EXACT SOLUTIONS FOR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS BY PROJECTIVE RICCATI EQUATION METHOD

Bin Zheng[1]

In this paper, the projective Riccati equation method is applied to find exact solutions for fractional partial differential equations in the sense of modified Riemann-Liouville derivative. Based on a nonlinear fractional complex transformation, a certain fractional partial differential equation can be turned into another ordinary differential equation of integer order. For illustrating the validity of this method, we apply it to solve the space-time fractional Whitham-Broer-Kaup (WBK) equations and the time fractional Sharma-Tasso-Olever (STO) equation, and as a result, some new exact solutions for them are established.

Keywords: Projective Riccati equation method; Fractional partial differential equation; Exact solution; Nonlinear fractional complex transformation; Fractional Whitham-Broer-Kaup equation; Fractional Sharma-Tasso-Olever equation.

MSC2010: 35Q 51, 35Q 53.

1. Introduction

Recently, fractional differential equations have gained much attention as they are widely used to describe various complex phenomena in many fields such as the fluid flow, signal processing, control theory, systems identification, biology and other areas. Among the investigations for fractional differential equations, research for seeking exact solutions and numerical solutions of fractional differential equations is an important topic, which can also provide valuable reference for other related research. Many powerful and efficient methods have been proposed to obtain numerical solutions and exact solutions of fractional differential equations so far. For example, these methods include the Adomian decomposition method [1,2], the variational iterative method [3-5], the homotopy perturbation method [6,7], the differential transformation method [8], the finite difference method [9], the finite element method [10], the fractional Riccati sub-equation method [11-13] and so on. Based on these methods, a variety of fractional differential equations have been investigated and solved.

In this paper, we apply the projective Riccati equation method [14,15] for solving fractional partial differential equations in the sense of modified Riemann-Liouville derivative by Jumarie [16]. In Section 2, we give some definitions and properties of Jumarie's modified Riemann-Liouville derivative and the description of the projective Riccati equation method for solving fractional partial differential equations. Then in Section 3 we apply the method to solve the space-time fractional Whitham-Broer-Kaup (WBK) equations and the time fractional Sharma-Tasso-Olever (STO) equation. Some conclusions are presented at the end of the paper.

[1]researcher, School of Sciences, Shandong University of Technology, Zibo, Shandong, 255049, China, e-mail: zhengbin2601@126.com zhengbin2601@163.com

2. **Jumarie's modified Riemann-Liouville derivative and description of the fractional projective Riccati equation method**

The Jumarie's modified Riemann-Liouville derivative of order α is defined by the following expression [16]:

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, \ 0<\alpha<1, \\ (f^{(n)}(t))^{(\alpha-n)}, \ n \le \alpha < n+1, \ n \ge 1. \end{cases}$$

For the modified Riemann-Liouville derivative, we have the following important properties (see [11-13,17,19]):

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \tag{1}$$

$$D_t^\alpha (f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \tag{2}$$

$$D_t^\alpha f[g(t)] = f'_g[g(t)]D_t^\alpha g(t) = D_g^\alpha f[g(t)](g'(t))^\alpha. \tag{3}$$

Suppose that a fractional partial differential equation, say in the independent variables $t, x_1, x_2, ..., x_n$, is given by

$$P(u_1, ...u_k, D_t^\alpha u_1, ..., D_t^\alpha u_k, \frac{\partial u_1}{\partial x_1}, ..., \frac{\partial u_k}{\partial x_1}, D_{x_2}^{2\alpha} u_1, ...D_{x_2}^{2\alpha} u_1, ...,$$
$$u_1 \frac{\partial u_1}{\partial x_{n-1}}, ..., u_k \frac{\partial u_k}{\partial x_{n-1}}, D_{x_n}^{3\alpha} u_1, ..., D_{x_n}^{3\alpha} u_k, ...) = 0, \tag{4}$$

where $u_i = u_i(t, x_1, x_2, ..., x_n), \ i = 1, ..., k$ are unknown functions, P is a polynomial in u_i and their various partial derivatives including fractional derivatives.

