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ON RESOLVABILITY IN DOUBLE-STEP CIRCULANT GRAPHS*

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In this paper, we study the metric dimension of double-step circulant graphs $C_n(1,2,k)$ *for any positive integer* $n \geq 13$ *and when* $k = 4$. *We prove that these double-step circulant graphs have constant metric dimension.*

Key Words: Metric dimension, basis, resolving set, double-step, circulant graph
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1. Introduction

Metric dimension is a parameter that has appeared in various applications of graph theory, as diverse as, pharmaceutical chemistry [5, 6], robot navigation [16], combinatorial optimization [19] and sonar and coast guard Loran [20], to name a few. Metric dimension is a generalization of affine dimension to arbitrary metric spaces (provided a resolving set exists).

In a connected graph G, the *distance* $d(u,v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* $r(v|W)$ of v with respect to W is the k-tuple $(d(v,w_1), d(v,w_2), d(v,w_3), \ldots, d(v,w_k))$. W is called a *resolving set* [6] or *locating set* [20] if every vertex of G is uniquely identified by its distances from the vertices of W, or equivalently, if distinct vertices of G have distinct representations with respect to W. A resolving set of minimum cardinality is called a *basis* for G and this cardinality is the *metric dimension* of G, denoted by $dim(G)$ [2]. The concepts of resolving set and metric basis have previously appeared in the literature (see [2-6, 9-23]).

For a given ordered set of vertices $W = \{w_1, w_2, \ldots, w_k\}$ of a graph G, the

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ith component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G) \setminus W$.

A useful property in finding $dim(G)$ is the following lemma: *[22]* Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u,w) = d(v,w)$ for all vertices $w \in V(G) \setminus \{u,v\}$, then $\{u,v\} \cap W \neq \emptyset$. Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension was introduced by Slater in [20, 21] and studied independently by Harary and Melter in [9]. Applications of this invariant to the navigation of robots in networks are discussed in [16] and applications to chemistry in [6] while applications to problem of pattern recognition and image processing, some of which involve the use of hierarchical data structures are given in [17].

Let $\mathcal{F}$ be a family of connected graphs $G_n : \mathcal{F} = (G_n)_{n \geq 1}$ depending on n as follows: the order $|V(G)| = \varphi(n)$ and $\lim_{n \to \infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $dim(G_n) \leq C$ for every $n \geq 1$ then we shall say that $\mathcal{F}$ has bounded metric dimension; otherwise $\mathcal{F}$ has unbounded metric dimension.

If all graphs in $\mathcal{F}$ have the same metric dimension (which does not depend on n), $\mathcal{F}$ is called a family with constant metric dimension [13]. A connected graph G has $dim(G) = 1$ if and only if G is a *path* [6]; *cycles* C_n have metric dimension 2 for every $n \geq 3$. Also generalized Petersen graphs $P(n,2)$, antiprisms A_n and circulant graphs $C_n(1,2)$ are families of graphs with constant metric dimension [13]. Recently some classes of regular graphs with constant metric dimension have been studied in [12].

Other families of graphs have unbounded metric dimension: if W_n denotes a *wheel* with n spokes and J_{2n} the graph deduced from the wheel W_{2n} by alternately deleting n spokes, then $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for every $n \geq 7$ [2] and $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$ [23] for every $n \geq 4$.

An example of a family which has bounded metric dimension is the family of *prisms*. In [3] it was proved that

$$dim(P_m \times C_n) = \begin{cases} 2, & \textit{if } n \textit{ is odd}; \\ 3, & \textit{otherwise.} \end{cases}$$

Since *prisms* D_n are the trivalent plane graphs obtained by the cross product of path P_2 with a cycle C_n, so prisms constitute a family of 3-*regular* *graphs* with bounded metric dimension. Also generlized Petersen graphs $P(n,3)$ have bounded metric dimension [10].

