A DRIVEN SIXTH-ORDER CAHN-HILLIARD EQUATION WITH CONCENTRATION DEPENDENT MOBILITY

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In this paper, we study a driven sixth-order Cahn-Hilliard equation, which as a continuum model for the formation of quantum dots and their faceting. Based on the Schauder type estimates and Campanato spaces, we prove the global existence of classical solutions.

Keywords: Sixth order Cahn-Hilliard equation, Campanato spaces, existence, concentration dependent mobility.

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1. Introduction

In this paper, we investigate the driven sixth-order Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} - D^2\left[m(u)(kD^4u - D^2A(u))\right] = \nu u Du, \text{ in } Q_T, \tag{1}$$

where $Q_T = (0,1)\times(0,T)$, $D = \frac{\partial}{\partial x}$ and $k > 0, \nu$ are constants. From the physical consideration, we prefer to consider a typical case of the potential $H(u)$, that is $H'(u) = A(u) = u^3 - u$, in the following form [7]

$$(H1) \qquad H(u) = \frac{1}{4}(u^2-1)^2,$$

namely, the well-known double well potential.

The equation (1) is supplemented by the boundary value conditions

$$u|_{x=0,1} = D^2u|_{x=0,1} = D^4u|_{x=0,1} = 0, \ t > 0, \tag{2}$$

and the initial value condition

$$u(x,0) = u_0(x). \tag{3}$$

The equation (1) arises naturally as a continuum model for the formation of quantum dots and their faceting, see [12]. Here $u(x,t)$ denotes the surface slope, and ν is proportional to the deposition rate. The high order derivatives are the result of the additional regularization energy which is required to form an edge between two plane surfaces with different orientations.

The sixth order parabolic equation with constant mobility has been intensively studied [11, 13]. Korzec, Evans, Münch and Wagner [7] studied the equation (1)

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with $m(u) = 1$. New types of stationary solutions of a one-dimensional driven sixth-order Cahn-Hilliard type equation (1) are derived by an extension of the method of matched asymptotic expansions that retains exponentially small terms. Liu and Liu [8] proved that the equation (1) with $m(u) = 1$ possesses a global attractor in the H^k ($k \geq 0$) space, which attracts any bounded subset of $H^k(\Omega)$ in the H^k-norm. Liu, Liu, Tang [9] based on Leray-Schauder fixed point theorem, proved the existence of time-periodic solutions for the equation (1) with $m(u) = 1$. However, only a few works have been devoted to the other sixth order parabolic equation with concentration dependent mobility [5]. Evans, Galaktionov and King [3, 4] considered the sixth-order thin film equation containing an unstable (backward parabolic) second-order term

$$\frac{\partial u}{\partial t} = \operatorname{div}\left[|u|^n \nabla \Delta^2 u\right] - \Delta(|u|^{p-1}u), n > 0, p > 1. \tag{4}$$

By a formal matched expansion technique, they show that, for the first critical exponent $p = p_0 = n + 1 + \frac{4}{N}$ for $n \in (0, \frac{5}{4})$, where N is the space dimension, the free-boundary problem with zero-height, zero-contact-angle, zero-moment, and zero-flux conditions at the interface admits a countable set of continuous branches of radially symmetric self-similar blow-up solutions $u_k(x,t) = (T-t)^{-\frac{N}{nN+6}} f_k(y)$, $y = \frac{x}{(T-t)^{\frac{1}{nN+6}}}$, where $T > 0$ is the blow-up time. Liu studied the equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\left[m(u)(k\nabla\Delta^2 u + \nabla(-a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + f(u)))\right] = 0,$$

and he proved the existence of classical solutions for two dimensions [10].

Our main purpose is to establish the global existence of classical solutions under much general assumptions. The main difficulties for treating the problem are caused by the nonlinearity of the principal part and the lack of maximum principle. The key step is to get a priori estimates on the Hölder norm of D^2u. The method used in [11] seems not applicable to the present situation. Our method is based on uniform Schauder type estimates for local in time solutions via the framework of Campanato spaces. To this purpose, we require some delicate local integral estimates rather than the global energy estimates used in the discussion for the Cahn-Hilliard equation with constant mobility.

Now, we state the main results in this paper.

Theorem 1.1. *Assume that*

$$(H2) \qquad m(s) \in C^{2+\alpha}(\mathbb{R}), \quad M_1 \leq m(s),$$

where M_1, α are positive constants, $u_0|_{\partial\Omega} = D^2u_0|_{\partial\Omega} = D^4u_0|_{\partial\Omega} = 0$. Then the problem (1)-(3) admits a unique classical solution $u \in C^{6+\alpha, 1+\alpha/6}(\overline{Q}_T)$ for any smooth initial data u_0, where $Q_T = \Omega \times (0,T)$.

This paper is organized as follows. We first present a key step for the priori estimates on the Hölder norm of solutions in Section 2, and then give the proof of our main theorem subsequently in Section 3.

2. **Hölder Estimates**

As an important step, in this section, we give the Hölder norm estimate on the local in time solutions. From the classical approach, it is not difficult to conclude that the problem admits a unique classical solution local in time. So, it is sufficient to make a priori estimates.

