MODULE MEAN FOR BANACH ALGEBRAS

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In this paper, the module (ϕ,φ)-amenable Banach algebras are characterized. Also, the relations of module (ϕ,φ)-amenability of a Banach algebra and their ideals are studied. Some mild conditions are found for a Banach algebra to possess a module (ϕ,φ)-mean of norm 1.

Keywords: Banach modules; Module character amenability; Module mean.

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1. Introduction

For a non-zero character φ on a Banach algebra $\mathcal{A}$, Kaniuth, Lau and Pym [11] introduced and studied the interesting notion of φ-amenability; see also [9, 13]. Precisely, a Banach algebra $\mathcal{A}$ is φ-amenable if there exists a bounded linear functional m on the dual space $\mathcal{A}^*$ such that $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a)m(f)$ for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. Bodaghi and Amini [5] introduced the notion of module (ϕ,φ)-amenability for a class of Banach algebras that are modules over another Banach algebra as follows:

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b,\ (ab) \cdot \alpha = a(b \cdot \alpha) \quad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Let $\Phi_{\mathfrak{A}}$ be the character space of $\mathfrak{A}$ and $\varphi \in \Phi_{\mathfrak{A}} \cup \{0\}$. Consider the linear map $\phi : \mathcal{A} \longrightarrow \mathfrak{A}$ such that

$$\phi(ab) = \phi(a)\phi(b), \quad \phi(a \cdot \alpha) = \phi(\alpha \cdot a) = \varphi(\alpha)\phi(a) \quad (a \in \mathcal{A},\ \alpha \in \mathfrak{A}).$$

We denote the set of all such maps by $\Omega_{\mathcal{A}}$. A bounded linear functional $m : \mathcal{A}^* \longrightarrow \mathbb{C}$ is called a *module* (ϕ,φ)*-mean* on $\mathcal{A}^*$ if $m(f\cdot a) = \varphi\circ\phi(a)m(f)$, $m(f\cdot\alpha) = \varphi(\alpha)m(f)$ and $m(\varphi \circ \phi) = 1$ for each $f \in \mathcal{A}^*, a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. We say $\mathcal{A}$ is *module* (ϕ,φ)*-amenable* if there exists a module (ϕ,φ)-mean on $\mathcal{A}^*$ [5]. We note that if $\mathfrak{A} = \mathbb{C}$ and φ is the identity map then the module (ϕ,φ)-amenability coincides with ϕ-amenability [11]. In [5], it is characterized the module (ϕ,φ)-amenability of a Banach algebra $\mathcal{A}$ through vanishing of the first Hochschild module cohomology group $\mathcal{H}^1_{\mathfrak{A}}(\mathcal{A}, X^*)$ for certain Banach $\mathcal{A}$-bimodules X (for modification of the first Hochschild module

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cohomology group $\mathcal{H}^1_{\mathfrak{A}}(\mathcal{A}, X^*)$, by using module homomorphisms between Banach algebras, refer to [4]).

In this paper, we characterize the module (ϕ, φ)-amenability of Banach algebras through the existence of a bounded net $(a_\gamma)_\gamma$ in $\mathcal{A}$ such that $\|aa_\gamma - \varphi\circ\phi(a)a_\gamma\| \to 0$ and $\|\alpha\cdot a_\gamma - \varphi(\alpha)a_\gamma\| \to 0$ for all $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. Then, we focus on (ϕ, φ)-means and establish various criteria for their existence.

2. **Main Results**

Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x,\ a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x,\ (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a)$$

for all $a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X$ and similarly for the right and two-sided actions. Then we say that X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If moreover $\alpha \cdot x = x \cdot \alpha$ for all $\alpha \in \mathfrak{A}$ and $x \in X$, then X is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module. Note that when $\mathcal{A}$ acts on itself by algebra multiplication, it is not in general a Banach $\mathcal{A}$-$\mathfrak{A}$-module. Indeed, if $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-module.

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions. An $\mathfrak{A}$-module map $D : \mathcal{A} \longrightarrow X$ is called a module derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in \mathcal{A}).$$

A module derivation D is called bounded if there exists $M > 0$ such that $\|D(a)\| \leq M\|a\|$, for every $a \in \mathcal{A}$. Note that boundedness of D implies its norm continuity while D can be non-linear. If X is a commutative $\mathcal{A}$-$\mathfrak{A}$-module, then each $x \in X$ defines an inner module derivation as $D_x(a) = a \cdot x - x \cdot a$ for all $a \in \mathcal{A}$. The Banach algebra $\mathcal{A}$ is called *module amenable* (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, each $\mathfrak{A}$-module derivation $D : \mathcal{A} \longrightarrow X^*$ is inner [1]; for other notions of module amenability for Banach algebras refer to [2], [6] and [14]. Note that if if $\mathfrak{A} = \mathbb{C}$, then the module amenability will absolutely overlap with Johnson's amenability [10] for a Banach algebra.

Theorem 2.1. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with compatible actions and $\phi \in \Omega_{\mathcal{A}}$ and $\varphi \in \Phi_{\mathfrak{A}}$ such that $\varphi \circ \phi \neq 0$. Then the following assertions are equivalent:*

(i) *$\mathcal{A}$ is module (ϕ, φ)-amenable;*
(ii) *$\mathcal{A}$ is $\varphi \circ \phi$-amenable.*