Note that the problem of determining whether $dim(G) < k$ is an NP-complete problem [8]. Some bounds for this invariant, in terms of the diameter of the graph, are given in [16] and it was shown in [6, 16, 17, 18] that the metric dimension of trees can be determined efficiently. It appears unlikely that significant progress can be made in determining the dimension of a graph unless it belongs to a class for which the distances between vertices can be described in some systematic manner.

The metric dimension of double-step circulant graphs $C_n(1,2)$ has been investigated in [13]. Recently, we have studies the metric dimension of double-step circulant graphs $C_n(1,2,3)$ [11]. In this paper, we extend this study to double-step circulant graphs $C_n(1,2,k)$ for any positive integer $n \geq 13$ when $k = 4$.

In what follows all indices i which do not satisfy inequalities $1 \leq i \leq n$ will be taken modulo n.

2. Upper bounds for the metric dimension of double-step circulant graphs $C_n(1,2,k)$ for any positive integer $n \geq 13$ and $k = 4$

The circulant graphs are an important class of graphs, which can be used in the design of local area networks [1]. Let n, m and $a_1, \cdots, a_m$ be positive integers, $1 \leq a_i \leq \lfloor \frac{n}{2} \rfloor$ and $a_i \neq a_j$ for all $1 \leq i < j \leq m$. An undirected graph with the set of vertices $V = \{v_1, \cdots, v_n\}$ and the set of edges $E = \{v_i v_{i+a_j} : 1 \leq i \leq n, 1 \leq j \leq m\}$, the indices being taken modulo n, is called a *circulant graph* and is denoted by $C_n(a_1, \cdots, a_m)$. The numbers $a_1, \cdots, a_m$ are called the generators and we say that the edge $v_i v_{i+a_j}$ is of type a_j.

It is easy to see that the circulant graph $C_n(a_1, \cdots, a_m)$ is a regular graph of degree r, where

$$r = \begin{cases} 2m-1, & if \ \frac{n}{2} \in \{a_1, \cdots, a_m\}; \\ 2m, & otherwise. \end{cases}$$

The metric dimension of circulant graphs $C_n(1,2)$ has been studied in [13] where it has been proved that $dim(C_n(1,2)) = 3$ for $n \equiv 0,2,3 (\bmod 4)$ and $dim(C_n(1,2)) \leq 4$ otherwise.

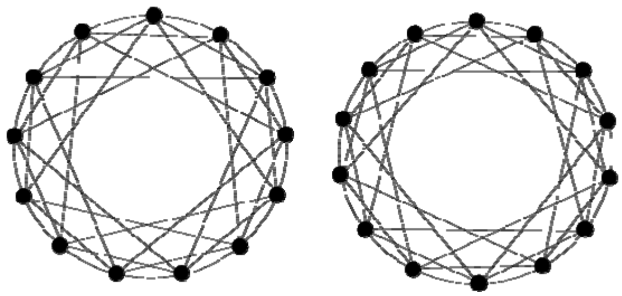

Fig. 1: double-step circulant graphs $C_{13}(1,2,4)$ and $C_{14}(1,2,4)$

The metric dimension of double-step circulant graphs $C_n(1,2,k)$ for any positive integer n and $k=3$ has been determined by Imran et al. (2012) [11]. In the next theorem, we give the upper bounds for the metric dimension of double-step circulant graphs $C_n(1,2,4)$. Note that the choice of an appropriate basis of vertices (also referred to as landmarks in [15]) is core of the problem.

Theorem 1. For the double-step circulant graphs $C_n(1,2,k)$ for any positive integer $n \geq 13$ and $k=4$, we have

$$dim(C_n(1,2,k)) \leq 4$$

Proof. Case (i) When $n \equiv 0 (\bmod 4)$

In this case, we can write $n = 4k, k \geq 4, k \in \mathbf{Z}^+$. Let $W = \{v_1, v_2, v_3, v_4\} \subset V(C_n(1,2,4))$, we show that W is a resolving set for $C_n(1,2,4)$ in this case. For this purpose, we give the representations of $V(C_n(1,2,4))$ with respect to $W = \{v_1, v_2, v_3, v_4\}$.