Step 1. For Eq. (4), suppose that

$$u_i(t, x_1, x_2, ..., x_n) = U_i(\xi),$$
$$\xi = \frac{ct^\alpha}{\Gamma(1+\alpha)} + k_1 x_1 + \frac{k_2 x_2^\alpha}{\Gamma(1+\alpha)} + ...k_1 x_{n-1} + \frac{k_n x_n^\alpha}{\Gamma(1+\alpha)} + \xi_0, \tag{5}$$

where $c, \ k_1, \ k_2, \ \xi_0$ are all nonzero constants. Based on the transformation above, for the terms in (5) containing fractional derivative, such as $D_t^\alpha u_1$, using the first equality in (3) one can obtain

$$D_t^\alpha u_1 = D_t^\alpha U_1(\xi) = U_1'(\xi) D_t^\alpha \xi = cU_1'(\xi).$$

for the terms in (5) containing derivative of integer order, such as $\frac{\partial u_1}{\partial x_1}$, using the first equality in (3) one can obtain

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial U_1}{\partial \xi} \xi'_{x_1} = k_1 U_1'(\xi).$$

So by this transformation for ξ, Eq. (4) can be turned into the following ordinary differential equation of integer order with respect to the variable ξ:

$$\widetilde{P}(U_1, ..., U_k, \ U_1', ..., U_k', \ U_1U_1', ..., U_kU_k', \ U_1'', ..., U_k'', \ ...) = 0. \tag{6}$$

Step 2. Suppose that the solution of (6) can be expressed by

$$U_i(\xi) = \sum_{i=0}^m a_i f^i + \sum_{j=1}^m b_j f^{j-1} g, \tag{7}$$

where $a_i, \ b_j$ are all constants to be determined later, and $f = f(\xi), \ g = g(\xi)$ satisfy the following fractional projective Riccati equations:

$$\left\{\begin{array}{l} f'(\xi) = -f(\xi)g(\xi), \\ g'(\xi) = 1 - g^2(\xi) - rf(\xi), \\ g^2(\xi) = 1 - 2rf(\xi) + (r^2+\varepsilon)f^2(\xi). \end{array}\right. \tag{8}$$

From the assumption one can see that the degree of U_i is m, while the degree of U_i' is $m+1$, and analogously, the degree of $U_i^{(k)}$ is $m+k$.

Based on (7) and (8), one polynomial in $f^i g^j$ can be constructed, where $i,\ j$ are integers. On one hand, the generation of the highest degree for $f^i g^j$ is due to the value of m, and is possibly due to other two factors, that is, the highest order derivative and the nonlinear term appearing in (6). On the other hand, in order to solve the undetermined constants $a_i,\ b_j$ in (7), one need to solve a set of algebraic equations for $a_i,\ b_j,\ k_1,\ k_2,\ ,...,k_n,\ c$ after equating each coefficient of this polynomial to zero. For the consideration of avoiding generating trivial solution, that is, $a_i,\ b_j,\ k_1,\ k_2,\ ,...,k_n,\ c$ are solved by zero, one need to generate more than one terms which contain the highest degree for $f^i g^j$ so as to be combined as terms in the same degree. Therefore, the positive integer m can be determined by considering the homogeneous balance between the highest order derivative and nonlinear term appearing in (6).

Step 3. Substituting (7) into (6) and using (8), the left-hand side of (6) is converted to a polynomial in $f^i g^j$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_i,\ b_j$.

Step 4. Solving the equations in Step 3, and by using the solutions of Eqs. (8), we can construct a variety of exact solutions for Eq. (4).

According to [14, Eqs. (7)-(9)], Eqs. (8) admits the following solutions:

When $\varepsilon = -1$:

$$f_1(\xi) = \frac{4}{5\cosh(\xi) + 3\sinh(\xi) + 4r},\ g_1(\xi) = \frac{5\sinh(\xi) + 3\cosh(\xi)}{5\cosh(\xi) + 3\sinh(\xi) + 4r}, \tag{9}$$

$$f_2(\xi) = \frac{1}{\cosh(\xi) + r},\ g_2(\xi) = \frac{\sinh(\xi)}{\cosh(\xi) + r}. \tag{10}$$

When $\varepsilon = 1$:

$$f_3(\xi) = \frac{1}{\sinh(\xi) + r},\ g_3(\xi) = \frac{\cosh(\xi)}{\sinh(\xi) + r}. \tag{11}$$

3. **Applications of the method**

In this section, we will apply the described method in Section 2 to some fractional partial differential equations.

