SOME RESULTS ON RICCI-HARMONIC BOURGUIGNON SOLITONS AND APPLICATIONS

Sakineh Hajiaghasi[1], Shahroud Azami[2], Jafar Ebrahimi[3]

We study some results for the almost Ricci-harmonic Bourguignon soliton which is a generalization of the Ricci-harmonic soliton. Next, we find some integral formulas for the compact almost gradient Ricci-harmonic Bourguignon soliton. As an important application, we prove that every almost gradient Ricci-harmonic Bourguignon soliton with dimension $n \geq 3$ is trivially rigid under certain condition.

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1. Introduction

Consider two complete Riemannian n-dimensional manifolds (M_1, g_1), and (M_2, g_2) with $\varphi : M_1 \longrightarrow M_2$, such that φ is critical point of the energy integral $E(\varphi) = \int_M |\nabla\varphi|_g^2 dv$, and M_2 is isometrically embedded in $\mathbb{R}^d$, $d \geq n$. By an one parameter family of Riemannian metrics $(g(x,t), \varphi(x,t))$, $t \in [0,T)$, and a family of smooth maps $\varphi(x,t)$, the Ricci-harmonic Bourguignon flow is defined by

$$\begin{cases} \dfrac{\partial}{\partial t} g(x,t) = -2\mathrm{Ric}(x,t) + 2\rho R(x,t) + 2\alpha\nabla\varphi(x,t) \otimes \nabla\varphi(x,t), g(0) = g_0, \\ \dfrac{\partial}{\partial t}\varphi(x,t) = \tau_g\varphi(x,t), \varphi(0) = \varphi_0. \end{cases} \tag{1}$$

Here α is a positive constant, and $\tau_g\varphi$ is the intrinsic Laplacian of map φ which denotes the tension field of map φ [3]. In (1), if we have $\alpha = 0$ or φ is a constant map, then it defines Ricci-Bourguignon flow. This concept was introduced in [7]. If $\rho = 0$, then (1) is just the Ricci-harmonic flow, and if $\alpha = \rho = 0$, then it reduces to the Ricci flow.

1.1. The Ricci-harmonic Bourguignon soliton

Let (M_1, g_1) be a smooth n-dimensional Riemannian manifold, and $Y : M_1 \to TM_1$ be a smooth vector field. Then the system $(M_1, g_1, Y, \lambda, \rho, \varphi)$ defines a Ricci-harmonic

[1]Corresponding author. Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran, e-mail: s.hajiaghasi@edu.ikiu.ac.ir

[2]Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran, e-mail: azami@sci.ikiu.ac.ir

[3]Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran, e-mail: jeffryebra@gmail.com

Bourguignon soliton (RHBS for short), when it satisfies in the following equations

$$\begin{aligned} &\mathrm{Ric} + \frac{1}{2}\mathcal{L}_Y g = \lambda g + \rho R g + \alpha \nabla\varphi \otimes \nabla\varphi, \\ &\tau_g \varphi - \mathcal{L}_Y \nabla\varphi = 0, \end{aligned} \tag{2}$$

where λ, α, and ρ are constants, R is scalar curvature, $\mathcal{L}_Y$ is Lie derivative along vector field Y, and $\varphi : (M_1, g_1) \to (M_2, g_2)$ is smooth function, where M_1, M_2 are static Riemannian manifolds. Here we provide some special examples of solitons:
(1) If in first equation of (2) $\alpha = \rho = 0$ or $\rho = 0$ and $\nabla\varphi = 0$, then M_1 is Ricci soliton

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_Y g = \lambda g.$$

(2) If $\alpha = 0$ or $\nabla\varphi = 0$, then M_1 is Ricci-Bourguignon soliton and equation becomes

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_Y g = \lambda g + \rho R g.$$

(3) If $\rho = 0$ then, M_1 is Ricci-harmonic soliton and equation changes as

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_Y g = \lambda g + \alpha \nabla\varphi \otimes \nabla\varphi.$$