$$r(v_{4i+1}|W) = \begin{cases} (i, i+1, i, i), & 1 \leq i \leq \lceil \frac{k-1}{2} \rceil; \\ (k-i, k-i+1, k-i+1, \lceil \frac{k+1}{2} \rceil - i), & \lceil \frac{k-1}{2} \rceil + 1 \leq i \leq k. \end{cases}$$

$$r(v_{4i+2}|W) = \begin{cases} (i+1, i, i+1, i), & 1 \leq i \leq \lceil \frac{k-1}{2} \rceil; \\ (k-i+1, k-i, k-i+1, k-i+1), & \lceil \frac{k-1}{2} \rceil + 1 \leq i \leq k. \end{cases}$$

$$r(v_{4i+3}|W) = \begin{cases} (i+1, i+1, i, i+1), & 1 \leq i \leq \lceil \frac{k-1}{2} \rceil; \\ (k-i, k-i+1, k-i, k-i+1), & \lceil \frac{k-1}{2} \rceil + 1 \leq i \leq k. \end{cases}$$

$$r(v_{4i}\,|\,W)=\begin{cases}(i+1,i,i,i+1), & 1\le i\le\lceil\frac{k-1}{2}\rceil;\\ (k-i+1,k-i+1,2\lceil\frac{k+1}{2}\rceil-i,2\lceil\frac{k+1}{2}\rceil-i+1), & \\ & \lceil\frac{k-1}{2}\rceil+1\le i\le k.\end{cases}$$

One can obviously verify that the set W can distinguish all the vertices of $C_n(1,2,4)$ which implies that $W=\{v_1,v_2,v_3,v_4\}$ is a resolving set for $C_n(1,2,4)$, thus showing that that $dim(C_n(1,2,4))\le 4$ in this case.

Case (ii) When $n\equiv 1(\mathrm{mod}\,4)$

In this case, we can write $n=4k+1, k\ge 3, k\in\mathbf{Z}^+$. Let $W=\{v_1,v_2,v_{4\lceil\frac{k}{2}\rceil-2},v_{4\lceil\frac{k}{2}\rceil-1}\}\subset V(C_n(1,2,4))$, we show that W is a resolving set for $C_n(1,2,4)$ in this case. For this purpose, we give the representations of $V(C_n(1,2,4))$ with respect to $W=\{v_1,v_2,v_{4\lceil\frac{k}{2}\rceil-2},v_{4\lceil\frac{k}{2}\rceil-1}\}$.

$$r(v_{4i-1}\,|\,W)=\begin{cases}(i,i,\lceil\frac{k+2}{2}\rceil-i,\lceil\frac{k}{2}\rceil-i), & 1\le i\le\lceil\frac{k}{2}\rceil-1;\\ (k-i+2,k-i+1,i-\lceil\frac{k}{2}\rceil+1,i-\lceil\frac{k}{2}\rceil), & \\ & \lceil\frac{k}{2}\rceil+1\le i\le k-1;\\ (2,1,\lceil\frac{k}{2}\rceil,\lceil\frac{k-1}{2}\rceil), & i=k.\end{cases}$$

$$r(v_{4i}\,|\,W)=\begin{cases}(i+1,i,\lceil\frac{k}{2}\rceil-i,\lceil\frac{k+2}{2}\rceil-i), & 1\le i\le\lceil\frac{k}{2}\rceil-1;\\ (\lceil\frac{k}{2}\rceil+1,\lceil\frac{k}{2}\rceil,1,1), & i=\lceil\frac{k}{2}\rceil;\\ (k-i+1,k-i+2,i-\lceil\frac{k}{2}\rceil+1,i-\lceil\frac{k}{2}\rceil+1), & \lceil\frac{k}{2}\rceil+1\le i\le k.\end{cases}$$

