PATA TYPE FIXED POINT THEOREMS OF MULTIVALUED OPERATORS IN ORDERED METRIC SPACES WITH APPLICATIONS TO HYPERBOLIC DIFFERENTIAL INCLUSIONS

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The purpose of this paper is to present some Pata type fixed point theorems for multivalued mappings on ordered complete metric spaces. Moreover, as an application of our main theorem, we give an existence theorem for the solution of a hyperbolic differential inclusion problem.

Keywords: Fixed point, Ordered metric space, Multivalued mapping, Hyperbolic differential inclusion.

1. **Introduction and preliminaries**

In 1922, the Polish mathematician Stefan Banach proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach's fixed point theorem or the Banach contraction principle[1]. Nadler [2] in 1969, proved a set-valued extension of the Banach's contraction principle in complete metric spaces. Afterward many fixed point theorems have been proved by various authors as generalization of the Nadler's theorem where the nature of contractive mapping is weakened along with some additional requirements, see for instance [3, 4, 5, 6, 7, 8, 9, 10]. Ran and Reurings [11] established the existence of unique fixed point for the monotone single valued mapping in partially ordered metric spaces. Their result was further extended in [12, 13, 4, 14, 15, 16, 17]. Recently, V. Pata [18] improve the Banach principal. In fact, Pata extended the Banach contraction principle with weaker hypotheses than those of the Banach contraction principle with an explicit estimate of the convergence rate. In this paper, using the idea of Pata, we prove some fixed point theorems on ordered complete metric spaces. As an application, we also obtain conditions which guarantee the existence of a solution for hyperbolic differential inclusions problem.

Let (M, d) be a metric space. Then 2^M is the class of all nonempty subsets of M and for $A, B \in 2^M$, let

$$H_d(A, B) = \max\{\sup_{b\in B} d(b, A), \sup_{a\in A} d(a, B)\},$$

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where

$$d(a,B) = \inf_{b\in B} d(a,b),$$

then H_d is called the Hausdorff-Pompeiu functional induced by d.

Definition 1.1. Let $(X, \leq)$ be a partially ordered set and $x, y \in X$. Elements x and y are said to be comparable elements of X if either $x \leq y$ or $y \leq x$.

Definition 1.2. [19] Let M be a metric space. A subset $N \subset M$ is said to be approximative if the multivalued mapping

$$\digamma_N(x) = \{y \in N : d(x,y) = d(N,x)\}, \quad \forall x \in M,$$

has nonempty values.

Definition 1.3. [19] The multivalued mapping $T : M \to 2^M$ is said to have approximative values, AV for short, if Tx is approximative for each $x \in M$.

Definition 1.4. [19] The multivalued mapping $T : M \to 2^M$ is said to have comparable approximative values, CAV for short, if T has approximative values and, for each $z \in M$, there exists $y \in \digamma_{Tz}(x)$ such that y is comparable to z.

Definition 1.5. [19] The multivalued mapping $T : M \to 2^M$ is said to have upper comparable approximative values, UCAV, for short(resp.lower comparable approximative values, LCAV for short) if T has approximative values and, for each $z \in M$, there exists $y \in \digamma_{Tz}(x)$ such that $y \geq z$(resp.$y \leq z$).

Definition 1.6. [19] For two subset X, Y of M, we denote $X \leq_r Y$ if for each $x \in X$ there exists $y \in Y$ such that $x \leq y$ and $X \leq Y$ if each $x \in X$ and each $y \in Y$ imply that $x \leq y$.

Definition 1.7. [19] A multivalued mapping $T : M \to 2^M$ is said to be r-nondecreasing(r-nonincreasing) if $x \leq y$ implies that $Tx \leq_r Ty(T_y \leq_r Tx)$ for all $x, y \in M$. T is said to be r-monotone if T is r-nondecreasing or r-nonincreasing.
The notation of nondecreasing(nonincreasing) is similarly defind by writing $\leq$ instead of the notation $\leq_r$.

2. Main Results

Let $(M, d, \leq)$ be a partial ordered complete metric space. The following hypothesis in M(which appear in [20]) will be applied:

(H_1) If $\{x_n\}$ is a non-decreasing(resp. non-increasing) sequence in M such that $x_n \to x$, then $x_n \leq x$(resp.$x_n \geq x$) for all $n \in \mathbb{N}$.

For a metric space (M, d), Selecting an arbitrary $x_0 \in M$ we denote

$$\| x \| = d(x, x_0) \text{ for all } x \in M.$$

Let $\psi : [0,1] \to [0,\infty)$ is an increasing function vanishing with continuity at zero. Also consider the vanishing sequence depending on $\alpha \geq 1$, $w_n(\alpha) = (\frac{\alpha}{n})^{\alpha} \sum_{k=1}^{n} \psi(\frac{\alpha}{k})$.

Theorem 2.1. *Let M be an ordered complete metric space and satisfy (H_1). Let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0,\alpha]$ be a fixed constants. Suppose that the multivalued map $T : M \to 2^M$ has UCAV and the inequality*

$$H_d(Tx,Ty) \leq (1-\epsilon)d(x,y) + \Lambda\epsilon^{\alpha}\psi(\epsilon)[1+\|x\|+\|y\|]^{\beta}, \tag{1}$$

is satisfied for every $\epsilon \in [0,1]$ and for all $x,y \in M$ with x and y comparable. Then T has a fixed point $x* \in Tx*$. Furthermore,

$$d(x*, Tx_{n-1}) \leq Kw_n(\alpha), \tag{2}$$

for some positive constant $K \leq \Lambda(1+4\|x*\|)^{\beta}$.