Proof. That (i) implies (ii) is trivial. So, it suffice to show that (ii) implies (i). To see this, suppose that $m \in \mathcal{A}^{**}$ is a $\varphi \circ \phi$-mean and $a_0 \in \mathcal{A}$ such that $\varphi \circ \phi(a_0) = 1$. Then, we set $n := a_0 \cdot m \in \mathcal{A}^{**}$. It is easy to see that $n(\varphi \circ \phi) = 1$. Moreover,

$$\langle n, f \cdot a\rangle = \langle m, f \cdot (aa_0)\rangle = m(f)\varphi \circ \phi(aa_0) = m(f)\varphi \circ \phi(a) = n(f)\varphi \circ \phi(a)$$

for all $a \in \mathcal{A}$, and

$$\langle n, f \cdot \alpha\rangle = \langle m, f \cdot (\alpha \cdot a_0)\rangle = m(f)\varphi \circ \phi(\alpha \cdot a_0) = m(f)\varphi(\alpha)\varphi \circ \phi(a_0) = n(f)\varphi(\alpha)$$

for all $a \in \mathcal{A}$. It follows that n is a module (ϕ, φ)-mean. □

Note that Theorem 2.1 does not tell us module (ϕ, φ)-amenability is equivalent to ϕ-amenability [11] because every character on $\mathcal{A}$ is not of the form $\varphi \circ \phi$ where $\phi \in \Omega_{\mathcal{A}}$ and $\varphi \in \Phi_{\mathfrak{A}}$.

We have the following analogue of a result in Gourdeau [8] on amenable Banach algebras. We bring the proof for the sake of completeness.

Theorem 2.2. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with compatible actions and $\phi \in \Omega_{\mathcal{A}}$, $\varphi \in \Phi_{\mathfrak{A}}$. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ is module (ϕ, φ)-amenable;*

(ii) *If X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module such that $a \cdot x = \phi(a) \cdot x$ and $\alpha \cdot x = x \cdot \alpha = \varphi(\alpha)x$ for all $x \in X, a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$, then any $\mathfrak{A}$-module derivation $D : \mathcal{A} \longrightarrow X^{**}$ is inner;*

(iii) *If X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module such that $a \cdot x = \phi(a) \cdot x$ and $\alpha \cdot x = x \cdot \alpha = \varphi(\alpha)x$ for all $x \in X, a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$, then any $\mathfrak{A}$-module derivation $D : \mathcal{A} \longrightarrow X$ is approximately inner; that is, there exists a bounded net (x_γ) in X such that $D(a) = \lim_\gamma (a \cdot x_\gamma - x_\gamma \cdot a)$ for all $a \in \mathcal{A}$.*

Proof. (i) $\Rightarrow$ (ii) It follows from the implicatiom (i) $\Rightarrow$ (ii) of [5, Theorem 2.1].

(ii) $\Rightarrow$ (iii) If $\iota : X \longrightarrow X^{**}$ is the canonical embeding, then $\iota \circ D$ is a module derivation from $\mathcal{A}$ into X^{**}. By assumption, there exists $\Lambda \in X^{**}$ with $(\iota \circ D)(a) = a \cdot \Lambda - \Lambda \cdot a$ for all $a \in \mathcal{A}$. Set $\sigma = \sigma(X^{**}, X^*)$, the weak* topology on X^{**}, $m = \|\Lambda\|$, and $U = \{x \in X : \lambda(x) \leq m\}$, where $\lambda \in X^*$. By [7, A.3.29 (i)], $\Lambda \in \overline{\iota(U)}^\sigma$. Now, fix $a_1, ..., a_n \in \mathcal{A}$. Then, the set $V = \prod_{j=1}^n (a_j \cdot U - U \cdot a_j)$ is a convex subset of X^n, and $(D(a_1), \cdots, D(a_n))$ belong to the weak closure of V. It follows from Mazur's theorem that $(D(a_1), \cdots, D(a_n))$ belongs to the norm closure of V. Thus, for each finite subset F of $\mathcal{A}$ and $\epsilon > 0$, there exists $x_{F,\epsilon} \in U$ such that

$$\|D(a) - (a \cdot x_{F,\epsilon} - x_{F,\epsilon \cdot a})\| < \epsilon \qquad (a \in F).$$

The family of such pairs (F, ϵ) is a directed for the partial order $\preceq$ given by $(F_1, \epsilon_1) \preceq (F_2, \epsilon_2)$ if $F_1 \subset F_2$ and $\epsilon_1 \geq \epsilon_2$. Obviously, $(x_{F,\epsilon})$ is the required net.

(iii) $\Rightarrow$ (i) Let $D : \mathcal{A} \longrightarrow X^*$ be a module derivation, and let (λ_α) be a bounded net in X^* such that $D(a) = \lim_\alpha (a \cdot \lambda_\alpha - \lambda_\alpha \cdot a)$ for all $\in \mathcal{A}$. By passing to a subnet, we suppose that $\lambda_\alpha \to \lambda$ in $(X^*, \sigma(X^*, X))$, and hence $D = D_\lambda$ is inner. Therefore, $\mathcal{A}$ is module (ϕ, φ)-amenable by [5, Theorem 2.1].

□

Let $\mathfrak{A}$ be a commutative Banach algebra. It is easy to see that each $\varphi \in \Phi_{\mathfrak{A}}$ induces a $\mathfrak{A}$-module structure on $\mathfrak{A}$ with actions $\alpha \cdot \beta = \beta \cdot \alpha = \varphi(\alpha)\beta$ for all $\alpha, \beta \in \mathfrak{A}$. Let $\phi \in \Omega_{\mathcal{A}}$. Then, we define an $\mathcal{A}$-module structure on $\mathfrak{A}$ by $a \cdot \alpha = \phi(a)\alpha$ and $\alpha \cdot a = \alpha\phi(a)$ for all $a \in \mathcal{A}$, and $\alpha \in \mathfrak{A}$. Then $\mathfrak{A}$ becomes a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module which is denoted by $\mathfrak{A}_{\phi,\varphi}$ and a bounded module derivation from $\mathcal{A}$ into $\mathfrak{A}_{\phi,\varphi}$ is called a *point module derivation* on $\mathcal{A}$ at (ϕ, φ). So, the following results follows immediately from Theorem 2.2 (see also [2] and [3]).