$$r(v_{4i+1}|W)=\begin{cases}(i,i+1,\lceil\frac{k}{2}\rceil-i,\lceil\frac{k}{2}\rceil-i), & 1\le i\le\lceil\frac{k}{2}\rceil-1;\\ (\lceil\frac{k}{2}\rceil,\lceil\frac{k+1}{2}\rceil,1,1), & i=\lceil\frac{k}{2}\rceil;\\ (k-i+1,k-i+1,i-\lceil\frac{k}{2}\rceil+2,i-\lceil\frac{k}{2}\rceil+1), & \\ & \lceil\frac{k}{2}\rceil+1\le i\le k-1;\\ (1,1,\lceil\frac{k}{2}\rceil,\lceil\frac{k+1}{2}\rceil), & i=k.\end{cases}$$

$$r(v_{4i+2}|W)=\begin{cases}(i+1,i,\lceil\frac{k}{2}\rceil-i-1,\lceil\frac{k}{2}\rceil-i), & 1\le i\le\lceil\frac{k}{2}\rceil-2;\\ (\lceil\frac{k-1}{2}\rceil,\lceil\frac{k}{2}\rceil,1,2), & i=\lceil\frac{k}{2}\rceil;\\ (k-i,k-i+1,i-\lceil\frac{k}{2}\rceil+1,i-\lceil\frac{k}{2}\rceil+2), & \lceil\frac{k}{2}\rceil+1\le i\le k;\\ (1,1,\lceil\frac{k}{2}\rceil,\lceil\frac{k+1}{2}\rceil), & i=k.\end{cases}$$

The set W can distinguish all the vertices of $C_n(1,2,4)$ which implies that $W=\{v_1,v_2,v_{4\lceil\frac{k}{2}\rceil-2},v_{4\lceil\frac{k}{2}\rceil-1}\}$ is a resolving set for $C_n(1,2,4)$, thus showing that that $dim(C_n(1,2,4))\le 4$ in this case too.

Case (iii) When $n\equiv 2(\bmod 4)$

In this case, we can write $n=4k+2, k\ge 3, k\in\mathbf{Z}^+$. Let $W=\{v_1,v_2,v_{4\lceil\frac{k+1}{2}\rceil-2},v_{4\lceil\frac{k+1}{2}\rceil-1}\}\subset V(C_n(1,2,3))$. Again we show that W is a resolving set for $C_n(1,2,4)$. For this purpose, we give the representations of $V(C_n(1,2,4))$ with respect to $W=\{v_1,v_2,v_{4\lceil\frac{k+1}{2}\rceil-2},v_{4\lceil\frac{k+1}{2}\rceil-1}\}$.

$$r(v_{4i-1}|W)=\begin{cases}(i,i,\lceil\frac{k+3}{2}\rceil-i,\lceil\frac{k+1}{2}\rceil-i), & 1\le i\le\lceil\frac{k+1}{2}\rceil-1;\\ (k-i+1,k-i+2,i-\lceil\frac{k+1}{2}\rceil+1,i-\lceil\frac{k+1}{2}\rceil+2), & \\ & \lceil\frac{k+1}{2}\rceil\le i\le k.\end{cases}$$

$$r(v_{4i}\mid W)=\begin{cases}(i+1,i,\lceil\frac{k+1}{2}\rceil-i,\lceil\frac{k+3}{2}\rceil-i), & 1\le i\le\lceil\frac{k+1}{2}\rceil-1;\\ (k-i+2,k-i+1,i-\lceil\frac{k+1}{2}\rceil+1,i-\lceil\frac{k+1}{2}\rceil+1), & \lceil\frac{k+1}{2}\rceil\le i\le k.\end{cases}$$

$$r(v_{4i+1}\mid W)=\begin{cases}(i,i+1,\lceil\frac{k+1}{2}\rceil-i,\lceil\frac{k+1}{2}\rceil-i), & 1\le i\le\lceil\frac{k+1}{2}\rceil-1;\\ (k-i+1,k-i+2,i-\lceil\frac{k+1}{2}\rceil+1,i-\lceil\frac{k+1}{2}\rceil+2), & \lceil\frac{k+1}{2}\rceil\le i\le k.\end{cases}$$

$$r(v_{4i+2}\mid W)=\begin{cases}(i+1,i,\lceil\frac{k-1}{2}\rceil-i-1,\lceil\frac{k+1}{2}\rceil-i), & 1\le i\le\lceil\frac{k+1}{2}\rceil-2;\\ (k-i+1,k-i+1,i-\lceil\frac{k+1}{2}\rceil+1,i-\lceil\frac{k+1}{2}\rceil+2), & \lceil\frac{k+1}{2}\rceil\le i\le k.\end{cases}$$