Proof. Given $x_0 \in M$, if $x_0 \in Tx_0$, proof is complete. Moreover from the fact that Tx_0 has UCAV it follows there exsists $x_1 \in Tx_0$ with $x_1 \neq x_0$ and $x_1 \geq x_0$ such that

$$d(x_0,x_1) = \inf_{x\in Tx_0} d(x,x_0) = d(Tx_0,x_0).$$

Continuing in this way obtain that, there exists $x_{n+1} \in Tx_n$ with $x_{n+1} \neq x_n$ and $x_{n+1} \geq x_n$ such that

$$d(x_n,x_{n+1}) = d(Tx_n,x_n), \qquad n = 1,2,\dots .$$

Moreover,

$$d(Tx_n,x_n) \leq \sup_{x\in Tx_{n-1}} d(Tx_n,x) \leq H_d(Tx_n,Tx_{n-1}),$$

therefore,

$$d(x_n,x_{n+1}) \leq H_d(Tx_{n-1},Tx_n) \qquad for\ n = 2,3,\dots .$$

Furthermore for $n = 1,2,\dots$ we set

$$C_n = \|x_n\| = d(x_n,x_0).$$

Since (1) is true for every $\varepsilon \in [0,1]$, setting $\varepsilon = 0$, we have the following relations

$$\begin{aligned} d(x_{n+1},x_n) \leq H_d(Tx_n,Tx_{n-1}) \leq d(x_n,x_{n-1}) &\leq H_d(Tx_{n-1},Tx_n) \\ &\leq d(x_{n-1},x_n) \\ &\vdots \\ &\leq H_d(Tx_1,Tx_0) \leq d(x_1,x_0) = C_1. \end{aligned} \tag{3}$$

By triangle inequality we have

$$d(x_{n+1},x_0) \leq d(x_{n+1},x_1) + d(x_1,x_0), \tag{4}$$

and

$$d(x_n,x_0) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_0). \tag{5}$$

Now, using (3), (4) and (5) we have

$$C_n \leq d(x_{n+1}, x_1) + 2C_1 \leq H_d(Tx_n, Tx_0) + 2C_1. \tag{6}$$

So for $\alpha \geq \beta$, there exist real numbers $E, D > 0$ such that

$$C_n \leq (1-\epsilon)C_n + \Lambda\epsilon^\alpha\psi(\epsilon)[1 + C_n]^\beta + 2C_1 \leq (1-\epsilon)C_n + E\epsilon^\alpha\psi(\epsilon)C_n^\alpha + D. \tag{7}$$

Accordingly,

$$\varepsilon C_n \leq E\varepsilon^\alpha\psi(\varepsilon)C_n^\alpha + D,$$

which holds by hypothesis for any $\varepsilon \in [0,1]$ taken for each $n \in \mathbb{N}$. If there is a subsequence $C_{n_k} \to \infty$, then the choice $\varepsilon_{n_k} = \min(1, \frac{1+D}{C_{n_k}})$, leads to the following contradiction

$$1 \leq E(1+D)^\alpha\psi(\varepsilon_{n_k}) \to 0 \text{ as } n_k \to \infty .$$

Then the sequence $\{C_n\}_{n=1}^\infty$ is bounded.

Now, we prove that sequence $\{x_n\}$ is Cauchy sequence:

$d(x_{n+m+1}, x_{n+1}) \leq H_d(Tx_{n+m}, Tx_n) \leq (1-\epsilon)d(x_{n+m}, x_n) + \Lambda\epsilon^\alpha\psi(\epsilon)[\|x_{n+m}\| + \|x_n\|]^\beta$.
For fixed m, set

$$K = \sup_{n\in N} \Lambda[1 + 2c_n]^\beta, \tag{8}$$

and $\varepsilon = 1 - (\frac{n}{n+1})^\alpha \leq \frac{\alpha}{n+1}$. So

$$(n+1)^\alpha d(x_{n+m+1}, x_{n+1}) \leq n^\alpha d(x_{n+m}, x_n) + K\alpha^\alpha\psi(\frac{\alpha}{n+1}).$$

Setting $r_n := n^\alpha d(x_{n+m}, x_n)$, we have

$$\begin{aligned} r_{n+1} &\leq r_n + K\alpha^\alpha\psi(\frac{\alpha}{n+1}) \\ &\leq r_{n-1} + K\alpha^\alpha\psi(\frac{\alpha}{n}) + K\alpha^\alpha\psi(\frac{\alpha}{n+1}) \\ &\leq \ldots \\ &\leq r_0 + K\alpha^\alpha\sum_{k=1}^{n+1}\psi(\frac{\alpha}{k}) = K\alpha^\alpha\sum_{k=1}^{n+1}\psi(\frac{\alpha}{k}). \end{aligned}$$

Therefore

$$d(x_{n+m}, x_n) \leq K(\frac{\alpha}{n})^\alpha\sum_{k=1}^{n}\psi(\frac{\alpha}{k}) = Kw_n(\alpha). \tag{9}$$

Taking limits as $n \to \infty$, we get $d(x_{n+m}, x_n) \to 0$. This implies that $\{x_n\}$ is Cauchy sequence in X. Since X is a complete metric space, there exists $x* \in M$ such that

$\lim_{n\to\infty} x_n = x*$.
Using (9) we have

$$0 \le d(x*, Tx_{n-1}) \le d(x*, x_n) = \lim_{m\to\infty} d(x_{n+m}, x_n) \le Kw_n(\alpha).$$

