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DYNAMIC ANALYSIS OF THE 3-3 STEWART PLATFORM

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Lucrarea prezentă stabileşte relaţii matriceale recursive pentru platforma Stewart cu şase actuatori prismatici. Controlat de şase forţe, prototipul manipulatorului este un sistem mecanic spaţial cu şase grade de libertate şi cu şase lanţuri cinematice ce conectează platforma mobilă. Cunoscând poziţia şi mişcarea generală a platformei, se dezvoltă mai întâi cinematica inversă şi se determină poziţia, viteza şi acceleraţia fiecărui element al manipulatorului. În continuare, problema dinamică inversă este rezolvată cu principiul lucrului mecanic virtual. În finalul lucrării sunt obţinute ecuaţii matriceale compacte şi grafice de simulare pentru forţele active.

Recursive matrix relations in kinematics and dynamics of the 3-3 Stewart platform having six prismatic actuators are established in this paper. Controlled by six forces, the parallel manipulator prototype is a spatial six-degrees-of-freedom mechanical system with six parallel legs connecting to the moving platform. Knowing the position and the general motion of the platform, we first develop the inverse kinematics problem and determine the position, velocity and acceleration of each manipulator's link. Further, the inverse dynamic problem is solved using an approach based on the principle of virtual work. Finally, compact matrix equations and graphs of simulation for the active forces are obtained

Keywords: dynamics modelling, kinematics, parallel mechanism, virtual work

1. Introduction

Parallel manipulators are closed-loop mechanisms presenting very good potential in terms of accuracy, rigidity and ability to manipulate large loads. In general, these manipulators consist of two main bodies coupled via numerous legs acting in parallel. One body is arbitrarily designated as fixed and is called the *base*, while the other is regarded as movable and hence is called the the *moving platform* of the manipulator. Several mobile legs or limbs, made up as serial robots, connect the movable platform to the fixed frame. The links of the robot are connected one to the other by spherical joints, universal joints, revolute joints or prismatic joints. Typically, the number of actuators is equal to the number of degrees of freedom such that every link is controlled at or near the fixed base [1].

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Compared with serial mechanisms, parallel manipulator is a complex mechanical structure, presenting some the special characteristics such as: greater rigidity, potentially higher kinematical precision, stabile capacity and suitable position of arrangement of actuators. However, they suffer the problems of relatively small useful workspace and design difficulties.

Parallel robots can be equipped with hydraulic or prismatic actuators. They have a robust construction and can move bodies of large dimensions with high speeds. That is the reason why the devices which produce translation or spherical motion to a platform, technically are based on the concept of parallel manipulator.

Parallel mechanisms can be found in practical applications, in which it is desired to orient a rigid body in space with high speed, such as aircraft simulators [2], positional tracker and telescopes [3], [4]. More recently, they have been used by many companies in the development of high precision machine tools [5], [6], such as Giddings & Lewis, Hexel, Geodetic and Toyoda.

Recently, considerable efforts have been devoted to the kinematics and dynamic analysis of fully parallel manipulators. Among these, the class of manipulators known as Stewart-Gough-like platform focused great attention (Stewart [2]; Merlet [7]; Parenti Castelli and Di Gregorio [8]). They are used in flight simulators and more recently for Parallel Kinematics Machines.

The prototype of Delta parallel robot (Clavel [9]; Staicu and Carp-Ciocardia [10]; Tsai and Stamper [11]) developed by Clavel at the Federal Polytechnic Institute of Lausanne and by Tsai and Stamper at the University of Maryland as well as the Star parallel manipulator (Hervé and Sparacino [12]) are equipped with three motors which train on the mobile platform in a three-degree-of-freedom general translation motion.

Angeles [13], Gosselin and Gagné [14], Wang and Gosselin [15] analysed the direct kinematics, dynamics and singularity loci of the Agile Wrist spherical parallel robot with three concurrent actuators.

The analysis of parallel manipulators is usually implemented trough analytical methods in classical mechanics, in which projection and resolution of vector equations on the reference axes are written in a considerable number of cumbersome, scalar relations and the solutions are rendered by large scale computations together with time consuming computer codes [16], [17].

In the present paper, a new recursive matrix method is introduced. It has been proved to reduce the number of equations and computation operations significantly by using a set of matrices for kinematics and dynamics models.

2. Inverse kinematics

A spatial 6-DOF parallel manipulator, which present in several applications including machine tools, is proposed in this paper. Since the pneumatic joints can easily achieve high accuracy and heavy loads, the majority of the 3-DOF or of the 6-DOF parallel mechanisms use the actuated prismatic joints.

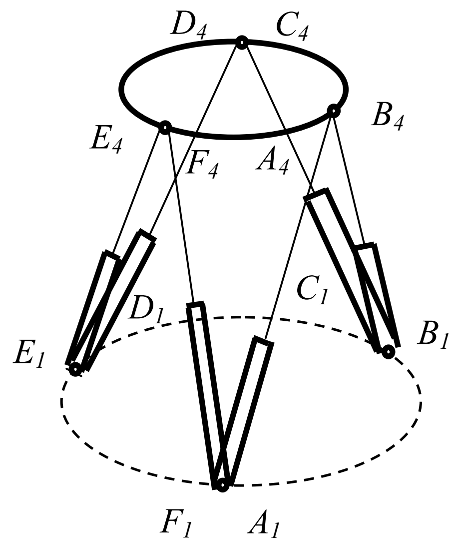

Fig. 1 General scheme of the 3-3 Stewart platform

The Stewart platform is a six-degrees-of-freedom fully spatial parallel mechanism in which a moving platform is connected together to a fixed base by six extensible legs from universal and spherical joints. Each leg is made up of two binary links that are connected by a prismatic joint. Ball screws or pneumatic jacks can be used to vary the lengths of the prismatic joints and to control the location of the platform.

