^*)=\left\langle x^*,x(t')\right\rangle+\sigma_{\int_t^{t'}F(\omega,r,x_r(.))dr}(x^*)$$
$$=\left\langle x^*,x(t')\right\rangle+\int_t^{t'}\sigma_{F(\omega,r,x_r(.))}(x^*)dr.$$

Notice that $(\omega,r)\to\sigma_{F(\omega,r,x_r(.))}(x^*)$ is measurable and for all $\omega\in\Omega$, $\sigma_{F(\omega,.,x_.(.))}(x^*)\in L^1(I,\mathbb{R})$. Hence we deduce that $\omega\to\int_t^{t'}\sigma_{F(\omega,r,x_r(.))}(x^*)dr$ is measurable which in turn implies that $\omega\to\sigma_{x(t')+\int_t^{t'}F(\omega,r,x_r(.))dr}(x^*)$ is measurable.

Applying Theorem III.37 of Castaing & Valadier ([2]) we deduce that the set-valued map $\omega \to x(t') + \int_t^{t'} F(\omega, r, x_r(.))dr$ is $\hat{\Sigma}$-measurable for all $t, t' \in I$, where $\hat{\Sigma}$ is the completion of Σ with respect to $\mu(.)$.

Next we prove that for all $\omega \in \Omega$, $(t, t', x(.)) \to \phi(\omega, t, t', x(.))$ is continuous from $I \times I \times C([t_0 - \Delta, T], X)$ to $\mathbb{R}_+$. For this purpose, let $(t^n, t'^n, x^n(.)) \underset{n\to\infty}{\to} (t, t', x(.))$. We have

$$\left|\phi(\omega, t^n, t'^n, x^n(.)) - \phi(\omega, t, t', x(.))\right|$$
$$= \left| d\left(x^n(t), x^n(t'^n) + \int_{t^n}^{t'^n} F(\omega, r, x_r^n(.))dr \right) - d\left(x(t), x(t') + \int_t^{t'} F(\omega, r, x_r(.))dr \right) \right|$$
$$\le \left\| x^n(t) - x(t) \right\| + d_H\left(x^n(t'^n) + \int_{t^n}^{t'^n} F(\omega, r, x_r^n(.))dr, x(t') + \int_t^{t'} F(\omega, r, x_r(.))dr \right)$$
$$\le \left\| x^n(t) - x(t) \right\| + \left\| x^n(t'^n) - x(t') \right\|$$
$$+ d_H\left(\int_I \chi_{[t^n, t'^n]}(r) F(\omega, r, x_r^n(.))\, dr, \int_I \chi_{[t,t']}(r) F(\omega, r, x_r(.))\, dr \right)$$
$$\le \left\| x^n(t) - x(t) \right\| + \left\| x^n(t'^n) - x(t') \right\|$$
$$+ \int_I \Big[d_H\left(\chi_{[t^n, t'^n]}(r) F(\omega, r, x_r^n(.)), \chi_{[t,t']}(r) F(\omega, r, x_r^n(.)) \right)$$
$$+ d_H\left(\chi_{[t,t']}(r) F(\omega, r, x_r^n(.)), \chi_{[t,t']}(r) F(\omega, r, x_r(.)) \right) \Big] dr$$
$$= \left\| x^n(t) - x(t) \right\| + \left\| x^n(t'^n) - x(t') \right\| + \int_I \Big[\left| \chi_{[t^n, t'^n]}(r) - \chi_{[t,t']}(r) \right| d_H\left(\{0\}, F(\omega, r, x_r^n(.)) \right)$$
$$+ \chi_{[t,t']}(r) d_H\left(F(\omega, r, x_r^n(.)), F(\omega, r, x_r(.)) \right) \Big] dr.$$

But $x^n(.) \subset C([t_0 - \Delta, T], X)$, hence for all $r \in I$ there exists $M(r) := \sup_{n\ge1} \left\| x_r^n(.) \right\|_\infty$.