Proposition 2.1. *Assume that (H1) holds, and u is a smooth solution of the problem (1)-(3). Then there exists a constant C depending only on the known quantities, such that for any $(x_1,t_1),(x_2,t_2)\in Q_T$,*

$$|u(x_1,t_1)-u(x_2,t_2)|\le C(|t_1-t_2|^{1/8}+|x_1-x_2|^{3/4}), \tag{5}$$

$$|Du(x_1,t_1)-Du(x_2,t_2)|\le C(|t_1-t_2|^{1/12}+|x_1-x_2|^{1/2}). \tag{6}$$

Proof. Let $z=kD^2u-A(u)$. Multiplying both sides of the equation (1) by z and then integrating the resulting relation with respect to x over Ω, we have

$$\int_0^1 \frac{\partial u}{\partial t}(kD^2u-A(u))dx-\int_0^1 D^2(m(u)D^2z)zdx$$
$$-\int_0^1 \frac{\nu}{2}Du^2zdx=0.$$

After integrating by parts, and using the boundary value conditions,

$$\frac{d}{dt}\int_0^1\left(\frac{k}{2}(Du)^2+H(u)\right)dx+\int_0^1 m(u)|D^2z|^2dx$$
$$=\int_0^1\frac{\nu}{2}u^2Dzdx,$$

using Hölder's inequality, we have

$$\frac{d}{dt}\int_0^1\left(\frac{k}{2}(Du)^2+H(u)\right)dx+\int_0^1 m(u)|Dz|^2dx$$
$$\le\frac{\nu^2}{4m_1}\int_0^1 u^4dx+m_1\int_0^1(Dz)^2dx$$
$$\le C_1\int_0^1 H(u)dx+m_1\int_0^1(Dz)^2dx+C_2.$$

Applying Poincaré's inequality and Friedrichs' inequality [2], we conclude

$$\int_0^1|z|^2dx\le\frac{1}{\pi}\int_0^1|Dz|^2dx\le\frac{1}{2\pi}\int_0^1|D^2z|^2dx.$$

Owning to the above inequality, we finally arrive at

$$\frac{d}{dt}\int_0^1\left(\frac{k}{2}(Du)^2+H(u)\right)dx+\frac{1}{2}\int_0^1 m(u)|D^2z|^2dx$$
$$\le C_1\int_0^1 H(u)dx+C_2.$$

The Gronwall inequality implies that

$$\iint_{Q_T} m(u)(D^4u)^2dxdt \le C, \tag{7}$$

$$\int_0^1 |Du|^2dx \le C, \quad 0 \le t \le T, \tag{8}$$

$$\int_0^1 u^4dx \le C, \quad 0 \le t \le T. \tag{9}$$

By Sobolev imbedding theorem,

$$\sup_{Q_T} |u| \le C. \tag{10}$$

By (8), (9) we have

$$|u(x_1,t) - u(x_2,t)| \le C|x_1 - x_2|^\alpha, \;\; 0 < \alpha < \frac{1}{2}. \tag{11}$$

Multiplying both sides of the equation (1) by D^4u and then integrating the resulting relation with respect to x over $(0,1)$, after integrating by parts, and using the boundary value conditions, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 (D^2u)^2dx + \int_0^1 m(u)|D^5u|^2dx \\
=&\int_0^1 m(u)(3u^2-1)D^3uD^5udx + 18\int_0^1 m(u)uDuD^2uD^5udx \\
&+ 6\int_0^1 m(u)(Du)^3D^5udx - \int_0^1 m'(u)kD^4uDuD^5udx \\
&+ 6\int_0^1 m'(u)u(Du)^3D^5udx + \int_0^1 m'(u)(3u^2-1)DuD^2uD^5udx.
\end{aligned}$$

Using (10) and the Hölder inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 (D^2u)^2dx + \int_0^1 m(u)|D^5u|^2dx \\
\le&\frac{k}{2}\int_0^1 (D^5u)^2dx + C\int_0^1 (D^3u)^2dx + C\int_0^1 |Du|^4dx \\
&+ C\int_0^1 |D^2u|^4dx + C\int_0^1 (Du)^6dx + C\sup|Du|^2\int_0^1 (D^4u)^2dx + C.
\end{aligned}$$

By (8) and the Hölder inequality, we see that

$$\int_0^1 (D^3u)^2dx = \int_0^1 DuD^5udx \le C\|D^5u\|.$$

On the other hand, using the Gagliardo-Nirenberg inequality, we have

$$\|Du\|_\infty \le C\|D^5u\|^{\frac{1}{8}}\|Du\|^{\frac{7}{8}} \le C\|D^5u\|^{\frac{1}{8}},$$

$$\int_0^1 (Du)^4dx \le C\left(\int_0^1 (D^5u)^2dx\right)^{\frac{1}{8}}\left(\int_0^1 (Du)^2dx\right)^{\frac{15}{8}} \le C\left(\int_0^1 (D^5u)^2dx\right)^{\frac{1}{8}},$$

$$\int_0^1 (Du)^6dx \le C\left(\int_0^1 (D^5u)^2dx\right)^{\frac{1}{4}}\left(\int_0^1 (Du)^2dx\right)^{\frac{11}{4}} \le C\left(\int_0^1 (D^5u)^2dx\right)^{\frac{1}{4}},$$

$$\int_0^1 (D^2u)^4dx \le C\left(\int_0^1 (D^5u)^2dx\right)^{\frac{5}{8}}\left(\int_0^1 (Du)^2dx\right)^{\frac{11}{8}} \le C\left(\int_0^1 (D^5u)^2dx\right)^{\frac{5}{8}}.$$