The first design for industrial purposes can be dated back to 1962, when Gough implemented a six-linear jack system for use as a Universal tire-testing machine. In fact, it was a huge force sensor, capable of measuring forces and torques on a wheel in all directions. Some years later, Stewart published a design of a platform robot for use as a flight simulator [2].

A special design, called the 3-3 Stewart platforms or the *octahedral platforms*, usually contains three concentric spherical joints at the moving platform and three concentric universal joints at the fixed base (Fig.1). This special construction makes closed-form direct kinematics solutions feasible.

We suppose a moving platform symbolically represented by three pairs of concentric spherical joints $A_4 = B_4$, $C_4 = D_4$, $E_4 = F_4$ and a fixed base represented by another three pairs of concentric universal joints located at the points $A_1 = F_1$, $B_1 = C_1$, $D_1 = E_1$. We assume also that the two sets of concentric joints forms two equilateral triangles.

For the purpose of analysis, we attach a Cartesian coordinate system $Ox_0y_0z_0(T_0)$ to the fixed base with its origin located at triangle centre O, the Oz_0 axis perpendicular to the base and the Ox_0 axis pointing along line OA_1. Another coordinate central frame $Gx_Gy_Gz_G$ could be linked just at the centre G of the moving platform.

In what follows we consider that the moving platform is initially located at a *central configuration*, where the moving platform is not rotated with respect to the fixed base and the mass centre G is at an elevation $OG = h$ above the centre of the fixed base.

The mechanism of the manipulator consists of six chains, including six variable lengths with identical topology, all connecting the fixed base to the moving platform. One of these identical active legs (leg $A \equiv A_1A_2A_3A_4$, for example) consists of a fixed Hooke joint, characterized by a mass m_1 and a tensor of inertia $\hat{J}_1$, which has the angular velocity $\omega_{10}^A = \dot{\varphi}_{10}^A$ and the angular acceleration $\varepsilon_{10}^A = \ddot{\varphi}_{10}^A$, and a moving cylinder of length l_2, mass m_2 and tensor of inertia $\hat{J}_2$, which has a relative rotation about $A_2y_2^A$ axis with the angle φ_{21}^A, so that $\omega_{21}^A = \dot{\varphi}_{21}^A$, $\varepsilon_{21}^A = \ddot{\varphi}_{21}^A$. An actuated prismatic joint is as well as a piston linked to the $A_3x_3^Ay_3^Az_3^A$ frame, having a relative displacement λ_{32}^A, velocity $v_{32}^A = \dot{\lambda}_{32}^A$ and acceleration $\gamma_{32}^A = \ddot{\lambda}_{32}^A$. It has the length l_3, mass m_3 and tensor of inertia $\hat{J}_3$. Finally, a ball-joint or a spherical joint is attached to the moving platform, which is schematised as a circle of radius l_0, mass m_p and inertia tensor $\hat{J}_p$ (Fig. 2).

At the central configuration, we also consider that all legs are extended at equal lengths and that the angles of orientation of universal joints and spherical joints are given by

$$\begin{aligned}
&\alpha_1^A = \alpha_1^F = 0, \alpha_1^B = \alpha_1^C = \frac{2\pi}{3}, \alpha_1^D = \alpha_1^E = -\frac{2\pi}{3} \\
&\alpha_2^A = \alpha_2^C = \alpha_2^E = -\frac{\pi}{3}, \alpha_2^B = \alpha_2^D = \alpha_2^F = \frac{\pi}{3} \\
&\alpha_3^A = \alpha_3^C = \alpha_3^E = \frac{\pi}{3}, \alpha_3^B = \alpha_3^D = \alpha_3^F = -\frac{\pi}{3} .
\end{aligned} \tag{1}$$

Assuming that the each leg is connected to the fixed base by universal joint such that it cannot rotate about the longitudinal axis, the orientation of the leg A with respect to the fixed base can be described by two Euler angles, namely a rotation angle φ_{10}^A about the $A_1x_1^A$ axis, followed by another rotation of angle φ_{21}^A

about the rotated $A_2 y_2^A$ axis. Pursuing the first leg A in the $OA_1A_2A_3A_4$ way, we obtain the following matrices of transformation [18]:

$$a_{10} = a_{10}^{\varphi} a_{2\alpha}^{A} a_{1\alpha}^{A}, \; a_{21} = a_{21}^{\varphi} a_{\beta}, \; a_{32} = \theta, \tag{2}$$

where

$$a_{i\alpha}^{A} = \begin{bmatrix} \cos\alpha_i^A & \sin\alpha_i^A & 0 \\ -\sin\alpha_i^A & \cos\alpha_i^A & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (i = 1, 2, 3)$$

$$a_{\beta} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}, \; \theta = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$a_{10}^{\varphi} = R(x, \varphi_{10}^{A}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_{10}^{A} & \sin\varphi_{10}^{A} \\ 0 & -\sin\varphi_{10}^{A} & \cos\varphi_{10}^{A} \end{bmatrix}, \; a_{21}^{\varphi} = R(y, \varphi_{21}^{A}) = \begin{bmatrix} \cos\varphi_{21}^{A} & 0 & -\sin\varphi_{21}^{A} \\ 0 & 1 & 0 \\ \sin\varphi_{21}^{A} & 0 & \cos\varphi_{21}^{A} \end{bmatrix}$$

$$a_{k0} = \prod_{j=1}^{k} a_{k-j+1,k-j}, \quad (k = 1, 2, 3).$$

Analogous relations can be written for other five legs of the mechanism.