One can see that the set W can distinguish all the vertices of $C_n(1,2,4)$ which implies that $W=\{v_1,v_2,v_{4\lceil\frac{k+1}{2}\rceil-2},v_{4\lceil\frac{k+1}{2}\rceil-1}\}$ is a resolving set for $C_n(1,2,4)$, thus showing that that $dim(C_n(1,2,4))\le 4$ in this case also.

Case (iv) When $n\equiv 3(\mathrm{mod}\,4)$

In this case, we can write $n=4k+3, k\ge 3, k\in\mathbf{Z}^+$. Let $W=\{v_1,v_2,v_{4\lceil\frac{k}{2}\rceil-2},v_{4\lceil\frac{k}{2}\rceil-1}\}\subset V(C_n(1,2,4))$. We show that W is a resolving set for $C_n(1,2,4)$. For this purpose, we give the representations of $V(C_n(1,2,4))$ with respect to $W=\{v_1,v_2,v_{4\lceil\frac{k}{2}\rceil-2},v_{4\lceil\frac{k}{2}\rceil-1}\}$.

$$r(v_{4i-1}|W)=\begin{cases}(i,i,\lceil\frac{k}{2}\rceil-i+1,\lceil\frac{k}{2}\rceil), & 1\le i\le\lceil\frac{k}{2}\rceil-1;\\ (k-i+2,k-i+2,i-\lceil\frac{k}{2}\rceil+1,i-\lceil\frac{k}{2}\rceil), & \lceil\frac{k}{2}\rceil+1\le i\le k-1;\\ (1,1,\lceil\frac{k}{2}\rceil,\lceil\frac{k}{2}\rceil), & i=k.\end{cases}$$

$$r(v_{4i}|W)=\begin{cases}(i+1,i,\lceil\frac{k}{2}\rceil-i,\lceil\frac{k}{2}\rceil-i+1), & 1\le i\le\lceil\frac{k}{2}\rceil-1;\\ (\lceil\frac{k}{2}\rceil,\lceil\frac{k+1}{2}\rceil,1,1), & i=\lceil\frac{k}{2}\rceil;\\ (k-i+2,k-i+1,i-\lceil\frac{k}{2}\rceil+2,i-\lceil\frac{k}{2}\rceil+1), & \lceil\frac{k}{2}\rceil+1\le i\le k.\end{cases}$$

$$r(v_{4i+1}|W)=\begin{cases}(i,i+1,\lceil\frac{k}{2}\rceil-i,\lceil\frac{k}{2}\rceil-i), & 1\le i\le\lceil\frac{k}{2}\rceil-1;\\ (\lceil\frac{k}{2}\rceil,\lceil\frac{k+1}{2}\rceil,1,1), & i=\lceil\frac{k}{2}\rceil;\\ (k-i+1,k-i+1,i-\lceil\frac{k}{2}\rceil+2,i-\lceil\frac{k}{2}\rceil+1), & \lceil\frac{k}{2}\rceil+1\le i\le k-1;\\ (1,1,\lceil\frac{k}{2}\rceil,\lceil\frac{k+1}{2}\rceil), & i=k.\end{cases}$$

$$r(v_{4i+2}|W)=\begin{cases}(i+1,i,\lceil\frac{k}{2}\rceil-i-1,\lceil\frac{k}{2}\rceil-i), & 1\le i\le\lceil\frac{k}{2}\rceil-2;\\ (\lceil\frac{k+1}{2}\rceil,\lceil\frac{k}{2}\rceil,1,2), & i=\lceil\frac{k}{2}\rceil;\\ (k-i+1,k-i+2,i-\lceil\frac{k}{2}\rceil+1,i-\lceil\frac{k}{2}\rceil+1), & \lceil\frac{k}{2}\rceil+1\le i\le k.\end{cases}$$

