APPLYING CUBIC B-SPLINE QUASI-INTERPOLATION TO SOLVE HYPERBOLIC CONSERVATION LAWS

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Numerical Solution of hyperbolic conservation laws is important in computational fluid dynamics. In this paper, we present a new numerical method to solve the hyperbolic conservation laws, which is constructed by using the derivative of the cubic B-spline quasi-interpolation to approximate the spatial derivative of the dependent variable and first order forward difference to approximate the time derivative of the dependent variable. Moreover, the method for advection equation and one-dimensional Burgers' equation (without viscosity) is verified with some numerical examples. The advantage of the resulting scheme is that the algorithm is very simple, so it is very easy to implement.

Keywords: Hyperbolic conservation laws; Numerical solution; Cubic B-splines; Quasi-interpolation

1. Introduction

The study of hyperbolic conservation laws, as described by

$$u_t + (f(u))_x = 0 \tag{1}$$

with initial condition $u(x,0) = u_0(x)$ is a classical topic in Computational Fluid Dynamics (CFD). Hyperbolic conservation laws arise as diverse as compressible gas dynamics, shallow water prediction, plasma modeling, rarefied gas dynamics and many others [1]. As is known, the solutions of hyperbolic equation laws may develop discontinuities in finite time even when the initial condition is smooth. A successful method should compute such discontinuities with the correct position and without spurious oscillations and retain high order of accuracy in smooth regions.

We know that high order linear schemes will generate spurious oscillations like standard finite-difference schemes (Lax and Wendroff [2] or MacCormack [3]) near discontinuities or sharp gradients of the solution from the theorem of Godunov [4, 5, 6]. These solutions can mask the physical solutions, even leading to code crashing. These deviations do not diminish with solution, in

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analogy with Gibbs phenomenon found in the Fourier series development of discontinuous functions. To avoid generating spurious oscillations, many researchers developed a lot of numerical schemes to solve the problem. Harten et. al [7, 8] presented the TVD and ENO methods. Balsara, Jiang and Shu [9, 10] introduced the WENO methods. More recently, Takakura, Titarev and Toro [11-17] gave ADER schemes for hyperbolic conservation laws and studied many different equations by ADER.

As the piecewise polynomial, spline, especially B-spline, has become a fundamental tool for numerical methods to get the solution of the differential equations. B-splines of various degrees in the collocation and Galerkin methods are introduced for the numerical solutions of the Burgers' equation in [18-25]. Chen and Wu [26, 27] studied the hyperbolic conservation laws by Multiquadric (MQ) quasi-interpolation.

From previous researches, we find both spline and quasi-interpolation are simple and effective for differential equations. In [25], Zhu and Wang presented a numerical scheme of the Burgers' equation based on cubic B-spline quasi-interpolation. In this paper, we provide a numerical scheme to solve hyperbolic conservation laws using the derivative of the cubic B-spline quasi-interpolation to approximate the spatial derivative of the differential equations and employ the first order accuracy forward difference for the approach of the temporal derivative as [25, 27]. Then we do not require solving any linear system of equation so that we do not meet the question of the ill-condition of the matrix. Therefore, we can save the computational time and decrease the numerical error.

The paper is organized as follows: In Sect. 2, we introduce the univariate B-spline quasi-interpolants. The numerical scheme using cubic B-spline quasi-interpolation to solve hyperbolic conservation laws is presented in Sect. 3. In Sect. 4, we perform a battery of standard tests in one space dimension, covering advection equation and Burgers equation (without viscosity). The numerical results are presented and compared with the exact solutions.

2. Univariate B-spline quasi-interpolants

For $I=[a,b]$, we denote by $S_d(X_n)$ the univariate spline space of degree d and C^{d-1} on the uniform partition $X_n=\{x_i=a+ih, i=0,\cdots,n\}$ with the meshlength $h=(b-a)/n$, where $b=x_n$. Let the B-spline basis of $S_d(X_n)$ be $\{B_j, j\in J\}$ with $J=\{1,2,\cdots,n+d\}$, which can be computed by the de Boor-Cox formula [28, 29]. With these notations, the support of B_j is $supp(B_j)=[x_j, x_{j+d+1}]$. As usual, we add multiple knots at the endpoints: $a=x_{-d}=x_{-d+1}=\cdots=x_0$ and $b=x_n=x_{n+1}=\cdots=x_{n+d}$.