When λ is a smooth function, we call the manifold endowed with (2), an almost Ricci-harmonic Bourguignon soliton.
In definition of RHBS if $Y = \nabla h$ for some smooth function h on M_1, then M_1 is called a gradient Ricci-harmonic Bourguignon soliton (GRHBS for short). Moreover, (2) changes as follows:

$$\begin{aligned} \mathrm{Ric} + \mathrm{Hess} h - \rho R g - \alpha \nabla\varphi \otimes \nabla\varphi &= \lambda g, \\ \tau_g \varphi - < \nabla\varphi, \nabla h > &= 0. \end{aligned} \tag{3}$$

The GRHBS is called steady, expanding or shrinking respectively, when $\lambda = 0$, $\lambda < 0$ or $\lambda > 0$.
In this paper, we denote $\mathrm{Ric} - \alpha\nabla\varphi \otimes \nabla\varphi$ by Ric_φ, its components in local coordinates by $R_\varphi^{ij} := R_{ij} - \alpha \nabla_i \varphi \nabla_j \varphi$, and the metric trace of R_φ^{ij} by $R_\varphi := R - \alpha|\nabla\varphi|^2$.
Some basic structural equations for compact almost Ricci-Bourguignon solitons, and almost Ricci-harmonic solitons were proved in [1] and [8]. In [8], Dwivedi showed that a compact gradient Ricci-Bourguignon soliton is isometric to an Euclidean sphere, if it has constant scalar curvature or its associated vector field is conformal. Abolarinwa [1] proved that a nontrivial gradient almost Ricci-harmonic soliton N^k with $k \geq 2$, and constant map φ is isometric to an k-Euclidean sphere, iff any of the following conditions hold:
(1) (N, g) has constant scalar curvature,
(2) $\int_N < \mathrm{Ric}_\varphi, \Delta h > dv = 0$,
(3) $\int_N \big(\mathrm{Ric}_\varphi(\nabla h, \nabla h) + (n-2) < \nabla\lambda, \nabla h > \big) dv \leq 0$,
(4) ∇h is a conformal vector field.
Here dv is the volume measure of N. Also, he obtained the main condition for a gradient almost Ricci-harmonic soliton $(RH)_\alpha$ to be rigid in the sense of triviality. See [4, 5, 6, 9, 10] for more study on rigidity and triviality of Ricci solitons.
Motivated by the mentioned articles, in this paper we succeeded in classifying Ricci-harmonic Bourguignon solitons. Also, we generalized the rigidity results in [1], for gradient Ricci-harmonic Bourguignon solitons under certain conditions. Here is our first result:

Theorem 1.1. *Let $(N, g, h, \lambda, \rho, \varphi)$ be a k-dimensional compact almost GRHBS. The following equation holds on N:*

$$\begin{aligned} 2\int_N |\mathrm{Hess}h - \frac{1}{k} g\Delta h|^2 dv &= (\frac{k-2}{k} + 2k\rho)\int_N g(\nabla R, \nabla h)dv \\ &-\frac{k-2}{k}\int_N \alpha g(\nabla|\nabla\varphi|^2, \nabla h)dv \\ &-2\alpha\int_N |g(\nabla\varphi, \nabla h)|^2 dv + \rho\int_N \Delta R dv, \end{aligned}$$

where dv is the volume element of N.

Studying on traviality results for Ricci-harmonic Bourguignon solitons, we provide the following classification:

Theorem 1.2. *Let $(N^k, g, Y, \lambda, \rho, \varphi)$ be a compact almost RHBS. For $k \geq 3$, If $\rho \neq \frac{1}{k}$ and*

$$\begin{aligned} \int_N [\mathrm{Ric}(Y, Y) + \frac{k\rho}{k\rho - 1}\nabla_Y \mathrm{div}Y - 2\rho g(\nabla R, Y) - \frac{(k-(2\rho+1)-2)}{k\rho - 1} g(\nabla\lambda, Y) \\ -2\alpha g(\tau_g(\varphi)\nabla\varphi, Y) - \frac{\alpha k\rho}{k\rho - 1} g(\nabla|\nabla\varphi|^2, Y)]dv \leq 0, \end{aligned}$$

thus Y is a Killing vector field, and M is a trivial RHBS.
We tried to know that under which conditions different types of GRHBS (i.e. steady, shrinking, expanding) have nonnegative scalar curvature, although we just proved the next theorem for steady Ricci-harmonic Bourguignon solitons.