3.1. **Space-time fractional Whitham-Broer-Kaup (WBK) equations**

We consider the space-time fractional Whitham-Broer-Kaup (WBK) equations [12]:

$$\left\{\begin{array}{l} D_t^{\alpha}u + uD_x^{\alpha}u + D_x^{\alpha}v + \beta D_x^{2\alpha}u = 0, \\ D_t^{\alpha}v + D_x^{\alpha}(uv) - \beta D_x^{2\alpha}v + \gamma D_x^{3\alpha}u = 0, \end{array}\right. \quad 0 < \alpha \le 1. \tag{12}$$

In [12], the authors solved Eqs. (12) by a proposed fractional sub-equation method based on the fractional Riccati equation, and established some exact solutions for them. Now we will apply the described method above to Eqs. (12). To begin with, we suppose $u(x,t) = U(\xi),\ v(x,t) = V(\xi)$, where $\xi = \frac{kx^{\alpha}}{\Gamma(1+\alpha)} + \frac{ct^{\alpha}}{\Gamma(1+\alpha)} + \xi_0,\ k,\ c,\ \xi_0$ are all constants with $k,\ c \neq 0$. Then by use of (1) we have $D_x^{\alpha}\xi = k,\ D_t^{\alpha}\xi = c$. Furthermore, by use of the first equality in (3), we obtain

$$\begin{cases} D_x^\alpha u = D_x^\alpha U(\xi) = U'(\xi) D_x^\alpha \xi = kU'(\xi), \\ D_t^\alpha u = D_t^\alpha U(\xi) = U'(\xi) D_t^\alpha \xi = cU'(\xi). \end{cases}$$

Similarly we have

$$\begin{cases} D_x^\alpha v = D_x^\alpha V(\xi) = V'(\xi) D_x^\alpha \xi = kV'(\xi), \\ D_t^\alpha v = D_t^\alpha V(\xi) = V'(\xi) D_t^\alpha \xi = cV'(\xi), \end{cases}$$

So Eqs. (12) can be turned into

$$\begin{cases} cU' + kUU' + kV' + \beta k^2 U'' = 0, \\ cV' + k(UV)' - \beta k^2 V'' + \gamma k^3 U''' = 0. \end{cases} \tag{13}$$

Suppose that the solutions of Eqs. (13) can be expressed by

$$\begin{cases} U(\xi) = \sum\limits_{i=0}^{m} a_i f^i + \sum\limits_{j=1}^{m} b_j f^{j-1} g, \\ V(\xi) = \sum\limits_{i=0}^{n} c_i f^i + \sum\limits_{j=1}^{n} d_j f^{j-1} g. \end{cases} \tag{14}$$

where $f = f(\xi),\ g = g(\xi)$ satisfy Eqs. (8).

Balancing the order between the highest order derivative term and nonlinear term in Eqs. (13), we can obtain $m = 1,\ n = 2$. So we have

$$\begin{cases} U(\xi) = a_0 + a_1 f + b_1 g, \\ V(\xi) = c_0 + c_1 f + c_2 f^2 + d_1 g + d_2 f g. \end{cases} \tag{15}$$

Substituting (15) along with (8) into (13) and collecting all the terms with the same power of $f^i g^j$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, yields:

Case 1:

$$a_0 = \frac{-k^{-1}c\sqrt{\beta^2k^4r^2 + \beta^2k^4\varepsilon + \gamma k^4 r^2 + \gamma k^4\varepsilon} \mp k^3\beta^2 r \mp k^3\gamma r}{\sqrt{\beta^2k^4r^2 + \beta^2k^4\varepsilon + \gamma k^4 r^2 + \gamma k^4\varepsilon}},$$

$$a_1 = \pm 2\sqrt{\beta^2k^4r^2 + \beta^2k^4\varepsilon + \gamma k^{4\alpha} r^2 + \gamma k^4\varepsilon}\,k^{-1},\ b_1 = 0,\ c_0 = -\frac{\varepsilon k^2(\gamma + \beta^2)}{(r^2 + \varepsilon)},$$

$$c_1 = 2rk^2(\gamma + \beta^2),\ c_2 = -2k^2(\gamma r^2 + \beta^2 r^2 + \gamma\varepsilon + \beta^2\varepsilon),\ d_1 = 0,$$

$$d_2 = \pm 2\sqrt{\beta^2k^4r^2 + \beta^2k^4\varepsilon + \gamma k^4 r^2 + \gamma k^4\varepsilon}\,\beta.$$