Hence, we obtain

$$\frac{d}{dt}\int_0^1 (D^2u)^2dx + \int_0^1 m(u)|D^5u|^2dx \le C.$$

Therefore

$$\int_0^1 |D^2u|^2dx \le C, \quad 0 \le t \le T, \tag{12}$$

$$\iint_{Q_T} m(u)(D^5u)^2dxdt \le C. \tag{13}$$

By Sobolev imbedding theorem,

$$\sup_{Q_T}|Du| \le C. \tag{14}$$

(8) and (9) imply that

$$|u(x_1,t) - u(x_2,t)| \le C|x_1 - x_2|^{3/4}. \tag{15}$$

Integrating the equation (1) with respect to x over $(y, y+(\Delta t)^{1/6}) \times (t_1, t_2)$, where $0 < t_1 < t_2 < T$, $\Delta t = t_2 - t_1$, we see that

$$\begin{aligned}
&\int_y^{y+(\Delta t)^{1/6}} [u(z,t_2) - u(z,t_1)]dz \\
&= \int_{t_1}^{t_2} [(m(u)(kD^5u - D^3A(u))(y',s) + m'(u)Du(kD^4u - D^2A(u))(y',s)) \\
&- (m(u)(kD^5u - D^3A(u))(y,s) + m'(u)Du(kD^4u - D^2A(u))(y,s))]ds. \qquad (16)
\end{aligned}$$

Set

$$\begin{aligned}
N(s,y) =&(m(u)(kD^5u - D^3A(u))(y',s) + m'(u)Du(kD^4u - D^2A(u))(y',s)) \\
&- (m(u)(kD^5u - D^3A(u))(y,s) + m'(u)Du(kD^4u - D^2A(u))(y,s)),
\end{aligned}$$

where $y' = y + (\Delta t)^{1/6}$.

Then (16) is converted into

$$(\Delta t)^{1/6}\int_0^1[u(y+\theta(\Delta t)^{1/6},t_2)-u(y+\theta(\Delta t)^{1/6},t_1)]d\theta$$

$$=\int_{t_1}^{t_2}N(s,y)ds.$$

Integrating the above equality with respect to y over $(x,x+(\Delta t)^{1/4})$, we get

$$(\Delta t)^{1/3}(u(x^*,t_2)-u(x^*,t_1))=\int_{t_1}^{t_2}\int_x^{x+(\Delta t)^{1/6}}N(s,y)dyds.$$

Here, we have used the mean value theorem, where $x^*=y^*+\theta^*(\Delta t)^{1/6}$, $y^*\in(x,x+(\Delta t)^{1/6})$, $\theta\in(0,1)$. Hence by Hölder's inequality and (10), (7), (12), (13), (14), we get

$$\Big|u(x^*,t_2)-u(x^*,t_1)\Big|\le C(\Delta t)^{1/8}.$$

Similar to the discussion above, we have

$$|Du(x_1,t_1)-Du(x_2,t_2)|\le C(|x_1-x_2|^{1/2}+|t_1-t_2|^{1/12}).$$

The proof is complete. □

3. Proof of the Result

To prove the Theorem 1.1, the key estimate is the Hölder estimate for D^2u. We consider the following linear problem

$$\frac{\partial u}{\partial t}-D^4(a(x,t)D^2u)=D^4f+D^2g, \tag{17}$$

$$u\big|_{x=0,1}=D^2u\big|_{x=0,1}=D^4u\big|_{x=0,1}=0, \tag{18}$$

$$u(x,0)=0. \tag{19}$$

Here we do not restrict the smoothness of the given functions $a(x,t)$, $b(x,t)$ and $f(x,t)$, but simply assume that they are sufficiently smooth. Our main purpose is to find the relation between the Hölder norm of the solution u and $a(x,t)$, $b(x,t)$, $f(x,t)$.

The crucial step is to establish the estimates on the Hölder norm of u. Let $(x_0,t_0)\in(0,1)\times(0,T)$ be fixed and define

$$\varphi(\rho)=\iint_{S_\rho}\left(|u-u_\rho|^2+\rho^6|D^3u|^2\right)dxdt,\quad(\rho>0),$$

where

$$S_\rho=B_\rho(x_0)\times(t_0-\rho^6,t_0+\rho^6),\quad u_\rho=\frac{1}{|S_\rho|}\iint_{S_\rho}u\,dxdt$$

and $B_\rho(x_0)=(x_0-\rho,x_0+\rho)$.

Let u be the solution of the problem (17), (18), (19). We split u on S_R into $u=u_1+u_2$, where u_1 is the solution of the problem

$$\frac{\partial u_1}{\partial t}-a(x_0,t_0)D^6u_1=0,\quad(x,t)\in S_R, \tag{20}$$

$$u_1 = u, \quad Du_1 = Du, \quad D^2u_1 = D^2u, \qquad x \in \partial B_R(x_0), \tag{21}$$

$$u_1 = u, \quad t = t_0 - R^6, \;\; x \in B_R(x_0), \tag{22}$$

and u_2 solves the problem

$$\begin{aligned} \frac{\partial u_2}{\partial t} - a(x_0,t_0)D^6u_2 = &-D^4\Big[(a(x_0,t_0) - a(x,t))D^2u\Big] \\ &+ D^4f + D^2g, \quad (x,t) \in S_R, \end{aligned} \tag{23}$$

$$u_2 = 0, Du_2 = 0, D^2u_2 = 0, \quad (x,t) \in \partial B_R(x_0) \times (t_0 - R^6, t_0 + R^6), \tag{24}$$

$$u_2 = 0, \quad t = t_0 - R^6, \;\; x \in B_R(x_0). \tag{25}$$

By classical linear theory, the above decomposition is uniquely determined by u.