Again

$$\begin{aligned} 0 \le d(x*, Tx*) = \lim_{n\to\infty} d(x_{n-1}, Tx*) &\le \lim_{n\to\infty} H_d(Tx_n, Tx*) \\ &\le (1-\epsilon)d(x_n, x*) + \Lambda\epsilon^\alpha\psi(\epsilon)[1 + \|x_n\| + \|x*\|]^\beta. \end{aligned}$$

Since the contractive condition (1) holds for any real constant $\varepsilon \in [0,1]$, we can replace ε, for each $n \in \mathbb{N}$, by a sequence $[0,1] \ni \varepsilon_n \to 0$ as $n \to \infty$. Then by letting $\varepsilon = \varepsilon_n \to 0$ as $n \to \infty$, we have

$$0 \le d(x*, Tx*) \le (1-\epsilon_n)d(x_n, x*) + \Lambda\epsilon_n^\alpha\psi(\epsilon_n)[1 + \|x_n\| + \|x*\|]^\beta \to 0.$$

So $d(x*, Tx*) = 0$ and $x* \in Tx*$.
Also, the convergence rate estimate stated in (2) is achieved from the following relations

$$\begin{aligned} d(x*, x_n) \le H_d(Tx*, Tx_{n-1}) \le d(x*, x_{n-1}) &\le H_d(Tx*, Tx_{n-2}) \\ &\le d(x*, x_{n-2}) \\ &\vdots \\ &\le H_d(Tx*, Tx_0) \\ &\le d(x*, x_0) = \|x*\|, \end{aligned}$$

which implies that

$$C_n = d(x_n, x_0) \le d(x_n, x*) + d(x*, x_0) \le \|x*\| + \|x*\| = 2\|x*\|.$$

From the last inequality and (8) we have $K \le \Lambda(1 + 4\|x*\|)^\beta$. □

Theorem 2.2. *Let M be an ordered complete metric space and satisfy (H_1). Let $\Lambda \ge 0$, $\alpha \ge 1$ and $\beta \in [0, \alpha]$ be a fixed constants. Suppose that the multivalued map $T : M \to 2^M$ has LCAV and the inequality*

$$H_d(Tx, Ty) \le (1-\epsilon)d(x, y) + \Lambda\epsilon^\alpha\psi(\epsilon)[1 + \|x\| + \|y\|]^\beta,$$

is satisfied for every $\epsilon \in [0,1]$ and for all $x, y \in M$ with x and y comparable. Then T has a fixed point $x* \in Tx*$. Furthermore,

$$d(x*, Tx_{n-1}) \le Kw_n(\alpha),$$

for some positive constant $K \le \Lambda(1 + 4\|x*\|)^\beta$.

Proof. Given $x_0 \in M$, if $x_0 \in Tx_0$, proof is complete. Moreover from the fact that Tx_0 has LCAV it follows there exsists $x_1 \in Tx_0$ with $x_1 \ne x_0$ and $x_1 \le x_0$ such that

$$d(x_0, x_1) = \inf_{x\in Tx_0} d(x, x_0) = d(Tx_0, x_0).$$

Continuing in this way obtain that, there exists $x_{n+1} \in Tx_n$ with $x_{n+1} \ne x_n$ and $x_{n+1} \le x_n$ such that

$$d(x_n, x_{n+1}) = d(Tx_n, x_n), \quad n = 1, 2, \dots .$$

Moreover,

$$d(Tx_n, x_n) \leq \sup_{x \in Tx_{n-1}} d(Tx_n, x) \leq H_d(Tx_n, Tx_{n-1}),$$

therefore,

$$d(x_n, x_{n+1}) \leq H_d(Tx_{n-1}, Tx_n) \quad for\ n = 2, 3, \dots\ .$$

Furthermore for $n = 1, 2, \dots$ we set

$$C_n = \|x_n\| = d(x_n, x_0).$$

Since (1) is true for every $\varepsilon \in [0, 1]$, setting $\varepsilon = 0$, we have the following relations

$$\begin{aligned} d(x_{n+1}, x_n) \leq H_d(Tx_n, Tx_{n-1}) \leq d(x_n, x_{n-1}) &\leq H_d(Tx_{n-1}, Tx_n) \\ &\leq d(x_{n-1}, x_n) \\ &\vdots \\ &\leq H_d(Tx_1, Tx_0) \leq d(x_1, x_0) = C_1. \end{aligned} \tag{10}$$

By triangle inequality we have

$$d(x_{n+1}, x_0) \leq d(x_{n+1}, x_1) + d(x_1, x_0), \tag{11}$$

and

$$d(x_n, x_0) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_0). \tag{12}$$

Now, using (10), (11) and (12) we have

$$C_n \leq d(x_{n+1}, x_1) + 2C_1 \leq H_d(Tx_n, Tx_0) + 2C_1. \tag{13}$$

So for $\alpha \geq \beta$, there exist real numbers $E, D > 0$ such that

$$C_n \leq (1-\epsilon)C_n + \Lambda\epsilon^\alpha\psi(\epsilon)[1 + C_n]^\beta + 2C_1 \leq (1-\epsilon)C_n + E\epsilon^\alpha\psi(\epsilon)C_n^\alpha + D. \tag{14}$$

Accordingly,

$$\varepsilon C_n \leq E\varepsilon^\alpha\psi(\varepsilon)C_n^\alpha + D,$$

which holds by hypothesis for any $\varepsilon \in [0, 1]$ taken for each $n \in \mathbb{N}$. If there is a subsequence $C_{n_k} \to \infty$, then the choice $\varepsilon_{n_k} = \min(1, \frac{1+D}{C_{n_k}})$, leads to the following contradiction

$$1 \leq E(1+D)^\alpha\psi(\varepsilon_{n_k}) \to 0 \text{ as } n_k \to \infty\ .$$

Then the sequence $\{C_n\}_{n=1}^\infty$ is bounded.