Corollary 2.1. *If $\mathfrak{A}$ is commutative and $\mathcal{A}$ is module (ϕ, φ)-amenable, then there is no non-zero bounded module point derivation on $\mathcal{A}$ at (ϕ, φ).*

Proposition 2.1. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with compatible actions and $\phi \in \Omega_{\mathcal{A}}$, $\varphi \in \Phi_{\mathfrak{A}}$. Then, $\mathcal{A}$ is module (ϕ, φ)-amenable if and only if there exists a bounded* *net (a_γ) in $\mathcal{A}$ such that $\varphi\circ\phi(a_\gamma)=1$ for all γ and*

$$\|aa_\gamma-\varphi\circ\phi(a)a_\gamma\|\to 0 \quad \textit{and} \quad \|\alpha\cdot a_\gamma-\varphi(\alpha)a_\gamma\|\to 0$$

for all $a\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$.

Proof. If m is a w^*-cluster point of (a_γ), then clearly m satisfies $m(\varphi\circ\phi)=1$, $m(f\cdot a)=\varphi\circ\phi(a)m(f)$ and $m(f\cdot\alpha)=\varphi(\alpha)m(f)$ for all $f\in\mathcal{A}^*$, $a\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$.

Conversely, let m be a module (ϕ,φ)-mean. Then, m is the w^*-limit of some net (b_γ) in $\mathcal{A}$ with $\|b_j\|\to\|m\|$. So, $\varphi\circ\phi(b_\gamma)\to m(\varphi\circ\phi)=1$, and w^*-continuity gives

$$ab_\gamma-\varphi\circ\phi(a)b_\gamma\to 0 \quad \text{and} \quad \alpha\cdot b_\gamma-\varphi(\alpha)b_\gamma\to 0$$

in the w^*-topology for all $a\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$. So, the nets $(ab_\gamma-\varphi\circ\phi(a)b_\gamma)$ and $(\alpha\cdot b_\gamma-\varphi(\alpha)b_\gamma)$ in $\mathcal{A}$, both converge to 0 weakly for all $a\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$. Now, take any finite subsets $F=\{a_1,...,a_k\}$ and $H=\{\alpha_1,...,\alpha_\ell\}$ of $\mathcal{A}$ and $\mathfrak{A}$, respectively. Let

$$C=\{((a_ib-\varphi\circ\phi(a_i)b)_{i=1}^k,(\alpha_j\cdot b-\varphi(\alpha_j)b)_{j=1}^\ell,\varphi\circ\phi(b)-1):b\in\mathcal{A}\}.$$

Then, in the Banach space $\mathcal{A}^{k+\ell}\times C$, 0 is in the weak closure of C and hence in the norm closure because C is convex. Thus, given $\varepsilon>0$, we can find $b_{F,H,\varepsilon}\in\mathcal{A}$ such that $\|b_{F,H,\varepsilon}\|\le 2\|m\|$, say, $|\varphi\circ\phi(b_{F,H,\varepsilon})-1|<\varepsilon$. Moreover, for each $a\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$ we have

$$\|ab_{F,H,\varepsilon}-\varphi\circ\phi(a)b_{F,H,\varepsilon}\|<\varepsilon \quad \text{and} \quad \|\alpha\cdot b_{F,H,\varepsilon}-\varphi(\alpha)b_{F,H,\varepsilon}\|<\varepsilon.$$

Finally, replace $b_{F,H,\varepsilon}$ by a scalar multiple $a_{F,H,\varepsilon}=\lambda_{F,H,\varepsilon}b_{F,H,\varepsilon}$ for which $\varphi\circ\phi(a_{F,H,\varepsilon})=1$. Hence, $|\lambda_{F,H,\varepsilon}|<\frac{1}{1-\varepsilon}$ and

$$\|aa_{F,H,\varepsilon}-\varphi\circ\phi(a)a_{F,H,\varepsilon}\|<\frac{\varepsilon}{1-\varepsilon} \quad \text{and} \quad \|\alpha\cdot a_{F,H,\varepsilon}-\varphi(\alpha)a_{F,H,\varepsilon}\|<\frac{\varepsilon}{1-\varepsilon}.$$

Therefore, the net $(a_{F,H,\varepsilon})$ is a bounded approximate module (ϕ,φ)-mean and m is the w^*-limit of $(a_{F,H,\varepsilon})$. □

Lemma 2.1. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with compatible actions, let I be a closed left ideal and $\mathfrak{A}$-submodule of $\mathcal{A}$ and let $\phi\in\Omega_\mathcal{A}$, $\varphi\in\Phi_\mathfrak{A}$ such that $I\nsubseteq\ker(\varphi\circ\phi)$. Then, the module $(\phi|_I,\varphi)$-amenability of I implies the module (ϕ,φ)-amenability of $\mathcal{A}$.*

Proof. Since I is module $(\phi|_I,\varphi)$-amenable, there is a net $(a_\gamma)\subseteq I$ with $\phi|_I(a_\gamma)=1$, $\|ba_\gamma-\phi|_I(b)a_\gamma\|\to 0$ and $\|\alpha\cdot a_\gamma-\varphi|_I(\alpha)a_\gamma\|\to 0$ for all $b\in I$ and $\alpha\in\mathfrak{A}$. Fix $\iota_0\in I$ such that $\varphi\circ\phi|_I(\iota_0)=1$ and set $\iota_\gamma:=\iota_0a_\gamma$ for all γ. So, $\varphi\circ\phi(\iota_\gamma)=\phi|_I(a_\gamma)=1$ and for each $a\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$ we have