Six displacements $\lambda_{32}^{A}, \lambda_{32}^{B}, \lambda_{32}^{C}, \lambda_{32}^{D}, \lambda_{32}^{E}, \lambda_{32}^{F}$ of the active links are the variables that gives the instantaneous position of the mechanism. But, in the inverse geometric problem, it can be considered that the coordinates of the mass centre G of the moving platform x_0^G, y_0^G, z_0^G and the three known Euler angles α_1, α_2, α_3 of successive rotations about the Gx_G, Gy_G, Gz_G axes give the position of the mechanism.

For convenience, we introduce three basic rotation matrices. Since all rotations take place successively about the moving coordinate axes, the resulting rotation matrix is obtained by multiplying three known rotation matrices:

$$R_1 = R(x, \alpha_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & \sin\alpha_1 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \end{bmatrix}, \; R_2 = R(y, \alpha_2) = \begin{bmatrix} \cos\alpha_2 & 0 & -\sin\alpha_2 \\ 0 & 1 & 0 \\ \sin\alpha_2 & 0 & \cos\alpha_2 \end{bmatrix}$$

$$R_3 = R(z, \alpha_3) = \begin{bmatrix} \cos\alpha_3 & \sin\alpha_3 & 0 \\ -\sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

Then, the general rotation matrix of the moving platform from $Ox_0y_0z_0(T_0)$ to $Gx_Gy_Gz_G$ reference system is given by

$$R = R_3R_2R_1 \,. \tag{5}$$

We suppose that the coordinates of the platform's centre G and the Euler angles α_1, α_2, α_3, which are expressed by following analytical functions

$$x_0^G = x_0^{G*}(1-\cos\frac{2\pi}{3}t)\,,\ y_0^G = y_0^{G*}(1-\cos\frac{2\pi}{3}t)\,,\ z_0^G = h - z_0^{G*}(1-\cos\frac{2\pi}{3}t)$$

$$\alpha_l = \alpha_l^*(1-\cos\frac{2\pi}{3}t),\quad (l=1,2,3)\,, \tag{6}$$

can describe the general absolute motion of the moving platform.

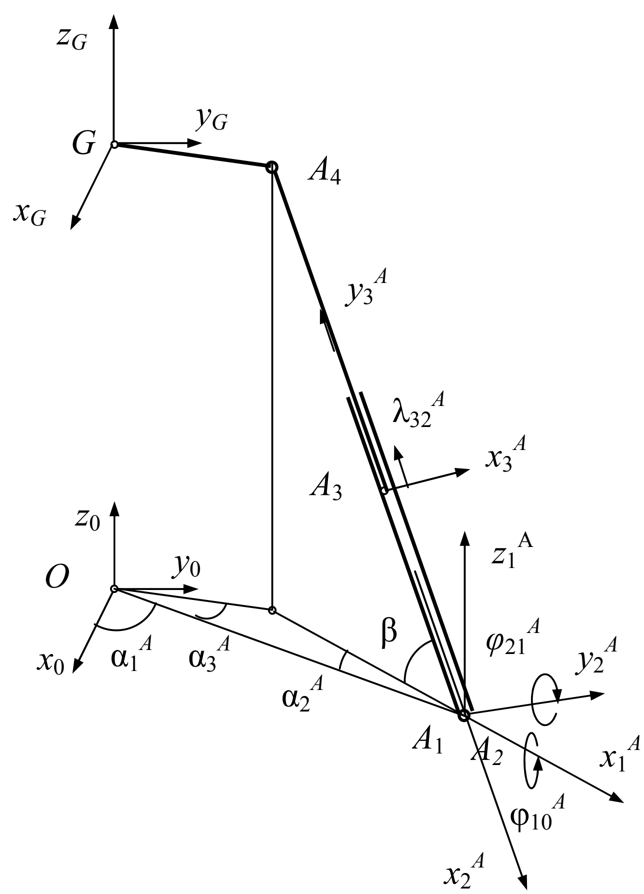

Fig. 2 Kinematical scheme of first leg A of the mechanism

The variables $\varphi_{10}^{A}, \varphi_{21}^{A}, \lambda_{32}^{A}, \ldots, \varphi_{10}^{F}, \varphi_{21}^{F}, \lambda_{32}^{F}$ will be determined by several vector-loop equations, as follows

$$\begin{aligned}
&\vec{r}_{10}^{\,A} + \sum_{k=1}^{3} a_{k0}^{T} \vec{r}_{k+1,k}^{\,A} - R^{T} \vec{r}_{G}^{\,A_4} = \\
&= \vec{r}_{10}^{\,B} + \sum_{k=1}^{3} b_{k0}^{T} \vec{r}_{k+1,k}^{\,B} - R^{T} \vec{r}_{G}^{\,B_4} = \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&= \vec{r}_{10}^{\,E} + \sum_{k=1}^{3} e_{k0}^{T} \vec{r}_{k+1,k}^{\,E} - R^{T} \vec{r}_{G}^{\,E_4} = \\
&= \vec{r}_{10}^{\,F} + \sum_{k=1}^{3} f_{k0}^{T} \vec{r}_{k+1,k}^{\,F} - a^{T} \vec{r}_{G}^{\,F_4} = \vec{r}_{0}^{\,G},
\end{aligned} \tag{7}$$

where

$$\vec{r}_{10}^{\,A} = l_0 a_{1\alpha}^{AT} \vec{u}_1,\ \vec{r}_{21}^{\,A} = \vec{0},\ \vec{r}_{32}^{\,A} = -(l_1 + \lambda_{32}^{A}) \vec{u}_1$$