Now, we prove that sequence $\{x_n\}$ is Cauchy sequence:

$$d(x_{n+m+1}, x_{n+1}) \leq H_d(Tx_{n+m}, Tx_n) \leq (1-\epsilon)d(x_{n+m}, x_n) + \Lambda\epsilon^\alpha\psi(\epsilon)[\|x_{n+m}\| + \|x_n\|]^\beta.$$

For fixed m, set

$$K = \sup_{n\in N} \Lambda[1+2c_n]^{\beta}, \tag{15}$$

and $\varepsilon = 1-(\frac{n}{n+1})^{\alpha} \leq \frac{\alpha}{n+1}$. So

$$(n+1)^{\alpha} d(x_{n+m+1}, x_{n+1}) \leq n^{\alpha} d(x_{n+m}, x_n) + K\alpha^{\alpha}\psi(\frac{\alpha}{n+1}).$$

Setting $r_n := n^{\alpha} d(x_{n+m}, x_n)$, we have

$$\begin{aligned} r_{n+1} &\leq r_n + K\alpha^{\alpha}\psi(\frac{\alpha}{n+1}) \\ &\leq r_{n-1} + K\alpha^{\alpha}\psi(\frac{\alpha}{n}) + K\alpha^{\alpha}\psi(\frac{\alpha}{n+1}) \\ &\leq ... \\ &\leq r_0 + K\alpha^{\alpha}\sum_{k=1}^{n+1}\psi(\frac{\alpha}{k}) = K\alpha^{\alpha}\sum_{k=1}^{n+1}\psi(\frac{\alpha}{k}). \end{aligned}$$

Therefore

$$d(x_{n+m}, x_n) \leq K(\frac{\alpha}{n})^{\alpha}\sum_{k=1}^{n}\psi(\frac{\alpha}{k}) = Kw_n(\alpha). \tag{16}$$

Taking limits as $n\to\infty$, we get $d(x_{n+m}, x_n)\to 0$. This implies that $\{x_n\}$ is Cauchy sequence in X. Since X is a complete metric space, there exists $x* \in M$ such that $\lim_{n\to\infty} x_n = x*$.
Using (16) we have

$$0 \leq d(x*, Tx_{n-1}) \leq d(x*, x_n) = \lim_{m\to\infty} d(x_{n+m}, x_n) \leq Kw_n(\alpha).$$

Again

$$\begin{aligned} 0 \leq d(x*, Tx*) = \lim_{n\to\infty} d(x_{n-1}, Tx*) &\leq \lim_{n\to\infty} H_d(Tx_n, Tx*) \\ &\leq (1-\epsilon)d(x_n, x*) + \Lambda\epsilon^{\alpha}\psi(\epsilon)[1+\|x_n\|+\|x*\|]^{\beta}. \end{aligned}$$

Since the contractive condition (1) holds for any real constant $\varepsilon \in [0,1]$, we can replace ε, for each $n\in\mathbb{N}$, by a sequence $[0,1]\ni\varepsilon_n \to 0$ as $n\to\infty$. Then by letting $\varepsilon = \varepsilon_n \to 0$ as $n\to\infty$, we have

$$0 \leq d(x*, Tx*) \leq (1-\epsilon_n)d(x_n, x*) + \Lambda\epsilon_n^{\alpha}\psi(\epsilon_n)[1+\|x_n\|+\|x*\|]^{\beta} \to 0.$$

So $d(x*, Tx*) = 0$ and $x* \in Tx*$.
Also, the convergence rate estimate stated in (2) is achieved from the following relations

$$\begin{aligned} d(x*, x_n) \leq H_d(Tx*, Tx_{n-1}) \leq d(x*, x_{n-1}) &\leq H_d(Tx*, Tx_{n-2}) \\ &\leq d(x*, x_{n-2}) \\ &\vdots \\ &\leq H_d(Tx*, Tx_0) \\ &\leq d(x*, x_0) = \|x*\|, \end{aligned}$$

which implies that

$$C_n = d(x_n, x_0) \leq d(x_n, x*) + d(x*, x_0) \leq \|x*\| + \|x*\| = 2\|x*\|.$$

From the last inequality and (15) we have $K \leq \Lambda(1 + 4\|x*\|)^{\beta}$. □

Theorem 2.3. *Let M be an ordered complete metric space and satisfy (H_1). Let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$ be a fixed constants. Suppose that the multivalued map $T : M \to 2^M$ has AV, is non-decreasing and the inequality*

$$H_d(Tx, Ty) \leq (1 - \epsilon)d(x, y) + \Lambda\epsilon^{\alpha}\psi(\epsilon)[1 + \|x\| + \|y\|]^{\beta},$$

is satisfied for every $\epsilon \in [0, 1]$ and for all $x, y \in M$ with x and y comparable. If there exists $x_0 \in M$ such that $\{x_0\} \leq Tx_0$, then T has a fixed point $x* \in Tx*$. Furthermore,

$$d(x*, Tx_{n-1}) \leq Kw_n(\alpha),$$

for some positive constant $K \leq \Lambda(1 + 4\|x*\|)^{\beta}$.