$$\begin{aligned}\|a\iota_\gamma-\varphi\circ\phi(a)\iota_\gamma\| &= \|a\iota_0a_\gamma-\varphi\circ\phi(a)\iota_0a_\gamma\|\\ &\le \|a\iota_0a_\gamma-\varphi\circ\phi(a)\varphi\circ\phi|_I(\iota_0)a_\gamma\|\\ &+ \|\varphi\circ\phi(a)\phi|_I(\iota_0)a_\gamma-\varphi\circ\phi(a)\iota_0a_\gamma\|\\ &= \|a\iota_0a_\gamma-\varphi\circ\phi|_I(a\iota_0)a_\gamma\|\\ &+ |\varphi\circ\phi(a)|\|\varphi\circ\phi|_I(\iota_0)a_\gamma-\iota_0a_\gamma\|\to 0,\end{aligned}$$

and

$$\begin{aligned}\|\alpha\cdot\iota_\gamma-\varphi(\alpha)\iota_\gamma\| &= \|\alpha\cdot(\iota_0 a_\gamma)-\varphi(\alpha)\iota_0 a_\gamma\|\\ &\le \|(\alpha\cdot\iota_0)a_\gamma-\varphi(\alpha)\varphi\circ\phi|_I(\iota_0)a_\gamma\|\\ &+\|\varphi(\alpha)\varphi\circ\phi|_I(\iota_0)a_\gamma-\varphi(\alpha)\iota_0 a_\gamma\|\\ &= \|(\alpha\cdot\iota_0)a_\gamma-\varphi\circ\phi|_I(\alpha\cdot\iota_0)a_\gamma\|\\ &+|\varphi(\alpha)|\|\varphi\circ\phi|_I(\iota_0)a_\gamma-\iota_0 a_\gamma\|\to 0.\end{aligned}$$

Thus, $\mathcal{A}$ is module (ϕ,φ)-amenable. □

The next result which follows from Lemma 2.1 and [5, Lemma 2.6], describes the interaction between character module-amenability of a Banach algebra and its closed ideals.

Proposition 2.2. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with compatible actions and let I be a closed left ideal which is a $\mathfrak{A}$-submodule of $\mathcal{A}$. If $\phi\in\Omega_{\mathcal{A}}$, $\varphi\in\Phi_{\mathfrak{A}}$ such that $I\nsubseteq\ker(\varphi\circ\phi)$, then the following statements are equivalent:*

(i) *I is module $(\phi|_I,\varphi)$-amenable;*
(ii) *$\mathcal{A}$ is module (ϕ,φ)-amenable.*

Proposition 2.3. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule with compatible actions and $\varphi\in\Phi_{\mathfrak{A}},\phi\in\Omega_{\mathcal{A}}$. Suppose that for each $f\in\mathcal{A}^{**}$ there exists $m_f\in\mathcal{A}^{**}$ such that $\|m_f\|=\langle m_f,\varphi\circ\phi\rangle=1$ and*

$$\langle m_f,f\cdot a\rangle=\varphi\circ\phi(a)\langle m_f,f\rangle,\ \ \langle m_f,f\cdot\alpha\rangle=\varphi(\alpha)\langle m_f,f\rangle$$

for all $a\in\mathcal{A},\alpha\in\mathfrak{A}$. Then, $\mathcal{A}$ has a module (ϕ,φ)-mean of norm 1.

Proof. Define

$$S=\{m\in\mathcal{A}^{**}:\|m\|=\langle m,\varphi\circ\phi\rangle=1\}=\{m\in\mathcal{A}^{**}:\|m\|\le 1;\langle m,\varphi\circ\phi\rangle=1\}.$$

It is easy to check that S is a semigroup with the first Arens product and w^*-compact subset of A^{**}. Let $\mathfrak{F}$ denote the collection of all finite subset F of $\mathcal{A}^*$. For every $F\in\mathfrak{F}$, we put

$$\begin{aligned}S_F=\{m\in S:&\langle m,f\cdot a\rangle=\varphi\circ\phi(a)\langle m,f\rangle,\\ &\langle m,f\cdot\alpha\rangle=\varphi(\alpha)\langle m,f\rangle,a\in\mathcal{A},\alpha\in\mathfrak{A},f\in F\}.\end{aligned}$$

Then, S_F is closed in S and $S_{F_1}\supseteq S_{F_2}$ whenever $F_1\subseteq F_2$. It is obvious that each $m\in\bigcap\{S_F: f\in\mathfrak{F}\}$ is a module (ϕ,φ)-mean with $\|m\|=1$. Now, if we show that $S_F\neq\emptyset$ for all $F\in\mathfrak{F}$, then $m\in\bigcap_{F\in\mathfrak{F}}S_F$ is the required module mean by finite intersection property. We argue this by induction on number of elements in F. Suppose that some $m_1\in S_F$ exists and consider $g\in\mathcal{A}^*\diagdown F$. Set $h=m_1\cdot g\in\mathcal{A}^*$. By assumption, there exists $m_2\in S_{\{h\}}$ such that $m=m_2m_1\in S$ (since S is a semigroup). For each $f\in F$ and $a,b\in A$, we have

$$\langle m_1\cdot(f\cdot a),b\rangle=\langle m_1,f\cdot(ab)\rangle=\varphi\circ\phi(a)\langle m_1,f\rangle\varphi\circ\phi(b).$$