Case 2:

$$a_0 = -k^{-1}c,\ a_1 = \pm\sqrt{\beta^2k^4r^2 + \beta^2k^4\varepsilon + \gamma k^{4\alpha} r^2 + \gamma k^4\varepsilon}\,k^{-1},$$

$$b_1 = \pm\sqrt{\beta^2k^4 + \gamma k^4}\,k^{-1},\ c_0 = 0,\ c_1 = r(\mp\beta\sqrt{\beta^2k^4 + \gamma k^4} + k^2\beta^2 + k^{2\alpha}\gamma),$$

$$c_2 = \pm\beta\sqrt{\beta^2k^{4\alpha} + \gamma k^4}\,r^2 \pm \beta\sqrt{\beta^2k^4 + \gamma k^{4\alpha}}\,\varepsilon - k^2\gamma r^2 - k^2\beta^2 r^2 - k^{2\alpha}\gamma\varepsilon - k^2\beta^2\varepsilon,$$

$$d_1 = 0,\ d_2 = \pm\sqrt{\beta^2k^4r^2 + \beta^2k^4\varepsilon + \gamma k^4 r^2 + \gamma k^4\varepsilon}\,(\mp k^{-2}\sqrt{\beta^2k^4 + \gamma k^{4\alpha}} + \beta).$$

Substituting the results above into Eqs. (15), and combining with the solutions of Eqs. (8) as denoted in (9)-(11) we can obtain the following six families of exact solutions to the space-time fractional Whitham-Broer-Kaup (WBK) equations.

From Case 1 and Eqs. (9)-(11) we obtain:

Family 1:

$$\begin{cases} u_1 = \dfrac{-k^{-1}c\sqrt{\beta^2k^4r^2-\beta^2k^4+\gamma k^4r^2-\gamma k^4}\mp k^3\beta^2 r\mp k^3\gamma r}{\sqrt{\beta^2k^4r^2-\beta^2k^4+\gamma k^4r^2-\gamma k^4}} \\ \qquad \pm\dfrac{8\sqrt{\beta^2k^4r^2-\beta^2k^4+\gamma k^4r^2-\gamma k^4}k^{-1}}{5\cosh(\xi)+3\sinh(\xi)+4r}, \\ v_1 = \dfrac{k^2(\gamma+\beta^2)}{(r^2-1)}+\dfrac{8rk^2(\gamma+\beta^2)}{5\cosh(\xi)+3\sinh(\xi)+4r}-\dfrac{32k^2(\gamma r^2+\beta^2r^2-\gamma-\beta^2)}{[5\cosh(\xi)+3\sinh(\xi)+4r]^2} \\ \qquad \pm\dfrac{40\sqrt{\beta^2k^4r^2-\beta^2k^4+\gamma k^4r^2-\gamma k^4}\beta[\sinh(\xi)+3\cosh(\xi)]}{[5\cosh(\xi)+3\sinh(\xi)+4r]^2}, \end{cases} \tag{16}$$

where $\xi=\dfrac{kx^\alpha}{\Gamma(1+\alpha)}+\dfrac{ct^\alpha}{\Gamma(1+\alpha)}+\xi_0$.

Family 2:

$$\begin{cases} u_2 = \dfrac{-k^{-1}c\sqrt{\beta^2k^4r^2-\beta^2k^4+\gamma k^4r^2-\gamma k^4}\mp k^3\beta^2 r\mp k^3\gamma r}{\sqrt{\beta^2k^4r^2-\beta^2k^4+\gamma k^4r^2-\gamma k^4}} \\ \qquad \pm\dfrac{2\sqrt{\beta^2k^4r^2-\beta^2k^4+\gamma k^4r^2-\gamma k^4}k^{-1}}{\cosh(\xi)+r}, \\ v_2 = \dfrac{k^2(\gamma+\beta^2)}{(r^2-1)}+\dfrac{2rk^2(\gamma+\beta^2)}{\cosh(\xi)+r}-\dfrac{2k^2(\gamma r^2+\beta^2r^2-\gamma-\beta^2)}{\cosh(\xi)+r]^2} \\ \qquad \pm\dfrac{2\sqrt{\beta^2k^4r^2-\beta^2k^4+\gamma k^4r^2-\gamma k^4}\beta\sinh(\xi)}{[\cosh(\xi)+r]^2}, \end{cases} \tag{17}$$

where $\xi=\dfrac{kx^\alpha}{\Gamma(1+\alpha)}+\dfrac{ct^\alpha}{\Gamma(1+\alpha)}+\xi_0$.