Proof. The first by using hypothesis of this theorem, we construct sequence $\{x_n\}$. If $x_0 \in Tx_0$, proof is complete. Moreover, according hypothesis, for any $x \in Tx_0$, we have $x \geq x_0$. Since T has AV, there exists $x_1 \in Tx_0$ with $x_1 \geq x_0$ and $d(x_0, x_1) = d(Tx_0, x_0)$. Continuing in this way obtain that, there exists $x_{n+1} \in Tx_n$ with $x_{n+1} \neq x_n$ and $x_{n+1} \geq x_n$ such that

$$d(x_n, x_{n+1}) = d(Tx_n, x_n), \quad n = 1, 2, \dots .$$

Moreover,

$$d(Tx_n, x_n) \leq \sup_{x \in Tx_{n-1}} d(Tx_n, x) \leq H_d(Tx_n, Tx_{n-1}),$$

therefore,

$$d(x_n, x_{n+1}) \leq H_d(Tx_{n-1}, Tx_n) \quad for\ n = 2, 3, \dots .$$

Furthermore for $n = 1, 2, \dots$ we set

$$C_n = \|x_n\| = d(x_n, x_0).$$

Since (1) is true for every $\varepsilon \in [0, 1]$, setting $\varepsilon = 0$, we have the following relations

$$\begin{aligned} d(x_{n+1}, x_n) \leq H_d(Tx_n, Tx_{n-1}) \leq d(x_n, x_{n-1}) &\leq H_d(Tx_{n-1}, Tx_n) \\ &\leq d(x_{n-1}, x_n) \\ &\vdots \\ &\leq H_d(Tx_1, Tx_0) \leq d(x_1, x_0) = C_1. \end{aligned} \tag{17}$$

By triangle inequality we have

$$d(x_{n+1}, x_0) \leq d(x_{n+1}, x_1) + d(x_1, x_0), \tag{18}$$

and

$$d(x_n, x_0) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_0). \tag{19}$$

Now, using (17), (18) and (19) we have

$$C_n \leq d(x_{n+1}, x_1) + 2C_1 \leq H_d(Tx_n, Tx_0) + 2C_1. \tag{20}$$

So for $\alpha \geq \beta$, there exist real numbers $E, D > 0$ such that

$$C_n \leq (1-\epsilon)C_n + \Lambda\epsilon^\alpha\psi(\epsilon)[1+C_n]^\beta + 2C_1 \leq (1-\epsilon)C_n + E\epsilon^\alpha\psi(\epsilon)C_n^\alpha + D. \tag{21}$$

Accordingly,

$$\varepsilon C_n \leq E\varepsilon^\alpha\psi(\varepsilon)C_n^\alpha + D,$$

which holds by hypothesis for any $\varepsilon \in [0,1]$ taken for each $n \in \mathbb{N}$. If there is a subsequence $C_{n_k} \to \infty$, then the choice $\varepsilon_{n_k} = \min(1, \frac{1+D}{C_{n_k}})$, leads to the following contradiction

$$1 \leq E(1+D)^\alpha\psi(\varepsilon_{n_k}) \to 0 \text{ as } n_k \to \infty \ .$$

Then the sequence $\{C_n\}_{n=1}^\infty$ is bounded.

Now, we prove that sequence $\{x_n\}$ is Cauchy sequence:

$d(x_{n+m+1}, x_{n+1}) \leq H_d(Tx_{n+m}, Tx_n) \leq (1-\epsilon)d(x_{n+m}, x_n) + \Lambda\epsilon^\alpha\psi(\epsilon)[\|x_{n+m}\| + \|x_n\|]^\beta$. For fixed m, set

$$K = \sup_{n \in N} \Lambda[1+2c_n]^\beta, \tag{22}$$

and $\varepsilon = 1 - (\frac{n}{n+1})^\alpha \leq \frac{\alpha}{n+1}$. So

$$(n+1)^\alpha d(x_{n+m+1}, x_{n+1}) \leq n^\alpha d(x_{n+m}, x_n) + K\alpha^\alpha\psi(\frac{\alpha}{n+1}).$$

Setting $r_n := n^\alpha d(x_{n+m}, x_n)$, we have

$$\begin{aligned} r_{n+1} &\leq r_n + K\alpha^\alpha\psi(\frac{\alpha}{n+1}) \\ &\leq r_{n-1} + K\alpha^\alpha\psi(\frac{\alpha}{n}) + K\alpha^\alpha\psi(\frac{\alpha}{n+1}) \\ &\leq ... \\ &\leq r_0 + K\alpha^\alpha\sum_{k=1}^{n+1}\psi(\frac{\alpha}{k}) = K\alpha^\alpha\sum_{k=1}^{n+1}\psi(\frac{\alpha}{k}). \end{aligned}$$

Therefore

$$d(x_{n+m}, x_n) \leq K(\frac{\alpha}{n})^\alpha\sum_{k=1}^{n}\psi(\frac{\alpha}{k}) = Kw_n(\alpha). \tag{23}$$

Taking limits as $n \to \infty$, we get $d(x_{n+m}, x_n) \to 0$. This implies that $\{x_n\}$ is Cauchy sequence in X. Since X is a complete metric space, there exists $x* \in M$ such that

$\lim_{n\to\infty} x_n = x*$.
Using (23) we have

$$0 \le d(x*, Tx_{n-1}) \le d(x*, x_n) = \lim_{m\to\infty} d(x_{n+m}, x_n) \le Kw_n(\alpha).$$