This shows that $m_1\cdot(f\cdot a)=(\varphi\circ\phi)(a)\langle m_1,f\rangle\varphi\circ\phi$. Similarly, $m_1\cdot f=\langle m_1,f\rangle\varphi\circ\phi$. So

$$\begin{aligned}\langle m,f\cdot a\rangle=\langle m_2m_1,f\cdot a\rangle&=\langle m_2,m_1\cdot(f\cdot a)\rangle=\varphi\circ\phi(a)\langle m_1,f\rangle\langle m_2,\varphi\circ\phi\rangle\\ &=\varphi\circ\phi(a)\langle m_2,\langle m_1,f\rangle\varphi\circ\phi\rangle=\varphi\circ\phi(a)\langle m_2,m_1\cdot f\rangle\\ &=(\varphi\circ\phi)(a)\langle m_2m_1,f\rangle=\varphi\circ\phi(a)\langle m,f\rangle,\end{aligned}$$

for all $f \in F$ and all $a \in \mathcal{A}$. Moreover,

$$\begin{aligned}\langle m, g\cdot a\rangle &= \langle m_2, (m_1\cdot g)\cdot a\rangle = \langle m_2, h\cdot a\rangle = \varphi\circ\phi(a)\langle m_2, h\rangle\\ &= \varphi\circ\phi(a)\langle m_2, m_1.g\rangle = \varphi\circ\phi(a)\langle m, g\rangle\end{aligned}$$

and similarly $\langle m, g\cdot\alpha\rangle = \varphi(\alpha)\langle m, g\rangle$ for all $a\in\mathcal{A}, \alpha\in\mathfrak{A}$. Also $\|m\| = \|m_2m_1\| = \|m_2\|\|m_1\| = 1$. Therefore, $m\in S_{F\bigcup\{g\}}$. This completes the proof. □

The upcoming theorem is the main result of the paper which shows that the existence of module (ϕ,φ)- mean with norm 1 is a pointwise property. In other words, it follows from the existence of an element of $\mathcal{A}^{**}$ associated with each of the elements of the ideal $\ker(\varphi\circ\phi)$.

Recall that a left Banach $\mathcal{A}$-module X is called *left essential* if the linear span of $\mathcal{A}\cdot X = \{a\cdot x : a\in\mathcal{A},\, x\in X\}$ is dense in X.

Theorem 2.3. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with compatible actions and $\varphi\in\Phi_{\mathfrak{A}}, \phi\in\Omega_{\mathcal{A}}$. Consider the following conditions.*

- (i) *There exists a module (ϕ,φ)- mean m such that $\|m\| = 1$;*
- (ii) *There exists a net $(u_j)_j$ in $\mathcal{A}$ such that $\varphi\circ\phi(u_j) = 1$, for all j, $\|u_j\|\to 1$ and $\|au_j\|\to|(\varphi\circ\phi)(a)|$, $\|\alpha\cdot u_j\|\to|\varphi(\alpha)|$ for all $a\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$;*
- (iii) *For each $a\in\ker(\varphi\circ\phi)$ and $b\in\ker(\phi)$, there exists $m_{a,b}\in\mathcal{A}^{**}$ with $\|m_{a,b}\|\le 1, \langle m_{a,b},\varphi\circ\phi\rangle = 1$ and $am_{a,b} = bm_{a,b} = 0, \alpha\cdot m_{a,b} = \varphi(\alpha)m_{a,b}$ for all $\alpha\in\mathfrak{A}$;*
- (iv) *For each $a\in\ker(\varphi\circ\phi), b\in\ker(\phi)$ and $\epsilon>0$, there exists $u\in\mathcal{A}$ such that $\|u\|\le 1+\epsilon, \|au\|\le\epsilon, \|bu\|\le\epsilon, \|\alpha\cdot u-\varphi(\alpha)u\|\le\epsilon$ and $\varphi\circ\phi(u) = 1$ for all $a\in\mathcal{A},\alpha\in\mathfrak{A}$.*

Then (iv)⇐(iii)⇐(i)⇒(ii)⇒(iv). *If, in addition, $\mathcal{A}$ is a left or right essential $\mathfrak{A}$-module, then all assertions are equivalent.*

Proof. (i)⇒(ii) Let there exists a module (ϕ,φ)-mean m such that $\|m\| = 1$. Then, by Proposition 2.1 there exists a net $(u_j)_j$ in $\mathcal{A}$ with the following properties:

$$\|u_j\|\to\|m\| = 1,\ \|au_j-(\varphi\circ\phi)(a)u_j\|\to 0,\ \|\alpha\cdot u_j-\varphi(\alpha)u_j\|\to 0$$

for all $a\in\mathcal{A},\alpha\in\mathfrak{A}$. Thus

$$\begin{aligned}|\|au_j\|-|(\varphi\circ\phi)(a)|| &\le |\|au_j\|-\|(\varphi\circ\phi)(a)u_j\||\\ &\quad+|\|(\varphi\circ\phi)(a)u_j\|-\|(\varphi\circ\phi)(a)\||\\ &\le\|au_j-(\varphi\circ\phi)(a)u_j\|+|(\varphi\circ\phi)(a)||\|u_j\|-1|\\ &\to 0\end{aligned}$$

and

$$\begin{aligned}|\|\alpha u_j\|-|\varphi(\alpha)|| &\le |\|\alpha u_j\|-\|\varphi(\alpha)u_j\||+|\|\varphi(\alpha)u_j\|-|\varphi(\alpha)||\\ &\le\|\alpha u_j-\varphi(\alpha)u_j\|+|\varphi(\alpha)|(\|u_j\|-1)\\ &\to 0\end{aligned}$$

Therefore, $\|au_j\|\to|(\varphi\circ\phi)(a)|$ and $\|\alpha u_j\|\to|\varphi(\alpha)|$ so (ii) holds.