We need several lemmas on u_1 and u_2.

Lemma 3.1. *Assume that*

$$|a(x,t) - a(x_0,t_0)| \le a_\sigma\Big(|t - t_0|^{\sigma/6} + |x - x_0|^\sigma\Big),$$

$$|b(x,t) - b(x_0,t_0)| \le b_\sigma\Big(|t - t_0|^{\sigma/6} + |x - x_0|^\sigma\Big),$$

where $(x,t) \in B_R(x_0) \times (t_0 - R^6, t_0 + R^6)$. *Then*

$$\begin{aligned} &\sup_{(t_0-R^6,t_0+R^6)} \int_{B_R(x_0)} u_2^2(x,t)\,dx + \iint_{S_R} (D^3u_2)^2\,dxdt \\ \le &CR^{2\sigma}\iint_{S_R}(D^3u)^2\,dxdt + C\iint_{S_R}|Df|^2dxdt + C\iint_{S_R}|g|^2dxdt. \end{aligned}$$

Proof. Multiply the equation (23) by u_2 and integrate the resulting relation over $(t_0 - R^6, t) \times B_R(x_0)$. Integrating by parts, we have

$$\begin{aligned} &\frac{1}{2}\int_{B_R} u_2^2dx + a(x_0,t_0)\int_{t_0-R^6}^t ds\int_{B_R}(D^3u_2)^2dx \\ = &\int_{t_0-R^6}^t ds\int_{B_R}[a(x_0,t_0) - a(x,s)]D^3uD^3u_2dx \\ &- \int_{t_0-R^6}^t ds\int_{B_R} Da(x,t)D^2uD^3u_2dx \\ &+ \int_{t_0-R^6}^t ds\int_{B_R} DfD^3u_2dx + \int_{t_0-R^6}^t ds\int_{B_R} gD^2u_2dx. \end{aligned}$$

Noticing that

$$\begin{aligned} &\left|\int_{t_0-R^6}^t ds\int_{B_R}[a(x_0,t_0) - a(x,s)]D^3uD^3u_2dx\right| \\ \le &\varepsilon\iint_{S_R}(D^3u_2)^2dxds + C_\varepsilon a_\sigma^2R^{2\sigma}\iint_{S_R}(D^3u)^2dxds, \end{aligned}$$

$$\begin{aligned}&\left|\int_{t_0-R^6}^{t}ds\int_{B_R}Da(x,t)D^2uD^3u_2dx\right|\\ \leq&\varepsilon\iint_{S_R}(D^3u_2)^2dxds+CR^{2\sigma}\iint_{S_R}(D^2u)^2dxds\\ \leq&\varepsilon\iint_{S_R}(D^3u_2)^2dxds+C_\varepsilon a_\sigma^2R^{2\sigma}\iint_{S_R}(D^3u)^2dxds,\end{aligned}$$

$$\left|\int_{t_0-R^6}^{t}ds\int_{B_R}DfD^3u_2dx\right|\leq\varepsilon\iint_{S_R}(D^3u_2)^2dxds+C_\varepsilon\iint_{S_R}|Df|^2dxdt,$$

and

$$\left|\int_{t_0-R^6}^{t}ds\int_{B_R}gD^2u_2dx\right|\leq\varepsilon\iint_{S_R}(D^3u_2)^2dxds+C_\varepsilon\iint_{S_R}|g|^2dxdt,$$

hence we obtain the estimate and the proof is complete. □

Lemma 3.2. *For any* (x_1,t_1), $(x_2,t_2)\in S_\rho$,

$$\begin{aligned}&\frac{|u_1(t_1,x_1)-u_1(t_2,x_2)|^2}{|t_1-t_2|^{1/6}+|x_1-x_2|}\\ \leq&C\sup_{(t_0-\rho^6,t_0+\rho^6)}\int_{B_\rho(x_0)}(Du_1(x,t))^2\,dx+C\iint_{S_\rho}(D^4u_1)^2\,dxdt.\end{aligned}$$

Proof. From the Sobolev embedding theorem, we have for any $(x_1,t),(x_2,t)\in S_\rho$,

$$\frac{|u_1(x_1,t)-u_1(x_2,t)|^2}{|x_1-x_2|}\leq C\sup_{(t_0-\rho^6,t_0+\rho^6)}\int_{B_\rho(x_0)}(Du_1(x,t))^2\,dx. \tag{26}$$

Integrating the equation with respect to x over $(y,y+(\Delta t)^{\frac{1}{6}})\times(t_1,t_2)$, where $0<t_1<t_2<T$, $\Delta t=t_2-t_1$, we see that

$$\int_y^{y+(\Delta t)^{\frac{1}{6}}}[u_1(z,t_2)-u_1(z,t_1)]dz+a(x_0,t_0)\int_{t_1}^{t_2}[D^5u_1(y',s)-D^5u_1(y,s)]ds=0,$$

where $y'=y+(\Delta t)^{1/6}$.