$$\vec{r}_{43}^{\,A} = l_3 \vec{u}_2,\ \vec{r}_{G}^{\,A_4} = l_0 a_{1\alpha}^{AT} a_{3\alpha}^{AT} \vec{u}_1.$$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{r}_0^{\,G} = \begin{bmatrix} x_0^G \\ y_0^G \\ z_0^G \end{bmatrix} \tag{8}$$

$$\tilde{u}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \tilde{u}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \tilde{u}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Knowing the general motion of the platform by the relations (6), we develop the inverse kinematical problem and determine the velocities $\vec{v}_{k0}^{\,A}$, $\vec{\omega}_{k0}^{\,A}$ and accelerations $\vec{\gamma}_{k0}^{\,A}$, $\vec{\varepsilon}_{k0}^{\,A}$ of each of the moving linksa follows.

First, we compute the linear and angular velocities of six legs in terms of the angular velocity of the moving platform

$$\vec{\omega}_{60}^{\circ G} = R^T \vec{\omega}_{60}^{G} = \dot{\alpha}_1 R_1^T \vec{u}_1 + \dot{\alpha}_2 R_1^T R_2^T \vec{u}_2 + \dot{\alpha}_3 R^T \vec{u}_3 \tag{9}$$

and the velocity of its centre G

$$\dot{\vec{r}}_0^{\,G} = R^T \vec{v}_{60}^{\,G} = [\dot{x}_0^G \quad \dot{y}_0^G \quad \dot{z}_0^G]^T. \tag{10}$$

The motions of the compounding elements of each leg (the leg A, for example) are characterized by the following skew symmetric matrices [19]:

$$\tilde{\omega}_{k0}^{A} = a_{k,k-1} \tilde{\omega}_{k-1,0}^{A} a_{k,k-1}^{T} + \tilde{\omega}_{k,k-1}^{A},\ (k = 1,2,3), \tag{11}$$

which are *associated* to the absolute angular velocities given by the recursive formulae

$$\vec{\omega}_{k0}^{A} = a_{k,k-1}\vec{\omega}_{k-1,0}^{A} + \vec{\omega}_{k,k-1}^{A} . \quad (12)$$

Following relations give the velocities $\vec{v}_{k0}^{A}$ of joints A_k :

$$\vec{v}_{k0}^{A} = a_{k,k-1}\vec{v}_{k-1,0}^{A} + a_{k,k-1}\tilde{\omega}_{k-1,0}^{A}\vec{r}_{k,k-1}^{A} + \vec{v}_{k,k-1}^{A} \quad (13)$$

$$\vec{v}_{\sigma,\sigma-1}^{A} = \vec{0} \quad (\sigma = 1, 2) .$$

Equations of geometrical constraints (7) can be derivate with respect to time to obtain the following *matrix conditions of connectivity* established for the characteristic relative velocities of leg A :

$$\omega_{10}^{A} \vec{u}_i^T a_{10}^T \tilde{u}_1 a_{21}^T \{\vec{r}_{32}^{A} + a_{32}^T \vec{r}_{43}^{A}\} + \omega_{21}^{A} \vec{u}_i^T a_{20}^T \tilde{u}_2 \{\vec{r}_{32}^{A} + a_{32}^T \vec{r}_{43}^{A}\} + v_{32}^{A} \vec{u}_i^T a_{30}^T \vec{u}_2 = \quad (14)$$

$$= \vec{u}_i^T \dot{\vec{r}}_0^{G} + \vec{u}_i^T R^T \tilde{\omega}_{60}^{G} \vec{r}_G^{A_4} , \quad (i = 1, 2, 3) ,$$

where

$$\tilde{\omega}_{60}^{\circ G} = R^T \tilde{\omega}_{60}^{G} R = \dot{\alpha}_1 R_1^T \tilde{u}_1 R_1 + \dot{\alpha}_2 R_1^T R_2^T \tilde{u}_2 R_2 R_1 + \dot{\alpha}_3 R^T \tilde{u}_3 R \quad (15)$$

denotes the skew-symmetric matrix associated to the angular velocity (9) of the moving platform [20]. From these equations, we obtain the relative velocities $\omega_{10}^{A}, \omega_{21}^{A}, v_{32}^{A}$ as functions of angular velocity of the platform and velocity of mass centre G . The Jacobian matrix of the robot, given from (14), is a fundamental element for the analysis of singularity loci and workspace of the robot.

Let us assume now that the manipulator has a first *virtual motion* determined by the following virtual velocities

$$v_{32a}^{Av} = 1 , v_{32a}^{Bv} = 0 , v_{32a}^{Cv} = 0 , v_{32a}^{Dv} = 0 , v_{32a}^{Ev} = 0 , \ v_{32a}^{Fv} = 0 . \quad (16)$$

The relations of connectivity (14) about the relative velocities express immediately the characteristic virtual velocities as function of the position of the manipulator. Other five sets of compatibility relations can be obtained, if we consider successively that $v_{32b}^{Bv} = 1 , \ v_{32c}^{Cv} = 1 , \ v_{32d}^{Dv} = 1 , \ v_{32e}^{Ev} = 1 , \ v_{32f}^{Fv} = 1$.