In [30], the univariate B-spline quasi-interpolant (abbr. QI) can be defined as an operator of the form

$$Q_d f = \sum_{j \in J} \mu_j(f) B_j \tag{2}$$

We denote by Π_d the space of polynomials of total degree at most d. In general, we impose that Q_d is exact on the space Π_d, i.e. $Q_d p = p$ for all $p \in \Pi_d$. As a consequence of this property, the approximation order of Q_d is $O(h^{d+1})$ on smooth functions. In this paper, the coefficient μ_j is a linear combination of discrete values of f at some points in the neighborhood of $supp(B_j)$ as introduced in [30].

The main advantage of QIs is that they have a direct construction without solving any system of linear equations. Moreover, they are local, in the sense that the value of $Q_d f(x)$ depends only on values of f in a neighborhood of x. Finally, they have a rather small infinity norm, so they are nearly optimal approximants [30]. The quasi-interpolation operators are also studied in the book of Schumaker [29]. In this paper, we use cubic B-spline quasi-interpolation since the cubic spline is used widely in numerical analysis.

Using the de Boor-Cox formula [28, 29], for $j \in J$, the cubic B-spline basis B_j can be computed. Let $f_i = f(x_i), i = 0,1,\cdots,n$. For the cubic B-spline QI defined as

$$Q_3 f = \sum_{j=1}^{n+3} \mu_j(f) B_j \,, \tag{3}$$

where the coefficients are listed as follows:

$$\begin{aligned}
&\mu_1(f) = f_0, \\
&\mu_2(f) = \frac{1}{18}(7f_0 + 18f_1 - 9f_2 + 2f_3), \\
&\mu_j(f) = \frac{1}{6}(-f_{j-3} + 8f_{j-2} - f_{j-1}),\ j = 3,\cdots,n+1, \\
&\mu_{n+2}(f) = \frac{1}{18}(2f_{n-3} - 9f_{n-2} + 18f_{n-1} + 7f_n), \\
&\mu_{n+3}(f) = f_n,
\end{aligned} \tag{4}$$

For $f \in C^4(I)$, we have the error estimate [30] as

$$\|f - Q_3 f\|_{\infty, I_k} \le \frac{8}{3} d_{\infty, I_k}(f, \Pi_3) \text{ for } 1 \le k \le n \Rightarrow \|f - Q_3 f\|_\infty = O(h^4), \tag{5}$$

where $I_k = [x_k, x_{k+1}], k = 0,1,\cdots,n-1.$

Differentiating interpolation polynomials leads to classical finite differences for the approximate computation of derivatives. Therefore, it seems natural to approximate derivatives of f by derivatives of $Q_d f\left(x\right)$ as long as it is possible, i.e. up to the order $d-1$. The general theory is developed elsewhere.

For $j \in J$, we can compute B_j' by the formula of cubic B-spine's derivatives as shown in [28]. Then we obtain the differential formulas for cubic B-spline QI as

$$f' \approx (Q_3 f)' = \sum_{j=1}^{n+3} \mu_j(f) B_j' . \tag{6}$$

3. Numerical scheme using cubic B-spline quasi-interpolation

In this paper, we solve mainly the one-dimensional scalar hyperbolic conservation laws

$$u_t + (f(u))_x = 0 \tag{7}$$

with initial condition

$$u(x,0)=u_0(x) \tag{8}$$

by using the cubic B-spline quasi-interpolation.

We can rewrite (7) as

$$u_t + a(u)u_x=0 \tag{9}$$

and discretize Equation (9) in time as

$$u_j^{n+1} = u_j^n - \tau \cdot (a(u))_j^n \cdot (u_x)_j^n , \tag{10}$$

where, $a(u) = (f(u))_u$, u_j^n is the approximation of the value of $u(x,t)$ at point (x_j, t_n) , $t_n = n\tau$; τ is the length of time step; $(a(u))_j^n$ is the value of the function $a(u)$ at $u = u_j^n$; and the $(u_x)_j^n$ is computed by Equation (6).

Since the scheme may be dispersed as discussed in [25, 26, 27], we define the switch function $g(x,t)$ to dump it as follows:

$$g_j^n = \max\{0, 1 + \min\{0, sign((u_x)_j^n \cdot (u_x)_k^n\}\}, \tag{11}$$

where $k = j - sign((a(u))_j^n)$. Thus the resulting numerical scheme is

$$u_j^{n+1} = u_j^n - \tau \cdot (a(u))_j^n \cdot (u_x)_j^n \cdot g_j^n \tag{12}$$

From the initial condition (7) and (8), we can compute the numerical solution of the hyperbolic conservation laws step by step using scheme (6) and (12).