That is

$$\begin{aligned}&(\Delta t)^{1/6}\int_0^1[u_1(y+\theta(\Delta t)^{1/6},t_2)-u_1(y+\theta(\Delta t)^{1/6},t_1)]d\theta\\ &+a(x_0,t_0)\int_{t_1}^{t_2}[D^5u_1(y+(\Delta t)^{1/6},s)-D^5u_1(y,s)]ds=0.\end{aligned}$$

Integrating the above equality with respect to y over $(x,x+(\Delta t)^{1/4})$, we get

$$\begin{aligned}&(\Delta t)^{1/3}(u_1(x^*,t_2)-u_1(x^*,t_1))\\ =&a(x_0,t_0)\int_{t_1}^{t_2}\int_x^{x+(\Delta t)^{1/6}}[D^5u_1(y+(\Delta t)^{1/6},s)-D^5u_1(y,s)]dyds.\end{aligned}$$

Hence,

$$\Big|u_1(x^*,t_2)-u_1(x^*,t_1)|\le C|t_1-t_2|^{1/6}\left[\iint_{S_\rho}(D^4u_1)^2\,dxds+\iint_{S_\rho}(D^2u_1)^2\,dxds\right],$$

where $x^*=y^*+\theta^*(\Delta t)^{1/6}$, $y^*\in(x,x+(\Delta t)^{1/6})$, $\theta\in(0,1)$. This and (26) yield the desired conclusion and the proof is complete. □

Lemma 3.3. *(Caccioppoli type inequality)*

$$\begin{aligned}&\sup_{(t_0-(R/2)^6,t_0+(R/2)^6)}\int_{B_{R/2}(x_0)}|u_1(x,t)-(u_1)_R|^2\,dx+\iint_{S_{R/2}}|D^3u_1|^2\,dxdt\\ \le&\frac{C}{R^6}\iint_{S_R}|u_1(x,t)-(u_1)_R|^2\,dxdt,\\ &\sup_{(t_0-(R/2)^6,t_0+(R/2)^6)}\int_{B_{R/2}(x_0)}|Du_1|^2\,dx+\iint_{S_{R/2}}|D^4u_1|^2\,dxdt\\ \le&\frac{C}{R^6}\iint_{S_R}|Du_1|^2\,dxdt\le\frac{C}{R^8}\iint_{S_{2R}}|u_1(x,t)-(u_1)_R|^2\,dxdt.\end{aligned}$$

Proof. For simplicity, we only prove the first inequality, since the other can be shown similarly. Choose a cut-off function $\chi(x)$ defined on (x_0-R,x_0+R) such that $\chi(x)=1$ in $(x_0-\frac{R}{2},x_0+\frac{R}{2})$ and

$$\begin{aligned}|D\chi|\le\frac{C}{R},\qquad\qquad |D^2\chi|\le\frac{C}{R^2},\\ |D^3\chi|\le\frac{C}{R^3},\qquad\qquad |D^4\chi|\le\frac{C}{R^4}.\end{aligned}$$

Let $g(t)\in C_0^\infty(t_0,+\infty)$ with $0\le g(t)\le 1$, $0\le g'(t)\le\frac{C}{R^6}$ and $g(t)=1$ for $t\ge t_0-(\frac{R}{2})^6$. Multiplying the equation (20) by $g(t)\chi^6[u_1(x,t)-(u_1)_R]$ and then integrating the resulting relation over $(t_0-R^6,t)\times(x_0-R,x_0+R)$, we have

$$\begin{aligned}&\int_{t_0-R^6}^t g(s)ds\int_{B_R(x_0)}\frac{\partial u_1}{\partial t}\chi^6[u_1(x,t)-(u_1)_R]dx+\\ &\quad-a(x_0,t_0)\int_{t_0-R^6}^t g(s)ds\int_{B_R(x_0)}D^6u_1\chi^6[u_1(x,t)-(u_1)_R]dx=0.\end{aligned}$$

It follows from integrating by parts and using the boundary value condition (21),

$$\begin{aligned}&\frac12\int_{B_R(x_0)}g(s)\chi^6|u_1(x,t)-(u_1)_R|^2\,dx\\ &+a(x_0,t_0)\int_{t_0-R^6}^t g(s)ds\int_{B_R(x_0)}D^5u_1D[\chi^6[u_1(x,t)-(u_1)_R]]\,dx\\ =&\frac12\int_{t_0-R^6}^t g'\,ds\int_{B_R(x_0)}\chi^6|u_1(x,t)-(u_1)_R|^2\,dx,\ t\in\left[t_0-\left(\frac{R}{2}\right)^6,t_0+\left(\frac{R}{2}\right)^6\right].\end{aligned}$$

Thus

$$
\begin{aligned}
&\frac{1}{2}\int_{B_R(x_0)} g(s)\chi^6|u_1(x,t)-(u_1)_R|^2\,dx\\
&+a(x_0,t_0)\int_{t_0-R^6}^t g(s)ds\int_{B_R(x_0)}\chi^6(D^3u_1)^2\,dx\\
&+a(x_0,t_0)\int_{t_0-R^6}^t g(s)ds\int_{B_R(x_0)}18\chi^5\chi' D^2u_1D^3u_1dx\\
&+a(x_0,t_0)\int_{t_0-R^6}^t g(s)ds\int_{B_R(x_0)}[(18\chi^5\chi''+90\chi^4\chi'^2)Du_1D^3u_1\\
&+(6\chi^5\chi'''+90\chi^4\chi'\chi''+120\chi^3\chi'^3)(u_1(x,t)-(u_1)_R)D^3u_1]\,dx\\
=&\frac{1}{2}\int_{t_0-R^6}^t g'\,ds\int_{B_R(x_0)}\chi^6|u_1-(u_1)_R|^2\,dx.
\end{aligned}
$$