As for the relative accelerations $\varepsilon_{10}^{A}, \varepsilon_{21}^{A}, \gamma_{32}^{A}$ of the elements of first leg A of the mechanism, following other *conditions of connectivity* are imposed

$$\varepsilon_{10}^{A} \vec{u}_i^T a_{10}^T \tilde{u}_1 a_{21}^T \{\vec{r}_{32}^{A} + a_{32}^T \vec{r}_{43}^{A}\} + \varepsilon_{21}^{A} \vec{u}_i^T a_{20}^T \tilde{u}_2 \{\vec{r}_{32}^{A} + a_{32}^T \vec{r}_{43}^{A}\} + \gamma_{32}^{A} \vec{u}_i^T a_{30}^T \vec{u}_2 =$$

$$= \vec{u}_i^T \ddot{\vec{r}}_0^{G} + \vec{u}_i^T R^T \{\tilde{\omega}_{60}^{G} \tilde{\omega}_{60}^{G} + \dot{\tilde{\omega}}_{60}^{G}\} \vec{r}_G^{A_4} -$$

$$- \omega_{10}^{A} \omega_{10}^{A} \vec{u}_i^T a_{10}^T \tilde{u}_1 \tilde{u}_1 a_{21}^T \{\vec{r}_{32}^{A} + a_{32}^T \vec{r}_{43}^{A}\} -$$

$$- \omega_{21}^{A} \omega_{21}^{A} \vec{u}_i^T a_{20}^T \tilde{u}_2 \tilde{u}_2 \{\vec{r}_{32}^{A} + a_{32}^T \vec{r}_{43}^{A}\} -$$

$$- 2\omega_{10}^{A} \omega_{21}^{A} \vec{u}_i^T a_{10}^T \tilde{u}_1 a_{21}^T \tilde{u}_2 \{\vec{r}_{32}^{A} + a_{32}^T \vec{r}_{43}^{A}\} -$$

$$- 2\omega_{10}^{A} v_{32}^{A} \vec{u}_i^T a_{10}^T \tilde{u}_1 a_{21}^T a_{32}^T \vec{u}_2 -$$

$$- 2\omega_{21}^{A} v_{32}^{A} \vec{u}_i^T a_{20}^T \tilde{u}_2 a_{32}^T \vec{u}_2 , \ (i = 1, 2, 3) , \quad (17)$$

where an useful square matrix is introduced

$$
\begin{aligned}
&\tilde{\omega}_{60}^{\circ G}\tilde{\omega}_{60}^{\circ G}+\dot{\tilde{\omega}}_{60}^{\circ G}=R^{T}(\tilde{\omega}_{60}^{G}\tilde{\omega}_{60}^{G}+\dot{\tilde{\omega}}_{60}^{G})R=\\
&=\ddot{\alpha}_1R_1^T\tilde{u}_1R_1+\ddot{\alpha}_2R_1^TR_2^T\tilde{u}_2R_2R_1+\ddot{\alpha}_3R^T\tilde{u}_3R+\\
&+\dot{\alpha}_1^2R_1^T\tilde{u}_1\tilde{u}_1R_1+\dot{\alpha}_2^2R_1^TR_2^T\tilde{u}_2\tilde{u}_2R_2R_1+\dot{\alpha}_3^2R^T\tilde{u}_3\tilde{u}_3R+\\
&+2\dot{\alpha}_1\dot{\alpha}_2R_1^T\tilde{u}_1R_2^T\tilde{u}_2R_2R_1+\\
&+2\dot{\alpha}_2\dot{\alpha}_3R_1^TR_2^T\tilde{u}_2R_3^T\tilde{u}_3R+\\
&+2\dot{\alpha}_3\dot{\alpha}_1R_1^T\tilde{u}_1R_2^TR_3^T\tilde{u}_3R\,.
\end{aligned}
\tag{18}
$$

The linear accelerations $\vec{\gamma}_{k0}^{A}$ of the joints A_k and the angular accelerations $\vec{\varepsilon}_{k0}^{A}$ are easily calculated with some recurrence relations, founded by the derivatives of the equations (12) and (13)

$$
\begin{aligned}
&\vec{\varepsilon}_{k0}^{A}=a_{k,k-1}\vec{\varepsilon}_{k-1,0}^{A}+\vec{\varepsilon}_{k,k-1}^{A}+a_{k,k-1}\tilde{\omega}_{k-1,0}^{A}a_{k,k-1}^{T}\vec{\omega}_{k,k-1}^{A}\\
&\tilde{\omega}_{k0}^{A}\tilde{\omega}_{k0}^{A}+\tilde{\varepsilon}_{k0}^{A}=a_{k,k-1}(\tilde{\omega}_{k-1,0}^{A}\tilde{\omega}_{k-1,0}^{A}+\tilde{\varepsilon}_{k-1,0}^{A})a_{k,k-1}^{T}+\\
&+\tilde{\omega}_{k,k-1}^{A}\tilde{\omega}_{k,k-1}^{A}+\tilde{\varepsilon}_{k,k-1}^{A}+2a_{k,k-1}\tilde{\omega}_{k-1,0}^{A}a_{k,k-1}^{T}\tilde{\omega}_{k,k-1}^{A}\\
&\vec{\gamma}_{k0}^{A}=a_{k,k-1}\vec{\gamma}_{k-1,0}^{A}+a_{k,k-1}(\tilde{\omega}_{k-1,0}^{A}\tilde{\omega}_{k-1.0}^{A}+\tilde{\varepsilon}_{k-1,0}^{A})\vec{r}_{k,k-1}^{A}+\\
&+2v_{k,k-1}^{A}a_{k,k-1}\tilde{\omega}_{k-1,0}^{A}a_{k,k-1}^{T}\vec{u}_3+\vec{\gamma}_{k,k-1}^{A}\\
&\vec{\gamma}_{\sigma,\sigma-1}^{A}=\vec{0}\qquad(\sigma=1,2)\,.
\end{aligned}
\tag{19}
$$

If other five kinematical chains of the manipulator are pursued, analogous relations can be easily obtained.