Let us define $\psi(\omega, r) := a(\omega, r) M(r) + b(\omega, r)$. From condition iii) in the Hypo-thesis 3.2 we obtain for all $z \in F(\omega, r, x_r^n(.))$:

$$\left\| z \right\| \le \psi(\omega, r) \text{ a.e. on } \Omega \times I.$$

This leads us to the following

$$\left|\phi(\omega, t^n, t'^n, x^n(.)) - \phi(\omega, t, t', x(.))\right|$$
$$\le \left\| x^n(t) - x(t) \right\| + \left\| x^n(t'^n) - x(t') \right\| + \int_I \left| \chi_{[t^n, t'^n]}(r) - \chi_{[t,t']}(r) \right| \psi(\omega, r) dr$$
$$+ \int_I \chi_{[t,t']}(r) d_H\left(F(\omega, r, x_r^n(.)), F(\omega, r, x_r(.)) \right) dr.$$

Passing to the limit as $n \to \infty$ we get

$$\int_I \left| \chi_{[t^n, t'^n]}(r) - \chi_{[t,t']}(r) \right| \psi(\omega, r) dr \to 0$$

and, because $F(\omega,t,.)$ is Hausdorff continuous,

$$\int_I \chi_{[t,t']}(r) d_H\left(F(\omega,r,x_r^n(.)),F(\omega,r,x_r(.))\right)dr \to 0 .$$

Finally, we obtain

$$\lim_{n\to\infty}\left|\phi(\omega,t^n,t'^n,x^n(.))-\phi(\omega,t,t',x(.))\right|=0$$

hence $(t,t',x(.))\to\phi(\omega,t,t',x(.))$ is continuous. Thus we deduce that $\phi(.,.,.,.)$ is $\hat{\Sigma}\times B(I)\times B(I)\times B(C([t_0-\Delta,T],X))$- measurable. Let $D\subset I$ be a dense subset of I and let us define $f(.,.):\Omega\times C([t_0-\Delta,T],X)$,

$$f(\omega,x(.))=\sup_{t,t'\in D}\phi(\omega,t,t',x(.)) .$$

Then $(\omega,x(.))\to f(\omega,x(.))$ is $\hat{\Sigma}\times B(C([t_0-\Delta,T],X))$-measurable. Note that

$$R_1(\omega)=\{x(.)\in C([t_0-\Delta,T],X): f(\omega,x(.))=0\},$$

which implies that $Graph(R_1(\omega))$ is $\hat{\Sigma}\times B(C([t_0-\Delta,T],X))$-measurable. On the other hand, it is clear that the function $g(.,.):\Omega\times C([t_0-\Delta,T],X)\to C([t_0-\Delta,t_0],X)$, $g(\omega,x(.))=x(.)|_{[t_0-\Delta,t_0]}$ is continuous with respect to the topology induced by sup-norm $\|.\|_\infty$. From Lemma 6 in [4], it follows that $(\omega,x(.))\to d(g(\omega,x(.)),G(\omega))$ is $\hat{\Sigma}\times B(C([t_0-\Delta,T],X))$-measurable and thus $R_2(.)$ has also measurable graph. Finally, $R(.)$ has measurable graph. Apply Aumann's selection theorem ([6, Theorem 5.2]) to find a $\hat{\Sigma}$- measurable selection $\hat{r}(.):\Omega\to C([t_0-\Delta,T],X)$ such that for all $\omega\in\Omega$, $\hat{r}(\omega)\in R(\omega)$. Let $r(.):\Omega\to C([t_0-\Delta,T],X)$ be Σ-measurable such that $r(\omega)(.)=\hat{r}(\omega)(.)$ for almost all $\omega\in\Omega$. Set

$$x(\omega,t)=r(\omega)(t),\ \ \forall(\omega,t)\in\Omega\times[t_0-\Delta,T].$$

By Theorem 2.1 and from the definition of $R(.)$, we conclude that $x(.,.)$ is a stochastic process which solves the Cauchy problem (6)-(7).

□

Remark 3.4. Several remarks are in order.

i) When there is no memory, i.e. $\Delta=0$ and $F(\omega,t,x_t(\omega,.))=F_1(\omega,t,x(\omega,t))$ Theorem 3.3 above yields Theorem 3.1. in [7] (see also [3] and [10, Theorem 4.1]).

ii) If $F(.,.,.)$ and $G(.)$ are constant with respect to the random parameter $\omega\in\Omega$ in the sense that $F(\omega,t,x_t(\omega,.))=F_1(t,x_t(.))$ and $G(\omega)=M\subset C([t_0-\Delta,t_0],X)$, then Theorem 3.3 yields Theorem 2.6 proved by Papageorgiou in [11].