By Cauchy inequality, we have

$$
\begin{aligned}
&\left|18\int_{t_0-R^6}^t\int_{B_R(x_0)} g(s)a(x_0,t_0)\chi^5\chi' D^2u_1D^3u_1dxds\right|\\
\le&\frac{1}{4}a(x_0,t_0)\int_{t_0-R^6}^t\int_{B_R(x_0)} g(s)\chi^6(D^3u_1)^2dxds\\
&\quad+C\int_{t_0-R^6}^t\int_{B_R(x_0)} g(s)\chi^4(\chi')^2(D^2u_1)^2dxds,
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int_{t_0-R^6}^t\int_{B_R(x_0)} g(s)a(x_0,t_0)(18\chi^5\chi''+90\chi^4\chi'^2)Du_1D^3u_1\,dxds\right|\\
\le&\frac{1}{4}a(x_0,t_0)\int_{t_0-R^6}^t\int_{B_R(x_0)} g(s)\chi^6(D^3u_1)^2dxds\\
&+C\int_{t_0-R^6}^t\int_{B_R(x_0)} g(s)\chi^4\chi''^2(Du_1)^2\,dxds\\
&+C\int_{t_0-R^6}^t\int_{B_R(x_0)} g(s)\chi^2\chi'^4(Du_1)^2\,dxds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_{t_0-R^6}^t\int_{B_R(x_0)} g(s)a(x_0,t_0)(6\chi^5\chi'''+90\chi^4\chi'\chi''+120\chi^3\chi'^3)\right.\\
&\quad\left.\cdot(u_1(x,t)-(u_1)_R)D^3u_1\,dxds\right|\\
\le&\frac{1}{4}a(x_0,t_0)\int_{t_0-R^6}^t\int_{B_R(x_0)} g(s)\chi^6(D^3u_1)^2dxds\\
&\quad+\frac{C}{R^6}\int_{t_0-R^6}^t\int_{B_R(x_0)}(u_1(x,t)-(u_1)_R)^2\,dxds.
\end{aligned}
$$

Combining the above these expressions yields

$$\begin{aligned}
&\int_{B_R(x_0)} g(s)\chi^6|u_1(x,t)-(u_1)_R|^2\,dx \\
&+\frac{1}{2}a(x_0,t_0)\int_{t_0-R^6}^t g(s)ds\int_{B_R(x_0)}\chi^6(D^3u_1)^2\,dx \\
\le&\int_{t_0-R^6}^t g'\,ds\int_{B_R(x_0)}\chi^6|u_1-(u_1)_R|^2\,dx \\
&+C\int_{t_0-R^6}^t\int_{B_R(x_0)}\chi^2\chi'^4(Du_1)^2\,dxds+C\int_{t_0-R^6}^t\int_{B_R(x_0)}\chi^4\chi''^2(Du_1)^2\,dxds \\
&+C\int_{t_0-R^6}^t\int_{B_R(x_0)}\chi^4(\chi')^2(D^2u_1)^2dxds \\
&+\frac{C}{R^6}\int_{t_0-R^6}^t\int_{B_R(x_0)}(u_1(x,t)-(u_1)_R)^2\,dxds \\
\equiv&\int_{t_0-R^6}^t g'\,ds\int_{B_R(x_0)}\chi^6|u_1-(u_1)_R|^2\,dx+C(I_1+I_2+I_3+I_4).
\end{aligned}$$

As for I_1, we get

$$\begin{aligned}
I_1=&-\int_{t_0-R^6}^t\int_{B_R(x_0)}u_1D(\chi^2\chi'^4Du_1)dxds \\
=&-\int_{t_0-R^6}^t\int_{B_R(x_0)}\chi^2\chi'^4u_1D^2u_1dxds-\int_{t_0-R^6}^t\int_{B_R(x_0)}(\chi^2\chi'^4)'u_1Du_1dxds \\
\le&\varepsilon_1I_3+C\int_{t_0-R^6}^t\int_{B_R(x_0)}\chi'^6u_1^2dxds+\frac{1}{2}\int_{t_0-R^6}^t\int_{B_R(x_0)}(\chi^2\chi'^4)''u_1^2dxds \\
\le&\varepsilon_1I_3+CI_4.
\end{aligned}\tag{27}$$