The relations (14) and (17) represent *the inverse kinematics model* of the 3-3 Stewart platforms.

3. Equations of motion

The dynamics of parallel manipulators is complicated by the existence of multiple closed-loop chains. Difficulties commonly encountered in dynamics modelling of parallel robots include problematic issues such as: complicated spatial kinematical structure with possess a large number of passive degrees of freedom, dominance of inertial forces over the frictional and gravitational components and the problem linked to the solution of the inverse dynamics.

In the recent years, many research works have been conducted on the dynamics of the Stewart manipulator [21], [22]. There are three methods, which can provide the same results concerning the determination of the inputs, which must be exerted by the actuators in order to produce a given motion of the end-effectors. The first one is using the Newton-Euler classic procedure, the second one applies the Lagrange equations and multipliers formalism and the third one is based on the fundamental principle of virtual work [23], [24], [25], [26], [27].

A lot of works have focused on the dynamics of Stewart platform. Dasgupta and Mruthyunjaya [16] used the Newton-Euler approach to develop some closed-form dynamic equations of Stewart platform, considering all dynamic and gravity effects as well as viscous friction at joints. Tsai [1] presented an algorithm to solve the inverse dynamics for a Stewart platform-type using Newton-Euler equations, which can be reduced to six if a proper sequence is taken. This classical approach requires computation of all constraint forces and moments between the links.

Geng [28] and Tsai [1] developed Lagrange equations of motion under some simplifying assumptions regarding the geometry and inertia distribution of the manipulator. The Lagrange formulation is well structured and can be expressed in closed form, but a large amount of symbolic computation is needed to find partial derivatives of the Lagrange's function, the analytical calculi involved are too long for each scheme of the manipulator.

Liu et al. [29] derived a set of differential equations for the forward dynamics of legs and moving platform, using the Huston form of Kane's equations [30].

In the inverse dynamic problem, in the present paper we applied the principle of virtual work in order to establish some recursive matrix relations for the forces of the six active systems.

Six independent pneumatic systems A, B, C, D, E, F, that generate six forces $\vec{f}_{32}^{A} = f_{32}^{A}\vec{u}_2$, $\vec{f}_{32}^{B} = f_{32}^{B}\vec{u}_2$, $\vec{f}_{32}^{C} = f_{32}^{C}\vec{u}_2$, $\vec{f}_{32}^{D} = f_{32}^{D}\vec{u}_2$, $\vec{f}_{32}^{E} = f_{32}^{E}\vec{u}_2$, $\vec{f}_{32}^{F} = f_{32}^{F}\vec{u}_2$, which are oriented along the axes $A_3y_3^A, B_3y_3^B, C_3y_3^C, D_3y_3^D, E_3y_3^E, F_3y_3^F$, control the motion of the moving sliders of the manipulator.

The force of inertia and the resultant moment of the forces of inertia of the T_k body are determined with respect to the centre of joint A_k. On the other hand, two vectors $\vec{f}_k^*$ and $\vec{m}_k^*$ evaluate the influence of the weight action $m_k\vec{g}$ and all other external and internal forces applied to the same T_k link.

Knowing the position and kinematics state of each link as well as the external forces acting on the robot, in the present paper we apply the principle of virtual powers in the inverse dynamic problem. The active forces required in a given motion of the moving platform will easily be computed using a recursive procedure.

The closed-loops can artificially be transformed in a set of open chain systems, which are subjected to the constraints. This is possible by cutting each joint for moving platform, and takes its effect into account by introducing the corresponding constraint conditions.

The fundamental principle of the virtual work [1], [13], [30] states that a mechanism is under dynamic equilibrium if and only if the virtual work developed by all external, internal and inertia forces vanish during any general virtual displacement, which is compatible with the constraints imposed on the mechanism. Assuming that frictional forces at the joints are negligible, the virtual work produced by the forces of constraint at the joints is zero.

Applying *the fundamental equations of the parallel robots dynamics* established in compact form by Staicu [26], [27], the following matrix relation results

$$\begin{aligned} f_{32}^{A} &= \vec{u}_2^{T}\vec{F}_3^{A} + \omega_{10a}^{Av}\vec{u}_1^{T}\vec{M}_1^{A} + \omega_{21a}^{Av}\vec{u}_2^{T}\vec{M}_2^{A} + \\ &+ \omega_{10a}^{Bv}\vec{u}_1^{T}M_1^{B} + \omega_{21a}^{Bv}\vec{u}_2^{T}M_2^{B} + \\ &+ \omega_{10a}^{Cv}\vec{u}_1^{T}\vec{M}_1^{C} + \omega_{21a}^{Cv}\vec{u}_2^{T}\vec{M}_2^{C} + \\ &+ \omega_{10a}^{Dv}\vec{u}_1^{T}\vec{M}_1^{D} + \omega_{21a}^{Dv}\vec{u}_2^{T}\vec{M}_2^{D} + \\ &+ \omega_{10a}^{Ev}\vec{u}_1^{T}\vec{M}_1^{E} + \omega_{21a}^{Ev}\vec{u}_2^{T}\vec{M}_2^{E} + \\ &+ \omega_{10a}^{Fv}\vec{u}_1^{T}\vec{M}_1^{F} + \omega_{21a}^{Fv}\vec{u}_2^{T}\vec{M}_2^{F} + \\ &+ v_{10a}^{Gv}\vec{u}_1^{T}\vec{F}_1^{G} + v_{21a}^{Gv}\vec{u}_2^{T}\vec{F}_2^{G} + v_{32a}^{Gv}\vec{u}_3^{T}F_3^{G} + \\ &+ \omega_{43a}^{Gv}\vec{u}_1^{T}\vec{M}_4^{G} + \omega_{54a}^{Gv}\vec{u}_2^{T}\vec{M}_5^{G} + \omega_{65a}^{Gv}\vec{u}_3^{T}\vec{M}_6^{G}, \end{aligned} \tag{20}$$