(i)⇒(iii) If m is module (ϕ,φ)-mean, we can choose $m_{a,b} = m$, for all $a\in\ker(\varphi\circ\phi), b\in\ker(\phi)$, and thus $\|m_{a,b}\|\le 1, \langle m_{a,b},\varphi\circ\phi\rangle = \langle m,\varphi\circ\phi\rangle = 1$. On the other hand, for all $f\in\mathcal{A}^*$, we get

$$\langle am_{a,b}, f\rangle = \langle m_{a,b}, f\cdot a\rangle = \langle m, f\cdot a\rangle = (\varphi\circ\phi)(a)\langle m, f\rangle = 0$$

and

$$\langle bm_{a,b}, f\rangle = \langle m, f \cdot b\rangle = \varphi \circ \phi(b)\langle m_{a,b}, f\rangle = \varphi(0)\langle m, f\rangle = 0.$$

Also, $\langle \alpha \cdot m_{a,b}, f\rangle = \langle m, f \cdot \alpha\rangle = \varphi(\alpha)\langle m_{a,b}, f\rangle$ for all $\alpha \in \mathfrak{A}$. The above relations imply that $am_{a,b} = bm_{a,b} = 0$ and $\alpha\cdot m_{a,b} = \varphi(\alpha)m_{a,b}$ for all $a \in$ker$(\varphi\circ\phi), b \in \ker(\phi)$.

(ii)⇒(iv) It is obvious.

(iii)⇒(iv) Fix $a \in$ker$(\varphi \circ \phi), b \in \ker(\phi)$ and take any net $(u_j)_j$ in $\mathcal{A}$ such that $\|u_j\| \leq 1$ in which $u_j \to m_{a,b}$ in w^*-topology. Then $(\varphi \circ \phi)(u_j) \to 1$. Replacing each u_j with the scaler multiple of itself and taking a coefficient subnet, we may arrange that $\|u_j\| \leq 1 + \epsilon$ and $(\varphi \circ \phi)(u_j) = 1$ for all j. We have

$$w^* - \lim_j au_j = am_{a,b} = 0,\ w^* - \lim_j bu_j = m_{a,b} = 0,$$

and $w^* - \lim_j(\alpha \cdot u_j - \varphi(\alpha)u_j) = 0$ for $\alpha \in \mathfrak{A}$. Thus, 0 is in the weak closure of sets $(au_j)_j, (bu_j)_j$ and $(\alpha \cdot u_j - \varphi(\alpha)u_j)_j$. Hence, 0 is the norm closure of convex hull of the mentioned sets. Thus, the set $(u_j)_j$ beings contained in the closed hyperplane $\{x \in \mathcal{A}; (\varphi \circ \phi)(x) = 1\}$, we easily arrive our conclusion.

(iv)⇒(i) We claim that for finite subset F, H of $\mathcal{A}, \mathfrak{A}$ and $\epsilon > 0$, there exists $u_{F,H,\epsilon}$ such that $(\varphi \circ \phi)(u_{F,H,\epsilon}) = 1, \|u_{F,H,\epsilon}\| \leq 1 + \epsilon$ and for all $a \in F, \alpha \in H$

$$\|a \cdot u_{F,H,\epsilon} - (\varphi \circ \phi)(a)u_{F,H,\epsilon}\| \leq \epsilon,\ \ \|\alpha \cdot u_{F,H,\epsilon} - \varphi(\alpha)u_{F,H,\epsilon}\| \leq \epsilon$$

Let $F = \{a_1, ..., a_k\}, H = \{\alpha_1, \alpha_2, ..., \alpha_k\}$, and choose $\delta > 0$ such that $(1 + \delta)^{k+1} \leq 1+\epsilon$, by hypothesis , there exists $u_0 \in \mathcal{A}$ such that $(\varphi\circ\phi)(u_0) = 1$ and $||u_0|| \leq 1+\delta$. Since $\mathcal{A}$ is a left or right essential $\mathfrak{A}$-module, it follows from the proof of [6, Theorem 3.14] that the map ϕ is $\mathbb{C}$-linear. Thus, $\alpha_1u_0 - \varphi(\alpha_1)u_0 \in \ker(\phi)$. On the other hand, $a_1u_0 - (\varphi \circ \phi)(a_1)u_0 \in$ker$(\varphi \circ \phi)$. Again by (iv) there exists $u_1 \in \mathcal{A}$ such that $(\varphi \circ \phi)(u_1) = 1, \|u_1\| \leq 1 + \delta$ and

$$\|(a_1u_0 - (\varphi \circ \phi)(a_1)u_0)u_1\| \leq \delta,\ \ \|(\alpha_1u_0 - \varphi(\alpha_1)u_0)u_1\| \leq \delta.$$

Similarly, $a_2u_0u_1 - (\varphi \circ \phi)(a_2)u_0u_1 \in$ker$(\varphi \circ \phi), \alpha_2u_0u_1 - \varphi(\alpha_2)u_0u_1 \in \ker(\phi)$ and hence there exists $u_2 \in \mathcal{A}$ such that $(\varphi \circ \phi)(u_2) = 1, \|u_2\| \leq 1 + \delta$ and

$$\|(a_2u_0u_1 - (\varphi \circ \phi)(a_2)u_0u_1)u_2\| \leq \delta,\ \ \|(\alpha_2u_0u_1 - \varphi(\alpha_2)u_0u_1)u_2\| \leq \delta.$$

Thus for $j = 1, 2$ we have $\|u_j\| \leq 1 + \delta$, $(\varphi \circ \phi)(u_j) = 1$ and

$$\|a_ju_0u_1u_2 - (\varphi \circ \phi)(a_j)u_0u_1u_2\| \leq \delta(1 + \delta),\ \ \|\alpha_ju_0u_1u_2 - \varphi(\alpha_j)u_0u_1u_2\| \leq \delta(1 + \delta).$$

Proceeding inductively, we see there exits u_j $(1 \leq j \leq k)$ such that $(\varphi \circ \phi)(u_j) = 1, \|u_j\| \leq 1 + \delta$ and for $i = 1, ..., j$

$$\|a_iu_0u_1...u_j - (\varphi \circ \phi)(a_i)u_0u_1...u_j\| \leq \delta(1 + \delta)^{j-1} \leq \epsilon,$$

$$\|\alpha_i \cdot u_0u_1...u_j - \varphi(\alpha_i)u_0u_1...u_j\| \leq \delta(1 + \delta)^{j-1} \leq \epsilon.$$