Next we pass to the study of random functional-differential inclusions with state constraints of the form

$$\frac{d}{dt}x(\omega,t)\in F(\omega,t,x_t(\omega,.)) \text{ a.e. } (I), \tag{8}$$

$$x(\omega,t)\in U,\ \ \forall(\omega,t)\in\Omega\times I, \tag{9}$$

$$x(\omega,.)\big|_{[t_0-\Delta,t_0]}=y(\omega,.) \text{ a.e. } (\Omega), \tag{10}$$

where $F(.,.,.):\Omega\times I\times C([t_0-\Delta,t_0],X)\to\mathcal{P}(X)$, $U\subset X$ is a nonempty set and $y(.,.):\Omega\times I\to X$ is measurable such that $y(\omega,.)\in C([t_0-\Delta,t_0],X)$ and $y(\omega,t_0)\in U$, $\forall\omega\in\Omega$.

The existence of solutions to the above problem leads us to what is known in applied mathematics as “viability theory”. More precisely, we are trying to select trajectories which are “viable”, in the sense that they always satisfy the constraints in (9).

We define the sets $C_0([t_0-\Delta,t_0],X):=\{\phi(.)\in C([t_0-\Delta,t_0],X):\ \phi(t_0)\in U\}$ and $C_U([t_0-\Delta,T],X):=\{\phi(.)\in C([t_0-\Delta,T],X):\ \phi_t(.)\in C_0([t_0-\Delta,t_0],X),\forall t\in I\}$.

Hypothesis 3.5. i) $F(.,.,.):\Omega\times I\times C([t_0-\Delta,t_0],X)\to\mathcal{P}(X)$ has nonempty, compact convex values and is jointly measurable;

ii) for all $(\omega,t)\in\Omega\times I$, the set-valued map $F(\omega,t,.)$ is Hausdorff continuous;

iii) there exists $a(.,.):\Omega\times I\to X$ measurable with $a(\omega,.)\in L^1(I,\mathbb{R}_+)$ for all $\omega\in\Omega$ and a subset $J\subset I$ with $\mu(I\setminus J)=0$ such that for all $(\omega,t,\phi(.))\in\Lambda\times J\times C_0([t_0-\Delta,t_0],X)$ one has

$$\|z\|\le a(\omega,t)(1+\|\phi(t_0)\|),\quad \forall z\in F(\omega,t,\phi(.))$$

$$F(\omega,t,\phi(.))\cap K^{+}_{\phi(t_0)}U\neq\varnothing \text{ a.e.}(\Omega).$$

Theorem 3.6. Assume that $F(.,.,.):\Omega\times I\times C_0([t_0-\Delta,t_0],X)\to\mathcal{P}(X)$ fulfills the assumptions in Hypothesis 3.5. Then the Cauchy problem (8)-(10) admits a viable trajectory.

Proof. As in the proof of Theorem 3.3 we consider the set-valued map $R(.):\Omega\to\mathcal{P}(C_U[t_0-\Delta,T],X))$ defined by

$$R(\omega)=\{x(.)\in C_U([t_0-\Delta,T],X): x'(t)\in F(\omega,t,x_t(.)) \text{ a.e. } (I),$$
$$x(.)\big|_{[t_0-\Delta,t_0]}=y(\omega,.)\}.$$

We prove that $R(.)$ has measurable graph. Letting

$$R_1(\omega)=\{x(.)\in C_U([t_0-\Delta,T],X): x(t)\in x(t')+\int_t^{t'} F(\omega,r,x_r(.))dr \text{ for all } t,t'\in I\},$$

$$R_2(\omega)=\{x(.)\in C([t_0-\Delta,T],X): x(.)\big|_{[t\ -\Delta,t_0]}=y(\omega,.)\},$$

we have $R(\omega)=R_1(\omega)\cap R_2(\omega)$. From Theorem 2.8 we obtain that for all $\omega\in\Omega$, $R(\omega)\neq\varnothing$ and with a same reasoning as in the proof of Theorem 3.1 in [9], we claim that it is closed. As before we can write

$$R_1(\omega)=\{x(.)\in C_U([t_0-\Delta,T],X): d(x(t),x(t')+\int_t^{t'} F(\omega,r,x_r(.))dr)=0$$
$$\text{for all } t,t'\in I\}.$$