4. Numerical experiments and accuracy tests

In this section, we test the proposed numerical scheme by means of some standard numerical experiments in one space dimension. We denote the present scheme by BSQI. The versatility and the accuracy of the proposed method are measured using the L_∞ error norms. The error norm is defined as

$$\|e\|_{L_\infty} = \max_i |e_i| = \max_i \left| u_i^{exact} - u_i^{num} \right| \ . \qquad (13)$$

4.1. *Advection equation*

Scalar advection equation is the simplest linear case, but it allows testing the propagation of arbitrary initial profiles, containing jump discontinuities and corner points, departing from smoothness in many different ways. This is the case of the Balsara-Shu profile [9], which will be evolved with periodic boundary conditions as

$$u_t + u_x = 0, \ -1 < x < 1 \qquad (14)$$

subject to the initial data $u(x,0) = \sin(\pi x)$. The accuracy of BSQI is computed at $t = 1, \tau = 0.0001$, and numerical results are presented in Table 1 and Fig. 1 versus the exact solution. In Table 1, the BSQI method is compared to different WENO methods which are presented in [9].

Table 1

The $L\infty$ error of the solution at $t = 1$ with $h = 1/N, \tau = 0.0001$

N	BSQI	WENO (r=5)	MPWENO (r=5)	MPWENO (r=3)
5	2.3128e-2	5.5930e-4	8.6886e-4	3.0224e-2
10	1.5460e-3	1.1927e-6	1.1927e-6	1.4569e-3
20	5.1867e-4	2.2653e-9	2.2653e-9	4.5939e-5
40	4.9439e-4	4.1460e-12	4.4160e-12	1.4783e-6

Table 2 shows the convergence studies for the advection equation (14) with initial condition $u(x,0) = \sin^4(\pi x)$. The results are computed at $t = 1$, $h = 1/N$, $\tau = 0.0001$. The BSQI method is also compared to different WENO methods which are introduced in [9].

From Table 1 and Table 2, we found that the presented method is not accurate than WENO methods, but we do not require solving any linear system of equation, so that we do not meet the question of the ill-condition of the matrix.

Table 2

The $L\infty$ error of the solution at $t = 1$ with $h = 1/N, \tau = 0.0001$

N	BSQI	WENO (r=5)	MPWENO (r=5)	MPWENO (r=3)
20	9.1127e-3	1.0711e-4	2.4370e-4	8.9043e-3
40	2.1422e-3	7.4607e-6	1.7941e-4	1.8086e-3
80	1.9788e-3	2. 8738e-8	2.1792e-5	1.7678e-4
100	1.9786e-3	1.2815e-10	1.8768e-6	1.6388e-5

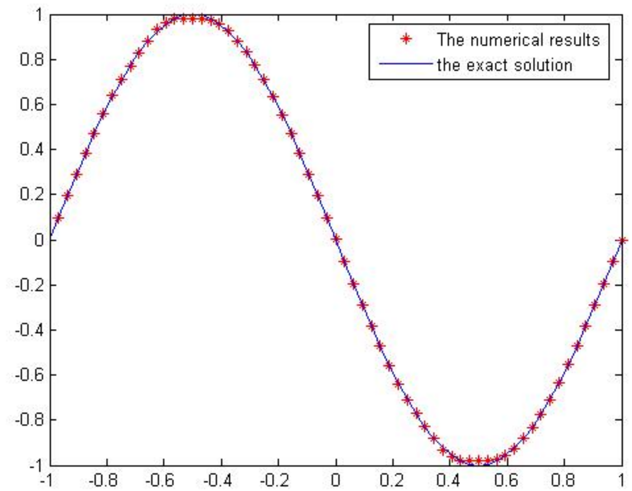

Fig. 1 Compute solutions of advection equation at t=1 ($h = 1/32, \tau = 0.0001$).

4.2. *Burgers' equation (without viscosity)*

Burgers equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \tag{15}$$

with the initial condition $u(x,0) = 1 + 1/2\sin(\pi x)$ provides a simple example of genuinely non-linear scalar equation. A true shock develops from smooth initial data. The accuracy of the computations was checked at t=0.33 with the accurate solution [8] and NT method which is presented by [1], that is, before the shock develops and the results are presented in Table 3. Fig. 2 shows the numerical solution at t=0.5. Moreover, the results at $t = 0.636 \approx \pi/5$, which is the time of the formation of the shock, are shown in Fig. 3, where $h = 1/32, \tau = 0.0001$.

Another example corresponds to initial condition $u(x,0) = \sin(\pi(x+1))$. The numerical results of BSQI algorithm, for $t = 0.1, 0.2, 0.3, 0.4$, with initial data are given in Fig. 4, where $h = 1/32, \tau = 0.0001$. Table 3 and Figs 2-4 show that the presented method for solving Burgers' equation (without viscosity) is effective.