In particular, when $j = k$, setting $u_{F,H,\epsilon} = \prod_{j=0}^k u_j$ gives us $(\varphi \circ \phi)(u_{F,H,\epsilon}) = \prod_{j=0}^k(\varphi \circ \phi)(u_j) = 1$ and

$$\|u_{F,H,\epsilon}\| \leq \|u_0\|\|u_1\|...\|u_k\| \leq (1 + \delta)^{k+1} \leq 1 + \epsilon.$$

Also, for each $a \in F, \alpha \in H$, we have

$$\|au_{F,H,\epsilon} - (\varphi \circ \phi)(a)u_{F,H,\epsilon}\| = \|au_0u_1...u_k - (\varphi \circ \phi)(a)u_0u_1...u_k\| \leq \delta(1 + \delta)^{k-1} \leq \epsilon,$$

and

$$\|\alpha \cdot u_{F,H,\epsilon} - \varphi(\alpha)u_{F,H,\epsilon}\| = \|\alpha u_0 u_1 ... u_k - \varphi(\alpha)u_0 u_1 ... u_k\| \leq \delta(1+\delta)^{k-1} \leq \epsilon.$$

This proves the above claim. Now, order the Triplet $(F, H, \epsilon), F \subseteq \mathcal{A}, H \subseteq \mathfrak{A}$ finite and $\epsilon > 0$, in the obvious manner manner, and let m be a w^*-cluster point of the net $(u_{F,H,\epsilon})_{F,H,\epsilon}$ in $\mathcal{A}^{**}$. Then, $\|m\| \leq 1$ and $\langle m, \varphi \circ \phi \rangle = 1$ and thus $\|m\| = 1$ and for all $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$, we get

$$\langle m, f \cdot a \rangle = \lim_{F,H,\epsilon} \langle u_{F,H,\epsilon}, f \cdot a \rangle = \lim_{F,H,\epsilon} \langle a \cdot u_{F,H,\epsilon}, f \rangle = (\varphi \circ \phi)(a) \langle m, f \rangle,$$

and similarly $\langle m, f \cdot \alpha \rangle = \varphi(\alpha)\langle m, f \rangle$. Therefore, m is required module (ϕ, φ)-mean. □

Remark 2.1. *Using similar methods to those employed in the proof of above Theorem, the following can be shown: Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with compatible actions and $\varphi \in \Phi_{\mathfrak{A}} \cup \{0\}, \phi \in \Omega_{\mathcal{A}}$. Then for the following conditions, we have the same implications as Theorem 2.3.*

(i) *$\mathcal{A}$ has a module (ϕ, φ)-mean of norm C;*
(ii) *$\mathcal{A}$ contain an approximate (ϕ, φ)-mean with norm bounded C;*
(iii) *For each $a \in ker(\varphi \circ \phi), b \in ker\phi$, there exists $m_{a,b} \in \mathcal{A}^{**}$ with $\|m_{a,b}\| = C, \langle m_{a,b}, \varphi \circ \phi \rangle = 1$ and $am_{a,b} = bm_{a,b} = 0$ and $\alpha \cdot m_{a,b} = \varphi(\alpha)m_{a,b}$;*
(iv) *There exists a net $(u_j)_j$ in $\mathcal{A}$ with $(\varphi \circ \phi)(u_j) = 1, \|u_j\| \to C$, for all j and $au_j \to 0$, for every $a \in ker(\varphi \circ \phi)$ and $\|\alpha \cdot u_j\| \to |\varphi(\alpha)|$ for every $\alpha \in \mathfrak{A}$.*

Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with compatible actions and $\varphi \in \Phi_{\mathfrak{A}} \cup \{0\}, \phi \in \Omega_{\mathcal{A}}$. Consider the set of all $f \in \mathcal{A}^*$ with the following property:
For each $\delta > 0$, there exists a sequence $(a_n)_n$ in $\mathcal{A}$ such that $(\varphi \circ \phi)(a_n) = 1$, $\|a_n\| \leq 1 + \delta$ for all n, and $\|f \cdot a_n\| \to 0$. We denote this set by $\mathcal{N}(\mathcal{A}, \varphi \circ \phi)$.

We have the following result which is analogous to Lemmas 2.6 and 2.7 from [12].

Lemma 2.2. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with compatible actions and $\varphi \in \Phi_{\mathfrak{A}} \cup \{0\}, \phi \in \Omega_{\mathcal{A}}$. Then, the following hold.*

(i) *$\varphi \circ \phi \notin \mathcal{N}(\mathcal{A}, \varphi \circ \phi)$.*
(ii) *$\mathcal{N}(\mathcal{A}, \varphi \circ \phi)$ is closed in $\mathcal{A}^*$ and closed under scaler multiplication.*
(iii) *If $\mathcal{A}$ is commutative, then $\mathcal{N}(\mathcal{A}, \varphi \circ \phi)$ is closed under addition.*
(iv) *If $\mathcal{A}$ admits a module (ϕ, φ)-mean of norm 1, then $\mathcal{N}(\mathcal{A}, \varphi \circ \phi)$ is subspace of $\mathcal{A}^*$.*

Proof. The proofs of [12, Lemma 2.6] and [12, Lemma 2.7] work verbatim if we put $\varphi \circ \phi$ instead of φ in their proofs. □

We now aim at a criterion for the existence of module $(\varphi \circ \phi)$-mean of norm 1 involving the set $\mathcal{N}(\mathcal{A}, \varphi \circ \phi)$.