As for I_2, we have

$$\begin{aligned}
I_2=&-\int_{t_0-R^6}^t\int_{B_R(x_0)}u_1D(\chi^4\chi''^2Du_1)dxds \\
=&\int_{t_0-R^6}^t\int_{B_R(x_0)}\chi'D(\chi^4\chi''u_1D^2u_1)dxds+\frac{1}{2}\int_{t_0-R^6}^t\int_{B_R(x_0)}(\chi^4\chi''^2)''u_1^2dxds \\
\le&\int_{t_0-R^6}^t\int_{B_R(x_0)}\chi'(\chi^4\chi'')'u_1D^2u_1dxds \\
&+\int_{t_0-R^6}^t\int_{B_R(x_0)}\chi^4\chi'\chi''(Du_1D^2u_1+u_1D^3u_1)dxds+CI_4 \\
=&\varepsilon_2I_3+CI_4+\varepsilon_3\int_{t_0-R^6}^t\int_{B_R(x_0)}\chi^6(D^3u_1)^2dxds \\
&-\frac{1}{2}\int_{t_0-R^6}^t\int_{B_R(x_0)}(\chi^4\chi'\chi'')'(Du_1)^2dxds
\end{aligned}$$

$$
\begin{aligned}
=&\varepsilon_2 I_3 + CI_4 + \varepsilon_3 \int_{t_0-R^6}^{t}\int_{B_R(x_0)} \chi^6 (D^3u_1)^2 dxds - \frac{1}{2} I_2 \\
&- \frac{1}{2}\int_{t_0-R^6}^{t}\int_{B_R(x_0)} (\chi^4\chi'\chi''' + 4\chi^3\chi'^2\chi'')(Du_1)^2 dxds,
\end{aligned}
$$

that is,

$$
\begin{aligned}
I_2 \le&\varepsilon_2 I_3 + CI_4 + \varepsilon_3 \int_{t_0-R^6}^{t}\int_{B_R(x_0)} \chi^6 (D^3u_1)^2 dxds \\
&- \frac{1}{3}\int_{t_0-R^6}^{t}\int_{B_R(x_0)} (\chi^4\chi'\chi''' + 4\chi^3\chi'^2\chi'')(Du_1)^2 dxds.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&- \int_{t_0-R^6}^{t}\int_{B_R(x_0)} (\chi^4\chi'\chi''' + 4\chi^3\chi'^2\chi'')(Du_1)^2 dxds \\
=&\int_{t_0-R^6}^{t}\int_{B_R(x_0)} (\chi^4\chi'\chi''' + 4\chi^3\chi'^2\chi'')u_1 D^2u_1 dxds \\
&+ \int_{t_0-R^6}^{t}\int_{B_R(x_0)} ((\chi^4\chi'\chi''')' + 4(\chi^3\chi'^2\chi'')')u_1 Du_1 dxds \\
\le&\varepsilon I_3 + CI_4.
\end{aligned}
$$

Combining the above two yields

$$
I_2 \le \varepsilon_4 I_3 + CI_4 + \varepsilon_3 \int_{t_0-R^6}^{t}\int_{B_R(x_0)} \chi^6 (D^3u_1)^2 dxds. \tag{28}
$$

Noticing that

$$
\begin{aligned}
I_3 =& - \int_{t_0-R^6}^{t}\int_{B_R(x_0)} \chi^4\chi'^2 Du_1 D^3u_1 dxds \\
&- \int_{t_0-R^6}^{t}\int_{B_R(x_0)} (4\chi^3\chi'^3 + 2\chi^4\chi'\chi'') Du_1 D^2u_1 dxds \\
\le&\varepsilon_5 \int_{t_0-R^6}^{t}\int_{B_R(x_0)} \chi^6 (D^3u_1)^2 dxds + C(\varepsilon_5) I_1 + \frac{1}{4} I_3 + CI_1 + \frac{1}{4} I_3 + CI_2,
\end{aligned}
$$

that is,

$$
I_3 \le 2C(\varepsilon_5) I_1 + CI_2 + 2\varepsilon_5 \int_{t_0-R^6}^{t}\int_{B_R(x_0)} \chi^6 (D^3u_1)^2 dxds. \tag{29}
$$

Finally, from (27), (28) and (29), choose $\varepsilon_1, \varepsilon_3, \varepsilon_4$ enough small, we see that

$$
I_i \le \varepsilon \int_{t_0-R^6}^{t}\int_{B_R(x_0)} \chi^6 (D^3u_1)^2 dxds + CI_4, \qquad i = 1, 2, 3.
$$

We obtain immediately the desired first inequality of the lemma and the proof is completed. □

Lemma 3.4. *Assume that*

$$|a(x,t)-a(x_0,t_0)| \le a_\sigma\Big(|t-t_0|^{\sigma/6}+|x-x_0|^{\sigma}\Big),$$
$$t\in(t_0-R^6,t_0+R^6),\quad x\in B_R(x_0).$$

Then for any $\rho\in(0,R)$,

$$\begin{aligned}&\frac{1}{\rho^8}\iint_{S_\rho}(|u_1-(u_1)_\rho|^2+\rho^6|D^3u_1|^2)\,dxdt\\ \le&\frac{C}{R^8}\iint_{S_R}(|Du_1-(Du_1)_R|^2+R^6|D^3u_1|^2)\,dxdt.\end{aligned}$$