Theorem 1.3. *For a compact almost RHBS $(N^k, g, h, \lambda, \rho, \varphi)$ with $\Delta\lambda \leq 0$, let $R_{min} := \min_N R_\varphi$, then*
(1) if N be steady with $R = 0$, then $R_{min} = 0$,
(2) if N be steady and $R > 0$, then $0 \leq R_{min} \leq k\rho R$,
(3) if N be steady and $R < 0$, then $k\rho R \leq R_{min} < 0$.

The next result is the generalization of [1] for $(RH)_\alpha$soliton to the almost GRHBS.

Theorem 1.4. *Let $(N^k, g, h, \lambda, \rho, \varphi)$ be an almost GRHBS with $k \geq 3$. If*

$$\int_N (\mathrm{Ric}_\varphi(\nabla h, \nabla h) + (n-2)g(\nabla\lambda, \nabla h) + 2(n-1)\rho g(\nabla R, \nabla h))dv \leq 0,$$

then N is trivially rigid.

2. **Preliminaries**

We need the following propositions to prove our main results.

Proposition 2.1. *For an almost GRHBS $(N^k, g, h, \lambda, \rho, \varphi)$, the following equations hold*

$$(1 - k\rho)R + \Delta h = k\lambda + \alpha|\nabla\varphi|^2, \tag{4}$$

$$(\frac{1}{2} - \rho(k-1))\nabla_i R = R_{il}\nabla_l h + (k-1)\nabla_i\lambda + \alpha\nabla_i|\nabla\varphi|^2 - \alpha\nabla_j\nabla_i\varphi\nabla_j\varphi, \tag{5}$$

$$\begin{aligned} \nabla_j R_{im} - \nabla_m R_{ij} &= R_{jmil}\nabla_l h + \rho(\nabla_j R g_{im} - \nabla_m R g_{ij}) + (\nabla_j\lambda g_{im} - \nabla_m\lambda g_{ij}) \\ &+\alpha(\nabla_j\nabla_i\varphi\nabla_m\varphi - \nabla_m\nabla_i\varphi\nabla_j\varphi), \end{aligned} \tag{6}$$

$$\nabla_i[(1-2\rho(k-1))R + |\nabla_l h|^2 - 2(k-1)\lambda - 2\alpha|\nabla\varphi|^2 + 2\alpha\nabla_i\varphi\nabla_j\varphi] \\ = (2\lambda + 2\rho R + 2\alpha\nabla_i\varphi\nabla_j\varphi)\nabla_i h. \tag{7}$$

Proof. For an almost GRHBS we have

$$R_{ij} + \nabla_i\nabla_j h = \lambda g_{ij} + \rho R g_{ij} + \alpha\nabla_i\varphi\nabla_j\varphi, \tag{8}$$

and taking the trace of this equation, we get the result for (4).
Taking the covariant derivative of (4) in a local orthonormal frame, we deduce that

$$(1-k\rho)\nabla_i R + \nabla_i\nabla_j\nabla_j h = k\nabla_i\lambda + \alpha\nabla_i|\nabla\varphi|^2.$$

Applying the consequence of contracted second Bianchi identity, we infer

$$(1-k\rho)\nabla_i R = R_{il}\nabla_l h + k\nabla_i\lambda + \alpha\nabla_i|\nabla\varphi|^2 - \nabla_j\nabla_i\nabla_j h.$$

If we substitute equation (8), in the above equation , we find

$$(1-k\rho)\nabla_i R = -\nabla_j(-R_{ij} + \lambda g_{ij} + \rho R g_{ij} + \alpha\nabla_i\varphi\nabla_j\varphi) + R_{il}\nabla_l h + n\nabla_i\lambda + \alpha\nabla_i|\nabla\varphi|^2.$$

Using the second Bianchi identity, we arrive at

$$(1-k\rho)\nabla_i R = \frac{1}{2}\nabla_i R - \nabla_i\lambda - \rho\nabla_i R - \alpha\nabla_j\nabla_i\varphi\nabla_j\varphi + R_{il}\nabla_l h + n\nabla_i\lambda + \alpha\nabla_i|\nabla\varphi|^2.$$