Theorem 2.4. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with compatible actions and $\varphi \in \Phi_{\mathfrak{A}} \cup \{0\}, \phi \in \Omega_{\mathcal{A}}$. Then the following four condition are equivalent:*

(i) *There exists a module (ϕ, φ)-mean with $\|m\| = 1$;*
(ii) *$\mathcal{N}(\mathcal{A}, \varphi \circ \phi)$ is subspace of $\mathcal{A}^*$ and $f \cdot a - f, f \cdot \alpha - f \in \mathcal{N}(\mathcal{A}, \varphi \circ \phi)$ for all $f \in \mathcal{A}^*$ and all $a \in \mathcal{A}, \alpha \in \mathfrak{A}$ with $(\varphi \circ \phi)(a) = 1$.*

Proof. Let (i) holds. By Lemma 2.2. $\mathcal{N}(\mathcal{A}, \varphi \circ \phi)$ is a subspace of $\mathcal{A}^*$. Let $f \in \mathcal{A}^*$ and $a \in \mathcal{A}, \alpha \in \mathfrak{A}$ with $(\varphi \circ \phi)(a) = 1, \varphi(\alpha) = 1$. By Theorem 2.3 there exists a net $(u_j)_j$ in $\mathcal{A}$ such that $(\varphi \circ \phi)(u_j) = 1, \|u_j\| \to 1$ and

$$\|a \cdot u_j - (\varphi \circ \phi)(a)u_j\| = \|a \cdot u_j - u_j\| \to 0, \ \|\alpha \cdot u_j - \varphi(\alpha)u_j\| = \|\alpha \cdot u_j - u_j\| \to 0.$$

Since $\|(f \cdot a - f) \cdot u_j\| \leq \|f\|\|au_j - u_j\|$ and $\|(f \cdot \alpha - f) \cdot u_j\| \leq \|f\|\|\alpha \cdot u_j - u_j\|$, it follows that $f \cdot a - f, f \cdot \alpha - f \in \mathcal{N}(\mathcal{A}, \varphi \circ \phi)$.

Conversely, suppose that $\mathcal{N}(\mathcal{A}, \varphi \circ \phi)$ is subspace of $\mathcal{A}^*$ and that (ii) holds. Since $\varphi \circ \phi \notin \mathcal{N}(\mathcal{A}, \varphi \circ \phi)$ and $||\varphi \circ \phi|| = 1$, by the Hahn-Banach theorem there exists $m \in \mathcal{A}^{**}$ such that $\|m\| = \langle m, \varphi \circ \phi \rangle = 1$ and $m|_{\mathcal{N}(\mathcal{A}, \varphi \circ \phi)} = 0$. By assumption, for each $a \in \mathcal{A}, \alpha \in \mathfrak{A}$ with $(\varphi \circ \phi)(a) = 1$ and $\varphi(\alpha) = 1$, we have

$$\langle m, f \cdot a \rangle = \langle a \cdot m, f \rangle = (\varphi \circ \phi)(a)\langle m, f \rangle,$$
$$\langle m, f \cdot \alpha \rangle = \langle \alpha \cdot m, f \rangle = \varphi(\alpha)\langle m, f \rangle$$

for all $a \in \mathcal{A}, \alpha \in \mathfrak{A}$ and $f \in \mathcal{A}^*$. This means that m is a module (ϕ, φ)-mean. □

3. Conclusions

Let Banach algebras $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras. If $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ and $\phi : \mathcal{A} \longrightarrow \mathfrak{A}$ are the classical character and module character, respectively, we showed when $\mathcal{A}$ is module (ϕ, φ)-amenable. Moreover, we found the relations of module (ϕ, φ)-amenability of $\mathcal{A}$ and its ideals.

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REFERENCES

[1] *M. Amini*, Module amenability for semigroup algebras, Semigroup Forum. **69** (2004), 243–254.

[2] *M. Amini and D. Ebrahimi Bagha*, Weak module amenability of semigroup algebras, Semigroup Forum. **71** (2005), 18–26.

[3] *A. Bodaghi*, Semigroup algebras and their weak module amenability, J. Appl. Func. Anal. **7**, No. 4 (2012), 332–338.

[4] *A. Bodaghi*, Module (φ, ψ)-amenability of Banach algebras, Arch. Math. (Brno). **46**, No. 4 (2010), 227–235.

[5] *A. Bodaghi and M. Amini*, Module character amenability of Banach algebras, Arch. Math. (Basel). **99** (2012), 353–365.

[6] *A. Bodaghi, M. Amini and R. Babaee*, Module derivations into iterated duals of Banach algebras, Proc. Romanian. Acad., Series A. **12**, No. 4 (2011), 277–284.

[7] *H. G. Dales*, Banach algebras and automatic continuity, London Mathematical Society Monographs 24, Clarendon Press, Oxford, 2000.

[8] *F. Gourdeau*, Amenablility and the second dual of a Banach algebras, Studia Math. **125** (1) (1997), 75–80.

[9] *A. T. Lau*, Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups, Fund. Math. **118** (1983), 161-175.

[10] *B. E. Johnson*, Cohomology in Banach Algebras, Memoirs Amer. Math. Soc. **127**, Providence, 1972.

[11] *E. Kaniuth, A. T. Lau, and J. Pym*, On φ-amenability of Banach algebras, Math. Proc. Camb. Soc. **144** (2008), 85–96.

[12] *E. Kaniuth, A.T. Lau, J. Pym*, On Character Amenability of Banach Algebras, J. Math. Anal. Appl. **344** (2008), 942–955.

[13] *M. S. Monfared*, Character amenability of Banach algebras, Math. Proc. Camb. Soc. **144** (2008), 697–706.

[14] *H. Pourmahmood and A. Bodaghi*, Module approximate amenability of Banach algebras, Bull. Iran. Math. Soc. **39**, No. 6 (2013), 1137–1158.