Again

$$\begin{aligned}0 \le d(x*, Tx*) = \lim_{n\to\infty} d(x_{n-1}, Tx*) &\le \lim_{n\to\infty} H_d(Tx_n, Tx*)\\ &\le (1-\epsilon)d(x_n, x*) + \Lambda\epsilon^{\alpha}\psi(\epsilon)[1 + \|x_n\| + \|x*\|]^{\beta}.\end{aligned}$$

Since the contractive condition (1) holds for any real constant $\varepsilon \in [0,1]$, we can replace ε, for each $n \in \mathbb{N}$, by a sequence $[0,1] \ni \varepsilon_n \to 0$ as $n \to \infty$. Then by letting $\varepsilon = \varepsilon_n \to 0$ as $n \to \infty$, we have

$$0 \le d(x*, Tx*) \le (1-\epsilon_n)d(x_n, x*) + \Lambda\epsilon_n^{\alpha}\psi(\epsilon_n)[1 + \|x_n\| + \|x*\|]^{\beta} \to 0.$$

So $d(x*, Tx*) = 0$ and $x* \in Tx*$.
Also, the convergence rate estimate stated in (2) is achieved from the following relations

$$\begin{aligned}d(x*, x_n) \le H_d(Tx*, Tx_{n-1}) \le d(x*, x_{n-1}) &\le H_d(Tx*, Tx_{n-2})\\ &\le d(x*, x_{n-2})\\ &\vdots\\ &\le H_d(Tx*, Tx_0)\\ &\le d(x*, x_0) = \|x*\|,\end{aligned}$$

which implies that

$$C_n = d(x_n, x_0) \le d(x_n, x*) + d(x*, x_0) \le \|x*\| + \|x*\| = 2\|x*\|.$$

From the last inequality and (22) we have $K \le \Lambda(1 + 4\|x*\|)^{\beta}$. □

3. Applications

In this section, as an application of the result of previous section, is concerned with the existence of solutions for hyperbolic differential inclusions

$$\begin{cases}\dfrac{\partial^2 u(t,x)}{\partial t\partial x} \in F(t,x,u(t,x)) & a.e.(t,x) \in J_a \times J_b,\\ u(t,0) = \xi(t),\ u(0,x) = \eta(x), & t \in J_a,\ x \in J_b,\end{cases} \tag{24}$$

where $J_a = [0,a]$, $J_b = [0,b]$, $F : J_a \times J_b \times \mathbb{R} \to 2^{\mathbb{R}}$ is multivalued mapping satisfying some hypotheses which will be specified later, $\xi \in C(J_a, \mathbb{R})$, $\eta \in C(J_b, \mathbb{R})$. In the present section, we will prove the existence of solutions for problem (24) based on Theorems 2.1. First of all, we introduce notations which are used throughout this section. $C(J_a \times J_b, \mathbb{R})$ is the Banach space consisting of all continuous functions from $J_a \times J_b$ into $\mathbb{R}$ with the norm

$$||u|| = \sup\{|u(t,x)| : (t,x) \in J_a \times J_b\}\ for\ u \in C(J_a \times J_b, \mathbb{R}).$$

For $u, v \in C(J_a \times J_b, \mathbb{R})$ we define that $u \leq v$ if and only if $u(t,x) \leq v(t,x)$ for each $(t,x) \in J_a \times J_b$. Let $P = \{u : u \in C(J_a \times J_b, \mathbb{R}), u \geq 0\}$ and

$$H = \{u \in C(J_a \times J_b, \mathbb{R}) : \frac{\partial^2 u}{\partial t \partial x} \; exists \; for \; each \; (t,x) \in J_a \times J_b\}.$$

$L^1(J_a \times J_b, \mathbb{R})$ stands for the Banach space consisting of measurable functions $u : J_a \times J_b \to \mathbb{R}$ which are Lebesgue integrable normed by

$$||u||_L = \int_0^a \int_0^b |u(t,x)| dx dt \qquad for \; u \in L^1(J_a \times J_b, \mathbb{R}).$$

Let $\mathcal{M} : J_a \times J_b \times \mathbb{R} \to 2^{\mathbb{R}}$ be a multivalued map with nonempty values. For each $u \in C(J_a \times J_b, \mathbb{R})$ define the set of selections of $\mathcal{M}$ by

$$S_{\mathcal{M},u} = \{v \in L^1(J_a \times J_b, \mathbb{R}) : v(t,x) \in \mathcal{M}(t, x, u(t,x)) \; a.e. \; (t,x) \in J_a \times J_b\},$$

and assign to $\mathcal{M}$ the multivalued operator $\mathfrak{M} : C(J_a \times J_b, \mathbb{R}) \to 2^{L^1(J_a \times J_b, \mathbb{R})}$ by letting

$$\mathfrak{M}(u) = \{w \in L^1(J_a \times J_b, \mathbb{R}) : w(t,x) \in \mathcal{M}(t, x, u(t,x)), \; (t,x) \in J_a \times J_b\}.$$

The operator $\mathfrak{M}$ is called the Niemytsky operator associated with $\mathcal{M}$ in the light of some of the current literature. In order to state and verify our results, we need the continuous map $\mathfrak{L} : L^1(J_a \times J_b, \mathbb{R}) \to C(J_a \times J_b, \mathbb{R})$ defined by

$$\mathfrak{L}u(t,x) = \int_0^t \int_0^x u(s,\tau) ds d\tau.$$

As an application of results included in Theorem 2.1, we first unify the following