This completes the proof of (5). From (8), we get

$$\begin{aligned}\nabla_j R_{im} - \nabla_m R_{ij} &= (\nabla_m\nabla_i\nabla_j h - \nabla_j\nabla_i\nabla_m h) + \rho(\nabla_j R g_{im} - \nabla_m R g_{ij}) \\ &\quad + (\nabla_j\lambda g_{im} - \nabla_m\lambda g_{ij}) + \alpha(\nabla_j\nabla_i\varphi\nabla_m\varphi - \nabla_m\nabla_i\varphi\nabla_j\varphi)\end{aligned}$$

and rotating indices, we infer that

$$\begin{aligned}\nabla_j R_{im} - \nabla_m R_{ij} &= R_{jmil}\nabla_l h + \rho(\nabla_j R g_{im} - \nabla_m R g_{ij}) + (\nabla_j\lambda g_{im} - \nabla_m\lambda g_{ij}) \\ &\quad + \alpha(\nabla_j\nabla_i\varphi\nabla_m\varphi - \nabla_m\nabla_i\varphi\nabla_j\varphi),\end{aligned}$$

which is equation (6).
According to the second equation, we can write

$$\begin{aligned}(1-2\rho(k-1))\nabla_i R &= 2\nabla_l h(-\nabla_i\nabla_l h + \lambda g_{il} + \rho R g_{il} + \alpha\nabla_i\varphi\nabla_l\varphi) \\ &\quad + 2(k-1)\nabla_i\lambda + 2\alpha\nabla_i|\nabla\varphi|^2 - 2\alpha\nabla_j\nabla_i\varphi\nabla_j\varphi \\ &= -2\nabla_l h\nabla_i\nabla_l h + 2\lambda\nabla_i h + 2\rho R\nabla_i h + 2\alpha\nabla_l h\nabla_i\varphi\nabla_l\varphi \\ &\quad + 2(k-1)\nabla_i\lambda + 2\alpha\nabla_i|\nabla\varphi|^2 - 2\alpha\nabla_j\nabla_i\varphi\nabla_j\varphi.\end{aligned}$$

Since $\nabla_i|\nabla_l h|^2 = 2\nabla_l h\nabla_i\nabla_l h$, we have

$$\begin{aligned}1-2\rho(k-1))\nabla_i R &= -\nabla_i|\nabla_l h|^2 + 2\lambda\nabla_i h + 2\rho R\nabla_i h + 2\alpha\nabla_l h\nabla_i\varphi\nabla_l\varphi \\ &\quad + 2(k-1)\nabla_i\lambda + 2\alpha\nabla_i|\nabla\varphi|^2 - 2\alpha\nabla_j\nabla_i\varphi\nabla_j\varphi,\end{aligned}$$

and hence

$$\nabla_i[(1-2\rho(k-1))R + |\nabla_l h|^2 - 2(k-1)\lambda - 2\alpha|\nabla\varphi|^2 + 2\alpha\nabla_i\varphi\nabla_j\varphi] \\ = (2\lambda + 2\rho R + 2\alpha\nabla_i\varphi\nabla_j\varphi)\nabla_i h,$$

this completes the proof of equation (7). □

Proposition 2.2. *For an almost GRHBS $(N^k, g, h, \lambda, \rho, \varphi)$, we have*

$$\mathrm{divRic}_\varphi = \frac{1}{2}\nabla R_\varphi - \alpha\tau_g(\varphi)\nabla\varphi, \tag{9}$$

$$g(\nabla R_\varphi, \nabla h) = 2(k-1)g(\nabla h, \nabla\lambda) + 2(k-1)\rho g(\nabla h, \nabla R) + 2\mathrm{Ric}_\varphi(\nabla h, \nabla h), \tag{10}$$

$$-\frac{1}{2}\nabla R_\varphi + \mathrm{Ric}_\varphi(\nabla h, .) + (k-1)\nabla\lambda + (k-1)\rho\nabla R = 0, \tag{11}$$

$$\nabla(R_\varphi + |\nabla h|^2) = 2(k-1)[\nabla\lambda + \rho\nabla R] + 2\lambda\nabla h + 2\rho R\nabla h, \tag{12}$$

$$\begin{aligned}\frac{1}{2}\Delta|\nabla h|^2 = |\mathrm{Hess}h|^2 - (k-2)g(\nabla\lambda, \nabla h) - 2(k-1)\rho g(\nabla R, \nabla h)\\ -\mathrm{Ric}_\varphi(\nabla h, \nabla h) + \alpha|g(\nabla\varphi, \nabla h)|^2.\end{aligned} \tag{13}$$