Table 3

The $L\infty$ error of the solution at $t = 0.33$ with $h = 1/N, \tau = 0.0001$

N	MNT1	MNT3	NT	BSQI	MQQI (t=0.3, τ=0.3h^2)
32	2.6644e-2	1.2722e-2	4.8091e-3	4.4431e-4	0.0046
64	8.4934e-3	3.4765e-3	1.8998e-3	2.6491e-4	0.0011
128	2.3333e-3	8.7231e-4	6.3009e-4	2.5634e-4	---
256	6.0026e-4	2.1341e-4	2.0354e-4	2.5575e-4	---

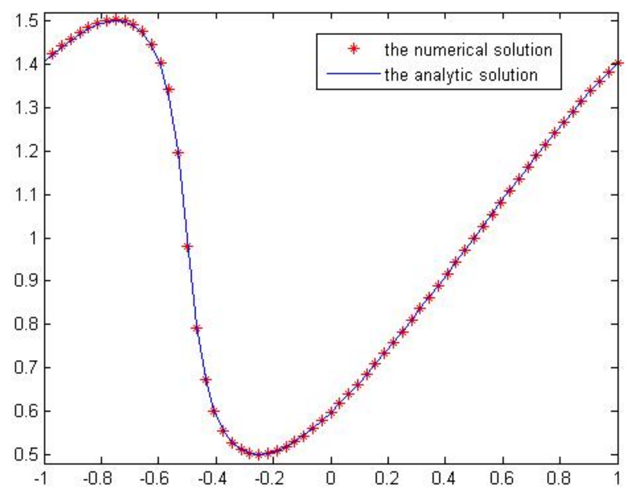

Fig. 2 Compute solutions of Burgers' equation at *t*=0.5 ($h = 1/32, \tau = 0.0001$).

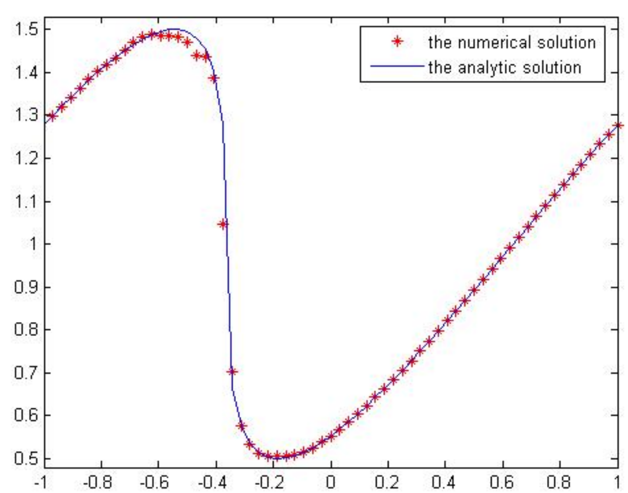

Fig. 3 Compute solutions of Burgers' equation at *t*=0.636 ($h = 1/32, \tau = 0.0001$).

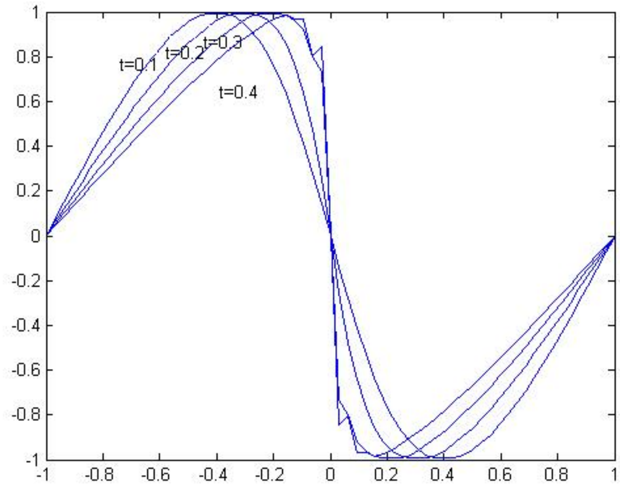

Fig. 4 Compute solutions of Burgers' equation at t=0.1, 0.2, 0.3, 0.4 ($h = 1/32, \tau = 0.0001$).

5. Conclusions and future work

In this article, we present a numerical scheme (BSQI) using cubic B-spline quasi-interpolation using switch function to deal with hyperbolic conservation laws. The numerical results illustrate the algorithm is more effective than Chen and Wu's MQ method [27]. Although it is not accurate than WENO methods in [9], it is simpler than WENO method. The WENO methods are perfect in theory, but they are difficult to implement. From the numerical experiments, we can say that the presented algorithm is feasible and the error is acceptable.

The algorithm can be generalized to other ordinary or partial differential equations and it is easy to implement. Unfortunately, the stability of BSQI scheme is unsolved. It is future work for the authors.

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