Theorem 3.1. *Suppose that the multivalued function $F : J_a \times J_b \times \mathbb{R} \to 2^{\mathbb{R}}$ satisfies the following conditions:*
(L1) $F(t,x,u)$ is compact subset for all $(t,x,u) \in J_a \times J_b \times C(J_a \times J_b, \mathbb{R})$. Moreover, $S_{F,u}$ is nonempty for each $u \in C(J_a \times J_b, \mathbb{R})$.
(L2) For any $u, v \in C(J_a \times J_b, \mathbb{R})$, if u, v are comparable then

$$H_d(F(t,x,u(t,x)), F(t,x,v(t,x))) \leq |\ell(t,x)|^2 |u(t,x) - v(t,x)|,$$

for almost each $(t,x) \in J_a \times J_b$, where $\ell \in L^1(J_a \times J_b, \mathbb{R})$ with $||\ell||_L \leq 1$.
(L3) For each $u \in C(J_a \times J_b, \mathbb{R})$ one has
$\{u(t,x) - \xi(t) - \eta(x) + \xi(0)\} \leq \mathfrak{L}v(t,x)$ for $(t,x) \in J_a \times J_b$ and $v \in S_{F,u}$. Then the problem (24) has at least a solution $u^* \in C(J_a \times J_b, \mathbb{R})$.

Proof. Let $M \subset C(J_a \times J_b, \mathbb{R})$ be a complete metric space. Then M satisfies the condition (H1). It is clear that problem (24) is equivalent to the integral inclusion

$$u(t,x) \in \Big\{h \in C(J_a \times J_b, \mathbb{R}) : h(t,x) = \xi(t) + \eta(x) - \xi(0) + \int_0^t \int_0^x v(s,\tau) ds d\tau, \; v \in S_{F,u}\Big\}.$$

Define the multivalued map $A : M \to 2^M$ by

$$(Au)(t,x) = \{h \in C(J_a \times J_b, \mathbb{R}) : h(t,x) = \xi(t) + \eta(x) - \xi(0) + \mathfrak{L}v(t,x), \; v \in S_{F,u}\}.$$

Clearly, the multivalued map A is well defined on in view of hypothesis (L1). We shall show that A satisfy all conditions of Theorem 2.1. We first show that A has compact values. It then suffices to prove that the composition map $\mathfrak{L} o S_F$ has compact values. Let $u \in C(J_a \times J_b, \mathbb{R})$ be arbitrary and let $\{u_n\}$ be a sequence in $S_{F,u}$. Then by the definition of $S_{F,u}, u_n(t,x) \in F(t,x,u(t,x))$ a.e for all $(t,x) \in J_a \times J_b$. Since $F(t,x,u(t,x))$ is compact, there exists a convergent subsequence of $\{u_n(t,x)\}$ (without loss of generality, we may assume it is $\{u_n(t,x)\}$ itself) that converges in measure to some $v(t,x) \in F(t,x,u(t,x))$ a.e for $(t,x) \in J_a \times J_b$. Now the continuity of $\mathfrak{L}$ guarantees that $\mathfrak{L}u_n(t,x) \to \mathfrak{L}v(t,x)$ pointwise on $J_a \times J_b$ provided $n \to \infty$. We shall prove that the convergence is uniform. To this end, we show that $\{\mathfrak{L}u_n\}$ is an equicontinuous sequence. Let $t_1, t_2 \in J_a$ with $t_1 < t_2$ and $x_1, x_2 \in J_b$ with $x_1 < x_2$, then

$$\begin{aligned}|\mathfrak{L}u_n(t_2,x_2) - \mathfrak{L}u_n(t_1,x_1)| &\leq \Big| \int_0^{t_2}\int_0^{x_2} u_n(s,\tau)dsd\tau - \int_0^{t_1}\int_0^{x_1} u_n(s,\tau)dsd\tau \Big| \\ &\leq \int_0^{t_1}\int_{x_1}^{x_2} |u_n(s,\tau)|dsd\tau + \int_{t_1}^{t_2}\int_0^{x_2} |u_n(s,\tau)|dsd\tau.\end{aligned}$$

Note that $u_n \in L^1(J_a \times J_b, \mathbb{R})$, we infer that the right-hand side of the above expression tends to zero as $t_2 \to t_1$ and $x_2 \to x_1$. Hence, $\{\mathfrak{L}u_n\}$ is equicontinuous and has a uniformly convergent subsequence by virtue of Arzela-Ascoli theorem. Obviously, this convergence is $\mathfrak{L}v$ and $\mathfrak{L}v \in \mathfrak{L}oS_{F,u}$. This shows that $\mathfrak{L}oS_{F,u}$ is compact. Therefore, A has compact values. As a result of this and (L3), we obtain that A has $UCAV$. Next we show that A satisfies relation (1) . Let $u, v \in K$ are comparable, say, $u \leq v$. Let $h_1 \in Au$. Then there exists $v_1 \in S_{F,u}$ such that

$$h_1(t,x) = \xi(t) + \eta(x) - \xi(0) + \int_0^t\int_0^x v_1(s,\tau)dsd\tau \quad for \quad (t,x) \in J_a \times J_b.$$

From (L2) there exists $w \in F(t,x,v(t,x))$ such that

$$|v_1(t,x) - w| \leq |\ell(t,x)|^2(v(t,x) - u(t,x)).$$

Define the multivalued map U by

$$U(t,x) = \{w \in \mathbb{R} : |v_1(t,x) - w| \leq |\ell(t,x)|^2(v(t,x) - u(t,x))\}.$$