Family 3:

$$\begin{cases} u_3 = \dfrac{-k^{-1}c\sqrt{\beta^2k^4r^2+\beta^2k^4+\gamma k^4r^2+\gamma k^4}\mp k^3\beta^2 r\mp k^3\gamma r}{\sqrt{\beta^2k^4r^2+\beta^2k^4+\gamma k^4r^2+\gamma k^4}} \\ \qquad \pm\dfrac{2\sqrt{\beta^2k^4r^2+\beta^2k^4+\gamma k^4r^2+\gamma k^4}k^{-1}}{\sinh(\xi)+r}, \\ v_3 = -\dfrac{k^2(\gamma+\beta^2)}{(r^2+1)}+\dfrac{2rk^2(\gamma+\beta^2)}{\sinh(\xi)+r}-\dfrac{2k^2(\gamma r^2+\beta^2r^2+\gamma+\beta^2)}{[\sinh(\xi)+r]^2} \\ \qquad \pm\dfrac{2\sqrt{\beta^2k^4r^2-\beta^2k^4+\gamma k^4r^2-\gamma k^4}\beta\cosh(\xi)}{[\sinh(\xi)+r]^2}, \end{cases} \tag{18}$$

where $\xi=\dfrac{kx^\alpha}{\Gamma(1+\alpha)}+\dfrac{ct^\alpha}{\Gamma(1+\alpha)}+\xi_0$.

From Case 2 and Eqs. (9)-(11) we obtain:

Family 4:

$$\begin{cases} u_4 = -k^{-1}c\pm\dfrac{4\sqrt{\beta^2k^4r^2-\beta^2k^4+\gamma k^4r^2-\gamma k^4}k^{-1}}{5\cosh(\xi)+3\sinh(\frac{\xi}{\Gamma(1+)})+4r} \\ \qquad \pm\dfrac{\sqrt{\beta^2k^4+\gamma k^4}k^{-1}[5\sinh(\xi)+3\cosh(\xi)]}{5\cosh(\xi)+3\sinh(\xi)+4r}, \\ v_4 = \dfrac{4r(\mp\beta\sqrt{\beta^2k^4+\gamma k^4}+k^2\beta^2+k^2\gamma)}{5\cosh(\xi)+3\sinh(\xi)+4r}+ \\ \qquad \dfrac{16(\pm\beta\sqrt{\beta^2k^4+\gamma k^4r^2}\pm\beta\sqrt{\beta^2k^4-\gamma k^4}-k^2\gamma r^2-k^2\beta^2r^2+k^2\gamma+k^2\beta^2)}{[5\cosh(\xi)+3\sinh(\xi)+4r]^2} \\ \qquad +\dfrac{4[\pm\sqrt{\beta^2k^4r^2-\beta^2k^4+\gamma k^4r^2-\gamma k^4}(\mp k^{-2}\sqrt{\beta^2k^4+\gamma k^4}+\beta)]}{[5\cosh(\xi)+3\sinh(\xi)+4r]^2}\times \\ \qquad [5\sinh(\xi)+3\cosh(\xi)], \end{cases} \tag{19}$$

where $\xi=\frac{kx^{\alpha}}{\Gamma(1+\alpha)}+\frac{ct^{\alpha}}{\Gamma(1+\alpha)}+\xi_0$.

Family 5:

$$\begin{cases} u_5=-k^{-1}c\pm\frac{\sqrt{\beta^2k^4r^2-\beta^2k^4+\gamma k^4r^2-\gamma k^4}k^{-1}}{\cosh(\xi)+r} \\ \quad\pm\frac{\sqrt{\beta^2k^4+\gamma k^4}k^{-1}\sinh(\xi)}{\cosh(\xi)+r}, \\ v_5=\frac{r(\mp\beta\sqrt{\beta^2k^4+\gamma k^4}+k^2\beta^2+k^2\gamma)}{\cosh(\xi)+r}+ \\ \quad\frac{(\pm\beta\sqrt{\beta^2k^4+\gamma k^4}r^2\pm\beta\sqrt{\beta^2k^4-\gamma k^4}-k^2\gamma r^2-k^2\beta^2r^2+k^2\gamma+k^2\beta^2)}{[\cosh(\xi)+r]^2} \\ \quad+\frac{[\pm\sqrt{\beta^2k^4r^2-\beta^2k^4+\gamma k^4r^2-\gamma k^4}(\mp k^{-2}\sqrt{\beta^2k^4+\gamma k^4}+\beta)]\sinh(\xi)}{[\cosh(\xi)+r]^2}, \end{cases} \tag{20}$$

where $\xi=\frac{kx^{\alpha}}{\Gamma(1+\alpha)}+\frac{ct^{\alpha}}{\Gamma(1+\alpha)}+\xi_0$.