Again, One can see that the set W can distinguish all the vertices of $C_n(1,2,4)$

which implies that $W=\{v_1,v_2,v_{4\lceil\frac{k}{2}\rceil-2},v_{4\lceil\frac{k}{2}\rceil-1}\}$ is a resolving set for $C_n(1,2,4)$, thus showing that that $dim(C_n(1,2,4))\le 4$ in this case, which completes the proof.

3. Metric dimension of double-step circulant graphs $C_n(1,2,k)$ for any positive integer $n\ge 13$ and $k=4$

In this section, we will prove that $dim(C_n(1,2,k))\ge 4$ for any positive integer $n\ge 13$ and $k=4$. For this purpose, we define the outer cycle as the cycle induced by $\{v_1,v_2,\ldots,v_n\}$. Due to the rotational symmetry of the circulant graphs $C_n(1,2,k)$ where $k=4$, we deduce that For any two vertices u_i and u_j ($i\ne j$) on the outer cycle induced by $V(C_n(1,2,4))$, we have $d(u_i;u_j)=d(u_{i+r};u_{j+r})$ for any $1\le r\le n-1$. For the concept of gaps and size of a gap (to be used later), we adopted the definitions and terminology used in [2]. Let C_n be a cycle with n vertices. We denote its vertices by $v_1,v_2,...,v_n$. Let k,l be positive integers, $1\le k<l\le n$. Then the vertices $v_{k+1},v_{k+2},...,v_{l-1}$ are the vertices in the gap determined by the vertices v_k and v_l and the size of the gap is $k-l-1$.

Theorem 2. For every positive integer $n\ge 13$, we have $dim((C_n(1,2,k))\ge 4$ when $k=4$.

Proof. Let $n=4k+l$ where $l\in\{0,1,2,3\}$. To prove this theorem, it suffices to show that there is no resolving set with 3 vertices for $V((C_n(1,2,4))$. Suppose to contrary that there exists a resolving set W with three vertices for $V(C_n(1,2,4))$.

Without loss of generality, we can assume that $W=\{v_1,v_i,v_j\}$ is a resolving set where $i\ne j$ and $i,j\ne 1$. We make the following claims.

Claim 1: *No two vertices with consecutive indices on outer cycle can appear in any resolving set with three elements for* $V(C_n(1,2,4))$.

Without loss of generality, we can suppose that $W'=\{v_1,v_2,v_j\}$ is a resolving set with two vertices having consecutive indices. By symmetry, we need only consider the case for $3\le j\le 3k+1$. Then

$$\begin{cases} r(v_5|W') = r(v_{n-1}|W'), & \text{if } j = 3 \text{ and for all } l; \\ r(v_3|W') = r(v_n|W'), & \text{if } j \equiv 0,2 \pmod 4 \text{ and for all } l; \\ r(v_4|W') = r(v_6|W'), & \text{if } j \equiv 1 \pmod 4,\ j \neq 4k+1 \text{ and for all } l; \\ r(v_{j-2}|W') = r(v_{j+1}|W'), & \text{if } j \equiv 3 \pmod 4,\ j \neq 4k+1 \text{ and for all } l; \\ r(v_{j-1}|W') = r(v_{j+1}|W'), & \text{if } j = 4k+1 \text{ and for all } l. \end{cases}$$

a contradiction.

Claim 2: *No first two gaps of same size between the indices of resolving vertices on outer cycle can appear in any resolving set with 3 elements for* $V(C_n(1,2,3))$.