where, for example, we denote

$$\begin{aligned} &\vec{F}_k^{A} = \vec{F}_{k0}^{A} + a_{k+1,k}^{T}\vec{F}_{k+1}^{A} \\ &\vec{M}_k^{A} = \vec{M}_{k0}^{A} + a_{k+1,k}^{T}\vec{M}_{k+1}^{A} + \tilde{r}_{k+1,k}^{A}a_{k+1,k}^{T}\vec{F}_{k+1}^{A} \\ &\vec{F}_{k0}^{A} = m_k^{A}\left[\vec{\gamma}_{k0}^{A} + \left(\tilde{\omega}_{k0}^{A}\tilde{\omega}_{k0}^{A} + \tilde{\varepsilon}_{k0}^{A}\right)\vec{r}_k^{CA}\right] - \vec{f}_k^{*A}\ . \\ &M_{k0}^{A} = m_k^{A}\tilde{r}_k^{CA}\vec{\gamma}_{k0}^{A} + \hat{J}_k^{A}\vec{\varepsilon}_{k0}^{A} + \tilde{\omega}_{k0}^{A}\hat{J}_k^{A}\vec{\omega}_{k0}^{A} - \vec{m}_k^{*A} \\ &\vec{f}_k^{*A} = -9.81 m_k^{A}a_{k0}\vec{u}_3,\ \vec{m}_k^{*A} = -9.81 m_k^{A}\tilde{r}_k^{CA}a_{k0}\vec{u}_3 . \end{aligned} \tag{21}$$

The relations (20) and (21) represent *the inverse dynamics model* of the 3-3 Stewart platforms, which can be easily transformed in a model for automatic command.

As applications let us consider a manipulator, which has the following characteristics

$$\alpha_1^* = 0\,,\ \alpha_2^* = 0\,,\ \alpha_3^* = \frac{15\pi}{180}$$

$$\beta = \frac{\pi}{3}\,,\ \Delta t = 3\,\text{s}$$

$$x_0^{G*} = 0.10\ \text{m},\ y_0^{G*} = 0\,\text{m},\ z_0^{G*} = 0.15\,\text{m}$$

$$m_1 = 1.5\,\text{kg}, m_2 = 15\,\text{kg}, m_3 = 10\,\text{kg}, m_p = m_6 = 50\,\text{kg}$$

$$l_0 = OA_1 = 2.50\,\text{m}, l_2 = 1.50\,\text{m}, l_3 = 3.00\ \text{m},\ l_4 = 2.50\ \text{m}$$

$$l_1 = \sqrt{l_0^2 + h^2} - l_3,\ OG = h = l_0 tg\beta$$

$$\hat{J}_1 = \begin{bmatrix} 0.2 & & \\ & 0.1 & \\ & & 0.1 \end{bmatrix},\ \hat{J}_2 = \begin{bmatrix} 5 & & \\ & 15 & \\ & & 15 \end{bmatrix},\ \hat{J}_3 = \begin{bmatrix} 30 & & \\ & 5 & \\ & & 30 \end{bmatrix},\ \hat{J}_p = \begin{bmatrix} 40 & & \\ & 40 & \\ & & 80 \end{bmatrix}.$$

4. Dynamics simulations

Based on the algorithm derived from above equations, a computer program was developed to solve the inverse dynamics of the Stewart platform, using the MATLAB software. To validate the dynamics modelling, it is assumed that the platform starts at rest from a central configuration and moves along or rotates about one of three orthogonal directions. Furthermore, at the initial location, the moving platform is assumed to be located $4.33\,m$ lower the fixed base, namely $t=0$: $x_0^G = 0,\ y_0^G = 0,\ z_0^G = 4.33\,m$.

Assuming that there are not external forces and moments acting on the moving platform, the time-history evolutions of the forces $f_{32}^A, f_{32}^B, f_{32}^C, f_{32}^D, f_{32}^E, f_{32}^F$ required by the three pneumatic active systems are shown for a period of three second of platform's motion.

The following examples are solved to illustrate the algorithm. For the first example, the moving platform moves along the *vertical* z_0 direction with variable acceleration while all the other positional parameters are held equal to zero.

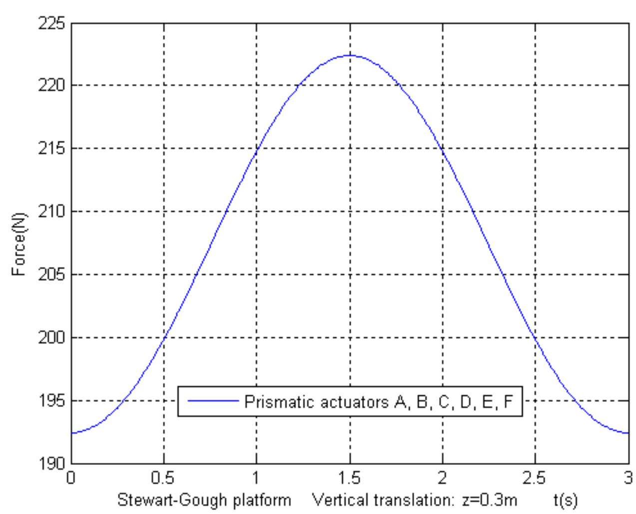

Fig. 3 Forces of the six actuators: vertical translation

As can be seen from Fig. 3, it is proved to be true that all input forces are permanently equal to one another. When the moving platform is going to the fixed base, the limbs become more horizontally oriented, therefore increasing the actuating forces.