Family 6:

$$\begin{cases} u_6=-k^{-1}c\pm\frac{\sqrt{\beta^2k^4r^2-\beta^2k^4+\gamma k^4r^2-\gamma k^4}k^{-1}}{\sinh(\xi)+r} \\ \quad\pm\frac{\sqrt{\beta^2k^4+\gamma k^4}k^{-1}\cosh(\xi)}{\sinh(\xi)+r}, \\ v_6=\frac{r(\mp\beta\sqrt{\beta^2k^4+\gamma k^4}+k^2\beta^2+k^2\gamma)}{\sinh(\xi)+r}+ \\ \quad\frac{(\pm\beta\sqrt{\beta^2k^4+\gamma k^4}r^2\pm\beta\sqrt{\beta^2k^4-\gamma k^4}-k^2\gamma r^2-k^2\beta^2r^2+k^2\gamma+k^2\beta^2)}{[\sinh(\xi)+r]^2} \\ \quad+\frac{[\pm\sqrt{\beta^2k^4r^2-\beta^2k^4+\gamma k^4r^2-\gamma k^4}(\mp k^{-2}\sqrt{\beta^2k^4+\gamma k^4}+\beta)]\cosh(\xi)}{[\sinh(\xi)+r]^2}, \end{cases} \tag{21}$$

where $\xi=\frac{kx^{\alpha}}{\Gamma(1+\alpha)}+\frac{ct^{\alpha}}{\Gamma(1+\alpha)}+\xi_0$.

Remark 1. Compared with the results in [12], the established solutions in Eqs. (16)-(21) are new exact solutions for the space-time fractional Whitham-Broer-Kaup (WBK) equations, and have not been reported by other authors in the literature.

3.2. Time-fractional Sharma-Tasso-Olever (STO) equation

We consider the time-fractional Sharma-Tasso-Olever (STO) equation [18,19] of the following form

$$D_t^{\alpha}u+3au_x^2+3au^2u_x+3auu_{xx}+au_{xxx}=0. \tag{22}$$

To begin with, we suppose $u(x,t)=U(\xi)$, where $\xi=kx+\frac{ct^{\alpha}}{\Gamma(1+\alpha)}+\xi_0$, $k,\ c,\ \xi_0$ are all constants with $k,\ c\neq 0$. Then by use of (1) and the first equality in (3), Eq. (22) can be turned into

$$cU'+3ak^2(U')^2+3akU^2U'+3ak^2UU''+ak^3U'''=0. \tag{23}$$

Suppose that the solution of Eq. (23) can be expressed by

$$U(\xi)=\sum_{i=0}^{m}a_if^i+\sum_{j=1}^{m}b_jf^{j-1}g, \tag{24}$$

where $f = f(\xi)$ satisfies Eqs. (8).

Balancing the order between the highest order derivative term and nonlinear term in Eq. (23), we can obtain $m = 1$. So we have

$$U(\xi) = a_0 + a_1 f + b_1 g, \tag{25}$$

Substituting (25) along with (8) into (23) and collecting all the terms with the same power of $f^i g^j$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, yields:

Case 1:

$$a_0 = 0,\ a_1 = a_1,\ b_1 = \pm\sqrt{\frac{1}{r^2+\varepsilon}}a_1,\ k = \pm\sqrt{\frac{1}{r^2+\varepsilon}}a_1,\ c = \mp\sqrt{\frac{1}{(r^2+\varepsilon)^3}}a_1^3 a.$$

Case 2:

$$a_0 = 0,\ a_1 = a_1,\ b_1 = \pm\sqrt{\frac{1}{r^2+\varepsilon}}a_1,\ k = \pm 2\sqrt{\frac{1}{r^2+\varepsilon}}a_1,\ c = \mp 2\sqrt{\frac{1}{(r^2+\varepsilon)^3}}a_1^3 a.$$

Case 3:

$$a_0 = a_0,\ a_1 = \pm\frac{1}{2}\sqrt{r^2+\varepsilon}k,\ b_1 = \frac{1}{2}k,\ k = k,\ c = -\frac{1}{4}ka(k^2 + 12a_0^2).$$

Substituting the results above into Eq. (25), and combining with Eqs. (9)-(11) we can obtain a rich variety of exact solutions to the nonlinear fractional Sharma-Tasso-Olever (STO) equation with space- and time-fractional derivatives.

From Cases 1-2 and Eqs. (9)-(11) we obtain:

Family 1:

$$u_1 = \frac{4a_1}{5\cosh(\xi) + 3\sinh(\xi) + 4r} \pm \sqrt{\frac{1}{r^2-1}}a_1[\frac{5\sinh(\xi) + 3\cosh(\xi)}{5\cosh(\xi) + 3\sinh(\xi) + 4r}], \tag{26}$$

where $\xi = \pm\sqrt{\frac{1}{r^2-1}}a_1 x \mp \sqrt{\frac{1}{(r^2-1)^3}}\frac{a_1^3 a t^\alpha}{\Gamma(1+\alpha)} + \xi_0$ or $\xi = \pm 2\sqrt{\frac{1}{r^2-1}}a_1 x \mp 2\sqrt{\frac{1}{(r^2-1)^3}}\frac{a_1^3 a t^\alpha}{\Gamma(1+\alpha)} + \xi_0$.