Let $W' = \{v_1, v_i, v_{2i-1}\}$ be a resolving set with 3 elements and having first two gaps of same size between the indices of resolving vertices. By symmetry, for $l = 2$, we consider the case for $3 \leq j \leq 3k+1$; for $l = 3,4$, we consider the case for $3 \leq j \leq 3k+2$; for $l = 5$, we consider the case for $3 \leq j \leq 3k+3$. Then

$$\begin{cases} r(v_{i-2}|W') = r(v_{i+1}|W'), & \text{if } j \equiv 0 \pmod 3 \text{ and for all } l; \\ r(v_{i+1}|W') = r(v_{i+3}|W'), & \text{if } j \equiv 1,2 \pmod 3 \text{ and for all } l. \end{cases}$$

a contradiction.

A consequence of Claim 1 and Claim 2 implies Claim 3.

Claim 3: *No first two gaps with different size between the indices of resolving vertices on outer cycle can appear in any basis with 3 elements for* $V(C_n(1,2,3))$.

Let $W' = \{v_1, v_i, v_j\}$ be a resolving set with three elements and having first two gaps of different size. For each fixed value of i, $3 \leq i \leq 2k$, we have variation of j as $i+3 \leq j \leq 4k+1$. We have the following possibilities.

(i) The first gap is of odd size and the second gap is of even size.

(ii) The first two gaps are of odd size.

(iii) The first gap is of even size and second gap is of odd size.

(iv) The first two gaps are of even size.

But in each of the above possibilities, we get either $r(v_{j-2}|W') = r(v_{j+1}|W')$ or $r(v_{j-1}|W') = r(v_{j-3}|W')$ or $r(v_{j-2}|W') = r(v_{j-4}|W')$ or $r(v_{j-3}|W') = r(v_{n-1}|W')$, or $r(v_{j+1}|W') = r(v_{j+3}|W')$ or $r(v_{j+2}|W') = r(v_{j+4}|W')$, or

$r(v_{j+3}|W') = r(v_{n-1}|W')$, or $r(v_{j+3}|W') = r(v_{j+5}|W')$, leading to a contradiction.

Hence, from above it follows that $dim(C_n(1,2,3)) \geq 4$ which completes the proof.

As an immediate consequence of Theorem 1 and Theorem 2, we deduce the following theorem.

Theorem 3. For circulant graphs $C_n(1,2,k))$ when $k = 4$, we have $dim(C_n(1,2,k)) = 4$ for every positive integer $n \geq 13$.

4. Conclusion

The metric dimension of double-step circulant graphs $C_n(1,2,k)$ for any positive integer $n \geq 12$ and $k = 3$ has been determined by Imran et al. in [11] where it was proved that the double-step circulant graphs $C_n(1,2,3)$ have metric dimension equal to 4 for $n \equiv 2,3,4,5(\mathrm{mod}\,6)$. For $n \equiv 0(\mathrm{mod}\,6)$ only 5 vertices appropriately chosen suffice to resolve all the vertices of $C_n(1,2,3)$, thus implying that $dim(C_n(1,2,3)) \leq 5$ except $n \equiv 1(\mathrm{mod}\,6)$ when $dim(C_n(1,2,3)) \leq 6$. In this paper, we have studied the metric dimension of double-step circulant graphs $C_n(1,2,k)$ for any positive integer $n \geq 13$ and $k = 4$. We proved that the metric dimension of these circulant graphs $C_n(1,2,k)$ when $k = 4$ is constant and does not depend on the number of vertices in the graphs. Moreover, only 4 vertices chosen appropriately suffice to resolve all the vertices of circulant graphs $C_n(1,2,k)$ when $k = 4$. We see that the behavior of metric dimension for circulant graphs $C_n(1,2,k)$ varies rapidly for each value of k, even for the class of circulant graphs of same degree, the nature of metric dimension is not the same. However we believe that the metric dimension of circulant graphs will never depend upon the number of vertices in the graphs.

5. Open Problem

Find the exact value of metric dimension or some good bounds in terms of other graphical parameters for double-step circulant graphs $C_n(1,2,k)$ when $k \geq 5$.

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