If the centre G moves along a *rectilinear trajectory* along the horizontal x_0 direction without rotation of the platform, the forces f_{32}^A, f_{32}^F (Fig. 4) and f_{32}^B, f_{32}^E (Fig. 5) required by the actuators are calculated by the program and plotted versus time in comparison with the input forces f_{32}^C, f_{32}^D of the prismatic actuators C, D.

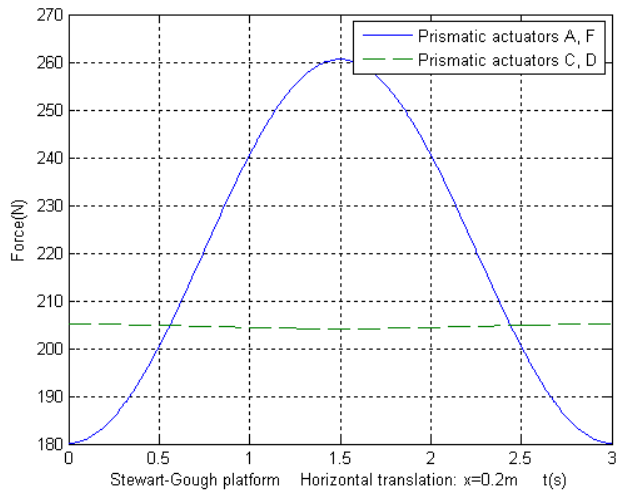

Fig. 4 Forces of four actuators: horizontal translation

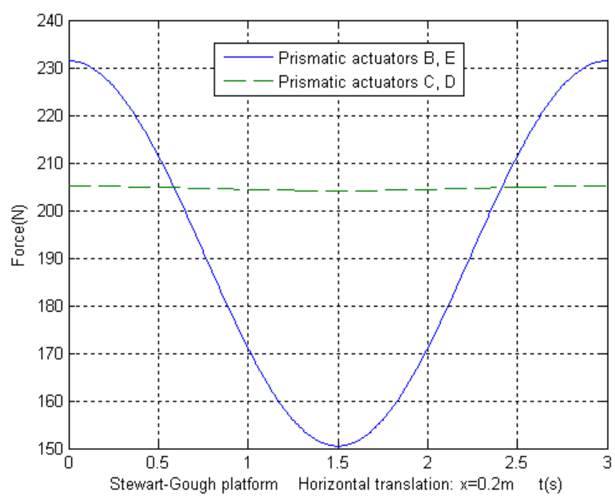

Fig. 5 Forces of four actuators: horizontal translation

For the third example we consider the *rotation motion* of the moving platform about the vertical z_0 axis with a variable angular acceleration $\ddot{\alpha}_3$. As can be seen from $f_{32}^{A}, f_{21}^{C}, f_{32}^{E}$ and $f_{32}^{B}, f_{32}^{D}, f_{32}^{F}$ (Fig. 6), we remark the anti-symmetrical distribution of the actuating forces.

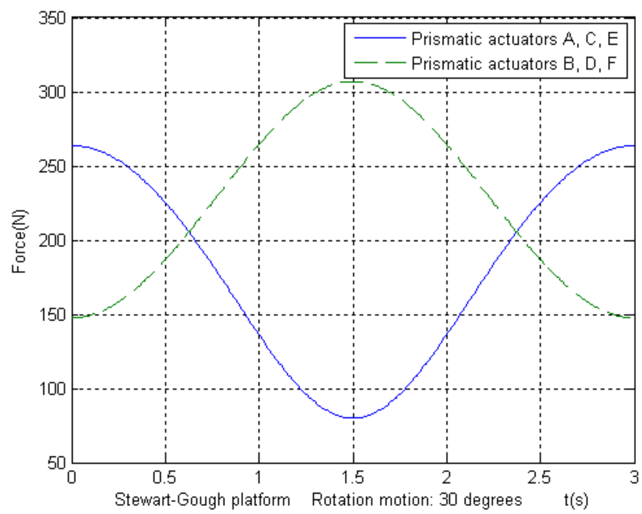

Fig. 6 Forces of six actuators: rotation about z_0 axis

5. Conclusions

Most of dynamical models based on the Lagrange formalism neglect the weight of intermediate bodies and take into consideration the active forces or moments only and the wrench of applied forces on the moving platform. The number of relations given by this approach is equal to the total number of the position variables and Lagrange multipliers inclusive. Also, the analytical calculations involved in these equations are very tedious, thus presenting an elevated risk of errors.

The commonly known Newton-Euler method, which takes into account the free-body-diagrams of the mechanism, leads to a large number of equations with unknowns including also the connecting forces in the joints. Finally, the actuating forces could be obtained.

Within the inverse kinematics analysis, the conditions of connectivity (14), (17) that give in real-time the relative velocity and acceleration of each element of the parallel robot have been established in the present paper. The dynamics model takes into consideration the mass, the tensor of inertia and the action of weight and inertia force introduced by each element of the manipulator.

Based on the principle of virtual work, the new approach is far more efficient, can eliminate all forces of internal joints and establishes a direct determination of the time-history evolution of input forces and active powers required by the three actuators. Also, the method described above is quite available in forward and inverse mechanics of serial and parallel mechanisms, the platform of which behaves in translation, spherical evolution or more general six-degree-of-freedom motion.

The recursive matrix relations (20), (21) represent a set of explicit equations of the dynamic simulation and, in a context of automatic command, can easily be transformed into a robust model for computerized control of the Stewart parallel manipulator.

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