Consider the multifunction $\Phi(\omega,t,t',x(.)):=x(t')+\int_t^{t'} F(\omega,r,x_r(.))dr$. Working with the support function and using similar arguments as in the proof of Theorem 3.3 we can state that $\omega\to\Phi(\omega,t,t',x(.))$ is $\hat{\Sigma}$-measurable. Now we show that $\Phi(\omega,.,.,.)$ is Hausdorff continuous. Let $(t^n,t'^n,x^n(.))\underset{n\to\infty}{\to}(t,t',x(.))$ and $I_n:=([t^n,t'^n]\setminus[t,t'])\cup([t,t']\setminus[t^n,t'^n])$; Using Hörmander's formula we have the following estimations

$$d_H(\Phi(\omega,t^n,t'^n,x^n(.)),\Phi(\omega,t,t',x(.)))$$
$$\leq \sup_{\|x^*\|\leq 1}\left|\int_{t^n}^{t'^n}\sigma_{F(\omega,r,x_r^n(.))}(x^*)dr-\int_t^{t'}\sigma_{F(\omega,r,x_r(.))}(x^*)dr\right|+\left\|x^n(t'^n)-x(t')\right\|$$
$$\leq \sup_{\|x^*\|\leq 1}\left[\left|\int_{t^n}^{t'^n}\sigma_{F(\omega,r,x_r^n(.))}(x^*)dr-\int_t^{t'}\sigma_{F(\omega,r,x_r^n(.))}(x^*)dr\right|\right.$$
$$\left.+\left|\int_t^{t'}\sigma_{F(\omega,r,x_r^n(.))}(x^*)dr-\int_t^{t'}\sigma_{F(\omega,r,x_r(.))}(x^*)dr\right|\right]+\left\|x^n(t'^n)-x(t')\right\|$$
$$\leq \sup_{\|x^*\|\leq 1}\int_{I_n}|\sigma_{F(\omega,r,x_r^n(.))}(x^*)|\,dr$$
$$+\sup_{\|x^*\|\leq 1}\int_t^{t'}|\sigma_{F(\omega,r,x_r^n(.))}(x^*)-\sigma_{F(\omega,r,x_r(.))}(x^*)|dr+\left\|x^n(t'^n)-x(t')\right\|.$$

Set $h(\omega,r)=\sup_{n\geq 1} d_H(\{0\},F(\omega,r,x_r^n(.)))$. We note that $h(.,.)$ is measurable and for all $\omega\in\Omega$, $h(\omega,.)\in L^1(I,\mathbb{R}_+)$. Hence we get that

$$d_H(\Phi(\omega,t^n,t'^n,x^n(.)),\Phi(\omega,t,t',x(.)))$$
$$\leq \int_{I_n} h(\omega,r)dr + \int_t^{t'} d_H(F(\omega,r,x_r^n(.)),F(\omega,r,x_r(.)))dr + \left\|x^n(t'^n)-x(t')\right\|.$$

Passing to the limit as $n\to\infty$ we get that $\Phi(\omega,.,.,.)$ is Hausdorff continuous, $\forall\omega\in\Omega$. We put $\phi(\omega,t,t',x(.)) = d(x(t),\Phi(\omega,t,t',x(.)))$ and we note that the function $(t,t',x(.))\to\phi(\omega,t,t',x(.))$ is continuous. Note that $R_1(.)$ may be written in the form

$$R_1(\omega)=\bigcap_{n\geq 1}\left\{x(.)\in C_U([t_0-\Delta,T],X):\ \phi(\omega,t^n,t'^n,x(.))=0\right\},$$

which implies that $R_1(.)$ has measurable graph. We note also that $R_2(.)$ has measurable graph. We conclude that $\mathrm{Graph}(R(.))$ is $\hat{\Sigma}\otimes B(C_U([t_0-\Delta,T],X))-$ measurable. Now the existence of the desired random viable trajectory can be obtained in a same manner as in the proceding theorem.

□

Remark 3.7. Several remarks are in order.

i) If $U=X$, the existence of random viable solutions reduces to the problem of existence of solutions to (6)-(7), hence the above theorem yields Theorem 3.3.

ii) When there is no memory, i.e. $\Delta=0$ and $F(\omega,t,x_t(\omega,.))=F_1(\omega,t,x(\omega,t))$, Theorem 3.6 yields Theorem 3.3. in [7].

iii) If $F(.,.,.)$ and $y(,..)$ are constant with respect to the random parameter $\omega\in\Omega$ in the sense that $F(\omega,t,x_t(\omega,.))=F_1(t,x_t(.))$ and $y(\omega,.)=y_1(.)\in C([t_0-\Delta,t_0],X)$, then Theorem 3.6 yields Theorem 4.1 proved by Gavioli and Malaguti in [5].

3. Conclusions

In this paper we extended the works of Kandilakis and Papageorgiou ([7], [10]) concerning random differential inclusions. Namely, it is about two existence theorems obtained for random functional-differential inclusions with memory on infinite separable Banach space. Within the family of functional-differential inclusions, our results may be interpreted as extensions to the random case, of the deterministic existence theorems of Gavioli, Malaguti ([5]) and Papageorgiou ([11]).

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