Family 2:

$$u_2 = \frac{a_1}{\cosh(\xi) + r} \pm \sqrt{\frac{1}{r^2-1}}a_1[\frac{\sinh(\xi)}{\cosh(\xi) + r}], \tag{27}$$

where $\xi = \pm\sqrt{\frac{1}{r^2-1}}a_1 x \mp \sqrt{\frac{1}{(r^2-1)^3}}\frac{a_1^3 a t^\alpha}{\Gamma(1+\alpha)} + \xi_0$ or $\xi = \pm 2\sqrt{\frac{1}{r^2-1}}a_1 x \mp 2\sqrt{\frac{1}{(r^2-1)^3}}\frac{a_1^3 a t^\alpha}{\Gamma(1+\alpha)} + \xi_0$.

Family 3:

$$u_3 = \frac{a_1}{\sinh(\xi) + r} \pm \sqrt{\frac{1}{r^2+1}}a_1[\frac{1}{\sinh(\xi) + r}], \tag{28}$$

where $\xi = \pm\sqrt{\frac{1}{r^2-1}}a_1 x \mp \sqrt{\frac{1}{(r^2-1)^3}}\frac{a_1^3 a t^\alpha}{\Gamma(1+\alpha)} + \xi_0$ or $\xi = \pm 2\sqrt{\frac{1}{r^2-1}}a_1 x \mp 2\sqrt{\frac{1}{(r^2-1)^3}}\frac{a_1^3 a t^\alpha}{\Gamma(1+\alpha)} + \xi_0$.

From Case 3 and Eqs. (9)-(11) we obtain:

Family 4:

$$u_4 = a_0 \pm \frac{2\sqrt{r^2-1}k}{5\cosh(\xi)+3\sinh(\xi)+4r} + \frac{1}{2}k[\frac{5\sinh(\xi)+3\cosh(\xi)}{5\cosh(\xi)+3\sinh(\xi)+4r}], \tag{29}$$

where $\xi = kx - \frac{ka(k^2+12a_0^2)t^\alpha}{4\Gamma(1+\alpha)} + \xi_0$.

Family 5:

$$u_5 = a_0 \pm \frac{\sqrt{r^2-1}k}{2[\cosh(\xi)+r]} + \frac{1}{2}k[\frac{\sinh(\xi)}{\cosh(\xi)+r}], \tag{30}$$

where $\xi = kx - \frac{ka(k^2+12a_0^2)t^\alpha}{4\Gamma(1+\alpha)} + \xi_0$.

Family 6:

$$u_6 = a_0 \pm \frac{\sqrt{r^2+1}k}{2[\sinh(\xi)+r]} + \frac{1}{2}k[\frac{\cosh(\xi)}{\sinh(\xi)+r}], \tag{31}$$

where $\xi = kx - \frac{ka(k^2+12a_0^2)t^\alpha}{4\Gamma(1+\alpha)} + \xi_0$.

Remark 2. In [18], Song et al. obtained a rational approximation solution as denoted in Eqs. (6.10), (6.11), (6.17), (6.18), (6.24), (6.25) in [18] for Eq. (22) by use of the variational iteration method, the Adomian decomposition method and the homotopy perturbation method respectively, while in [19], Lu obtained some exact solutions as denoted in Eqs. (53)-(56), (58) in [19] for Eq. (22) by use of the first integral method. We note first that our results are different from those in [18], since the solution obtained in [18] is an approximation solution, while our results are direct exact solutions. Furthermore, as a substantial different method was used here, the solutions established in Eqs. (26)-(31) here are of different forms from those in [19], and are new exact solutions for the nonlinear fractional Sharma-Tasso-Olever equation so far in the literature.