Proof. One needs only to check the inequality for $\rho\le\frac{R}{2}$. From Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned}&\frac{1}{\rho^8}\iint_{S_\rho}|u_1-(u_1)_\rho|^2\,dxdt\\ \le&C\sup_{(t_0-(\frac{R}{2})^4,t_0+(\frac{R}{2})^4)}\int_{B_{\frac{R}{2}}(x_0)}|Du_1|^2\,dx+C\iint_{S_{\frac{R}{2}}}|D^4u_1|^2\,dxdt\\ \le&\frac{C}{R^8}\iint_{S_R}|u_1-(u_1)_R|^2\,dxdt.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\iint_{S_\rho}\rho^6|D^3u_1|^2\,dxdt\\ \le&C_1\iint_{S_\rho}\rho^8(D^4u_1)^2dxdt+C_2\iint_{S_\rho}\rho^2(Du_1)^2dxdt\\ \le&C_1\rho^8\iint_{S_{\frac{R}{2}}}(D^4u_1)^2dxdt+C_2\rho^8\sup_{(t_0-(\frac{R}{2})^6,t_0+(\frac{R}{2})^6)}\int_{B_{\frac{R}{2}}(x_0)}(Du_1)^2dx\\ \le&C\left(\frac{\rho}{R}\right)^8\iint_{S_{\frac{R}{2}}}R^2(Du_1)^2dxdt\\ \le&C\left(\frac{\rho}{R}\right)^8\left[\iint_{S_R}R^6(D^3u_1)^2dxdt+\iint_{S_R}(Du_1-(Du_1)_R)^2dxdt\right].\end{aligned}$$

The conclusion of the lemma follows at once. □

Lemma 3.5. *For* $\lambda\in(5,6)$,

$$\varphi(\rho)\le C_\lambda\left(\varphi(R_0)+\sup_{S_{R_0}}|f|^2\right)\rho^\lambda,\qquad \rho\le R_0=min\Big(dist(x_0,\partial\Omega),t_0^{1/4}\Big),$$

where C_λ *depends on* λ, R_0 *and the known quantities.*

Proof. By Lemma 3.4,

$$\varphi(\rho)=\iint_{S_\rho}(|u-(u)_\rho|^2+\rho^6|D^3u|^2)\,dxdt$$

$$
\begin{aligned}
&= \iint_{S_\rho} (|u_1 - (u_1)_\rho|^2 + \rho^6 |D^3 u_1|^2)\, dxdt \\
&\quad + \iint_{S_\rho} (|u_2 - (u_2)_\rho|^2 + \rho^6 |D^3 u_2|^2)\, dxdt \\
&\le C \left(\frac{\rho}{R}\right)^8 \iint_{S_R} (|u - (u)_R|^2 + R^6 |D^3 u|^2)\, dxdt \\
&\quad + C \iint_{S_R} (|u_2|^2 + R^6 |D^3 u_2|^2)\, dxdt \\
&\le C \left[\left(\frac{\rho}{R}\right)^8 + R^{2\sigma}\right] \varphi(R) + C \left(\iint_{S_R} |Df|^2 dxdt + \iint_{S_R} |g|^2 dxdt\right) R^6.
\end{aligned}
$$

The conclusion follows immediately from [6]. □

Similar to the discussion about the Campanato spaces in [6], we first conclude from Lemma 3.5 that

Theorem 3.1. *Let $f(x,t)$ be appropriately smooth function, and u be the smooth solution of the problem (17)-(19). Then for any $\alpha \in (0, \frac{1}{2})$, there exists a coefficient K depending only on α, a_σ, b_σ, $\iint_{Q_T} u^2 dxdt$ and $\iint_{Q_T} (D^3 u)^2 dxdt$, such that*

$$
\begin{aligned}
&|u(x_1, t_1) - u(x_2, t_2)| \\
&\le K \left(\iint_{S_R} |Df|^2 dxdt + \iint_{S_R} |g|^2 dxdt\right) (|x_1 - x_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{6}}).
\end{aligned} \tag{30}
$$

Proof of Theorem 1.1. Let $w = D^2 u - D^2 u_0$. w satisfies

$$
\begin{cases}
\dfrac{\partial w}{\partial t} - D^4(a(x,t) D^2 w) = D^4 f + D^2 g, \\
w|_{x=0,1} = D^2 w|_{x=0,1} = D^4 w|_{x=0,1} = 0, \\
u(x,0) = 0,
\end{cases}
$$

where $a(x,t) = km(u)$, $f = m(u)(kD^4 u_0 - A'(u)(D^2 u - D^2 u_0) - A''(u)|Du|^2)$ and $g(x,t) = \nu u Du$. Hence, using (11)-(13) and Theorem 3.1, we conclude that

$$
|D^2 u(x_1, t_1) - D^2 u(x_2, t_2)| \le C(|x_1 - x_2|^{\alpha/2} + |t_1 - t_2|^{\alpha/12}). \tag{31}
$$

The conclusion follows immediately from the classical theory, since we can transform the equation (1) into the form

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + a_1(x,t) D^6 u + b_1(x,t) D^5 u + a_2(x,t) D^4 u + b_2(x,t) D^3 u \\
&\qquad + a_3(x,t) D^2 u + b_3(x,t) Du = 0,
\end{aligned}
$$

where the Hölder norms on

$$a_1(x,t) = -km(u)(x,t), \quad b_1(x,t) = -2km'(u(x,t))Du(x,t),$$
$$a_2(x,t) = m(u(x,t))A'(u(x,t)) - km(u)D^2u - km'(u)(Du)^2,$$
$$b_2(x,t) = 4m(u)A''(u)Du + 2m'(u)A'(u)Du,$$
$$a_3(x,t) = m(u)(3A''(u)D^2u + 6A'''(u)(Du)^2)$$
$$+ m'(u)(A'(u)D^2u + 7A''(u)(Du)^2) + m''(u)A'(u)(Du)^2,$$
$$b_3(x,t) = m''(u)A''(u)(Du)^3 - \nu u$$

have been estimated in the above discussion. The proof is complete. □

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