Then the multivalued map $V(t,x) = U(t,x) \cap S_{F,v}$ has nonempty values and is measurable (see [21]). Then there exists a function v_2 which is a measurable selection for V. Clearly, $v_2(t,x) \in F(t,x,v(t,x))$ for each $(t,x) \in J_a \times J_b$ satisfying

$$|v_1(t,x) - v_2(t,x)| \leq |\ell(t,x)|^2(v(t,x) - u(t,x)).$$

Let us define for each $(t,x) \in J_a \times J_b$,

$$h_2(t,x) = \xi(t) + \eta(x) - \xi(0) + \int_0^t\int_0^x v_2(s,\tau)dsd\tau.$$

It follows that $h_2 \in Av$ and

$$|h_2(t,x) - h_1(t,x)| \leq \int_0^t\int_0^x |v_2(s,\tau) - v_1(s,\tau)|dsd\tau.$$

Thus, we have

$$\begin{aligned}\|h_2 - h_1\| &\le \|\ell\|_L^2 \|v-u\| \\ &\le \|\ell\|_L^2 \|v-u\| + \|\ell\|_L \|v-u\| \\ &= \|\ell\|_L \|v-u\| - \|\ell\|_L^2 \|v-u\| + 2\|\ell\|_L^2 \|v-u\| \\ &= \|\ell\|_L \|v-u\| (1 - \|\ell\|_L) + 2\|\ell\|_L^2 \|v-u\| \\ &\le (1 - \|\ell\|_L)\|v-u\| + 2\|\ell\|_L^2 [1 + \|u\| + \|v\|].\end{aligned}$$

Put $\psi(\|\ell\|_L) = \|\ell\|_L$, $\alpha = 1$, $\Lambda = 2$ and $\beta = 1$. Then we have

$$\|h_2 - h_1\| \le (1 - \|\ell\|_L) d(u,v) + \Lambda \|\ell\|_L^\alpha \psi(\|\ell\|_L)[1 + \|u\| + \|v\|]^\beta.$$

By an analogous relation, obtained by interchanging the roles of u and v, it follows that

$$H_d(Au, Av) \le (1 - \|\ell\|_L) d(u,v) + \Lambda \|\ell\|_L^\alpha \psi(\|\ell\|_L)[1 + \|u\| + \|v\|]^\beta.$$

Therefore, A satisfies Theorem 2.1 with respect to $\psi(\epsilon) = \epsilon$, $\alpha = 1$, $\Lambda = 2$ and $\beta = 1$. Now Theorem 2.1 guarantees that (24) has a desired solution and this proof is completed. □

REFERENCES

[1] *S. Banach.,* Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math., **3**(1922), 133-181.

[2] *S.B. Nadler,* Multivalued contraction mappings, Pacific J Math, **30**(1969), 475-488.

[3] *I. Beg and A. Azam,* Fixed points of asymptotically regular multivalued mappings, J. Austral. Math. Soc. (Series-A), **53**(1992), 313-326.

[4] *P.Z. Daffer,* Fixed points of generalized contractive multivalued mappings, J. Math. Anal. Appl., **192**, 655-666.

[5] *P.Z. Daffer , H. Kaneko H and W. Li,* On a conjecture of S. Reich, Proc. Amer. Math. Soc., **124**, 3159-3162.

[6] *Y. Feng and S. Liu,* Fixed point theorems for multivalued contractive mappings and multivaled Caristi type mappings, J. Math. Anal. Appl., **317**(2006), 103–112.

[7] *D. Klim and D. Wardowski,* Fixed point theorems for set-valued contractions in complete metric spaces, J. Math. Anal. Appl., **334**, 132-139.

[8] *C.Y. Qing,* On a fixed point problem of Reich, Proc. Amer. Math. Soc., **124**(1996), 3085-3088.

[9] *S. Reich,* Fixed points of contractive functions, Boll. Unione. Mat. Ital., **4**(1972), 26-42.

[10] *E. Zeidler,* Nonlinear Functional Analysis and its Application, New York, Springer Verlag, 1985.

[11] *A.C.M. Ran and M.C.B. Reurings,* A fixed point theorm in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc., **132**(2004), 1435-1443.

[12] *I. Beg and A.R. Butt,* Fixed point for set-valued mappings satisfying an implicit relation in partially ordered metric spaces, Nonlinear Anal, **71**(2009), 3699-3704.

[13] *I. Beg and A.R. Butt,* Common fixed point for generalized set-valued contractions satisfying an implicit relation in partially ordered metric spaces, Math. Commun., **15**(2010), 65-76.

[14] *J. Harjani and K. Sadarangani,* Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Anal, **71**(2009), 3403-3410.

[15] *A. Petrusel and I.A. Rus,* Fixed point theorems in ordered L-spaces, Proc. Amer. Math. Soc., **134**(2005), 411-418.

[16] *D. O'Regan and A. Petrusel,* Fixed point theorems in ordered metric spaces, J. Math. Anal. Appl., **341**(2008), 1241-1252.

[17] *A. Nicolae, D. O'Regan and A. Petrusel,* Fixed point theorems for singlevalued and multivalued generalized contractions in metric spaces endowed with a graph, Georgian. Math. Soc., **18**(2011), 307-327.

[18] *V. Pata,* A fixed point theorem in metric spaces, J Fixed Point Theory Appl, **10**(2011), 299-305.

[19] *SH. Hong,* Fixed points of multivalued operators in ordered metric spaces with applications, Nonlinear Anal, **72**(2009), 3929-3942.

[20] *J.J. Nieto and R. Rodríguez-Lôpez,* Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order, **72**(2005), 223-239.

[21] *A. Belarbi and M. Benchohra,* Existence theory for perturbed impulsive hyperbolic differential inclusions with variable times, J. Math. Anal. Appl., **372**(2007), 1116-1129.