Remark 3. The method used above can also be used to obtain solutions to some certain initial or boundary value problems, that is, solutions to initial or boundary value problems are particular cases of the solutions obtained above. For example, in Eq. (22), if we add the initial value condition

$$u(x,0) = \coth(x+1).$$

Then after substituting this initial value condition to the hyperbolic solutions obtained in Eq. (31) and fulfilling some basic comparison and computation, one can see that $k=2,\ r=0,\ a_0=0,\ \xi_0=2$. So we obtain the solution to initial value problem as

$$u(x,t) = \frac{1}{\sinh(2x - \frac{2at^\alpha}{\Gamma(1+\alpha)} + 2)} + \coth(2x - \frac{2at^\alpha}{\Gamma(1+\alpha)} + 2).$$

4. **Conclusions**

We have applied the projective Riccati equation method for solving fractional partial differential equations, and applied it to find exact solutions of the space-time fractional Whitham-Broer-Kaup (WBK) equations and the time fractional Sharma-Tasso-Olever (STO) equation. The most important point of this approach lies in that a nonlinear fractional complex transformation from $x,\ t$ to ξ is used here, which ensures that a certain fractional partial differential equation can be turned into another ordinary differential equation of integer order, whose solutions can be expressed by a polynomial in the solutions of the projective Riccati equations. Finally, we note this method can also be applied to solve other fractional differential equations.

5. Acknowledgements

This work was partially supported by Natural Science Foundation of Shandong Province (in China) (grant No. ZR2013AQ009), Doctoral initializing Foundation of Shandong University of Technology (in China) (grant No. 4041-413030).

The author would like to thank the anonymous reviewers very much for their valuable suggestions on improving this paper.

REFERENCES

[1] *A.M.A. El-Sayed and M. Gaber*, The Adomian decomposition method for solving partial differential equations of fractal order in finite domains, Phys. Lett. A, **359**(2006), 175-182.

[2] *A.M.A. El-Sayed, S.H. Behiry and W.E. Raslan*, Adomian's decomposition method for solving an intermediate fractional advection-dispersion equation, Comput. Math. Appl., **59**(2010), 1759-1765.

[3] *J.H. He*, A new approach to nonlinear partial differential equations, Commun. Nonlinear Sci. Numer. Simul., **2**(1997), 230-235.

[4] *G. Wu and E.W.M. Lee*, Fractional Variational Iteration Method And Its Application, Phys. Lett. A, **374**(2010), 2506-2509.

[5] *S. Guo and L. Mei*, The fractional variational iteration method using He's polynomials, Phys. Lett. A, **375**(2011), 309-313.

[6] *J.H. He*, Homotopy perturbation technique, Comput. Methods Appl. Mech. Engrg., **178**(1999), 257-262.

[7] *J.H. He*, A coupling method of homotopy technique and a perturbation technique for non-linear problems, Inter. J. Non-Linear Mech., **35**(2000), 37-43.

[8] *Z. Odibat and S. Momani*, Fractional Green function for linear time-fractional equations of fractional order, Appl. Math. Lett., **21**(2008), 194-199.

[9] *M. Cui*, Compact finite difference method for the fractional diffusion equation, J. Comput. Phys., **228**(2009), 7792-7804.

[10] *Q. Huang, G. Huang and H. Zhan*, A finite element solution for the fractional advection-dispersion equation, Adv. Water Resour., **31**(2008), 1578-1589.

[11] *S. Zhang and H.Q. Zhang*, Fractional sub-equation method and its applications to nonlinear fractional PDEs, Phys. Lett. A, **375**(2011), 1069-1073.

[12] *S.M. Guo, L.Q. Mei, Y. Li and Y.F. Sun*, The improved fractional sub-equation method and its applications to the space-time fractional differential equations in fluid mechanics, Phys. Lett. A, **376**(2012), 407-411.

[13] *B. Lu*, Bäcklund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations, Phys. Lett. A, **376**(2012), 2045-2048.

[14] *D.J. Huang and H.Q. Zhang*, Exact travelling wave solutions for the Boiti-Leon-Pempinelli equation, Chaos, Solitons and Fractals, **22**(2004), 243-247.

[15] *D.J. Huang and H.Q. Zhang*, Variable-coefficient projective Riccati equation method and its application to a new (2+1)-dimensional simplified generalized Broer-Kaup system, Chaos, Solitons and Fractals, **23**(2005),601-607.

[16] *G. Jumarie*, Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results, Comput. Math. Appl., **51**(2006),1367-1376.

[17] *B. Zheng*, (G'/G)-Expansion Method for Solving Fractional Partial Differential Equations in the Theory of Mathematical Physics, Commun. Theor. Phys. (Beijing, China), **58**(2012),623-630.

[18] *L.N. Song, Q. Wang and H.Q. Zhang*, Rational approximation solution of the fractional Sharma-Tasso-Olever equation, J. Comput. Appl. Math., **224**(2009),210-218.

[19] *B. Lu*, The first integral method for some time fractional differential equations, J. Math. Anal. Appl., **395**(2012),684-693.