Proof. Using the second Bianchi identity $\mathrm{divRic} = \frac{1}{2}\nabla R$, and the identity $\mathrm{div}(\nabla\varphi \otimes \nabla\varphi) = \tau_g(\varphi)\nabla\varphi + \frac{1}{2}\nabla|\nabla\varphi|^2$, we obtain

$$\begin{aligned}\frac{1}{2}\nabla R &= \mathrm{divRic}\\ &= \mathrm{divRic}_\varphi + \alpha\mathrm{div}(\nabla\varphi \otimes \nabla\varphi)\\ &= \mathrm{divRic}_\varphi + \alpha\tau_g(\varphi)\nabla\varphi + \frac{\alpha}{2}\nabla|\nabla\varphi|^2,\end{aligned}$$

this proves (9).
Using $R_\varphi^{ij} = R_{ij} - \alpha\nabla_i\varphi\nabla_j\varphi$ and (3), we have

$$R_\varphi^{ij} = \rho R g_{ij} + \lambda g_{ij} - h_{ij}.$$

Taking the covariant derivative, leads

$$R_\varphi^{ijm} = \rho\nabla_m R g_{ij} + \nabla_m\lambda g_{ij} - h_{ij,m}, \tag{14}$$

and tracing with respect to j, m, concludes

$$R_\varphi^{imm} = \rho\nabla_i R + \nabla_i\lambda - h_{im,m}.$$

By Ricci identity $h_{im,m} - h_{mm,i} = R_{im}h_m$, and (9), we infer

$$\frac{1}{2}R_\varphi^i - \alpha\tau_g(\varphi)\nabla\varphi = \rho\nabla_i R + \nabla_i\lambda - h_{mm,i} - R_{im}h_m,$$

here $\tau_g(\varphi) = trace\nabla d\varphi$. Taking the trace of equation (14) with respect to i, j, we obtain

$$R_\varphi^i = k\rho\nabla_i R + k\nabla_i\lambda - h_{mm,i},$$

so using the last two above equations for an almost GRHBS, we arrive at

$$R_\varphi^i = 2(k-1)\rho\nabla_i R + 2(k-1)\nabla_i\lambda + 2R_{im}h_m - 2\alpha\tau_g(\varphi)\nabla\varphi,$$

and by $\mathrm{Ric} = \mathrm{Ric}_\varphi + \alpha\nabla\varphi \otimes \nabla\varphi$, we get

$$g(\nabla R_\varphi, \nabla h) = 2(k-1)g(\nabla\lambda, \nabla h) + 2(k-1)\rho g(\nabla R, \nabla h) + 2\mathrm{Ric}_\varphi(\nabla h, \nabla h).$$

For the next identity note that $\rho R g + \lambda g = \mathrm{Hess}h + \mathrm{Ric}_\varphi$, so by taking divergence, we conclude

$$\mathrm{div}((\rho R + \lambda)g) = \mathrm{divHess}h + \mathrm{divRic}_\varphi.$$

Since $\mathrm{divRic}_\varphi = \frac{1}{2}\nabla R_\varphi - \alpha\tau_g(\varphi)\nabla\varphi$ and $\mathrm{divHess}h = \nabla\Delta h + \mathrm{Ric}(\nabla h, .)$, we obtain

$$\mathrm{div}((\rho R + \lambda)g) = \nabla\Delta h + \mathrm{Ric}(\nabla h, .) + \frac{1}{2}\nabla R_\varphi - \alpha\tau_g(\varphi)\nabla\varphi. \tag{15}$$

Applying (4) in (15), we infer

$$\rho\nabla R + \nabla\lambda = k\rho\nabla R + k\nabla\lambda - \frac{1}{2}\nabla R_\varphi + \mathrm{Ric}(\nabla h, .) - \alpha\tau_g(\varphi)\nabla\varphi,$$

This completes the proof of (11).
From the second identity, we infer

$$\nabla R_\varphi = 2(k-1)\nabla\lambda + 2(k-1)\rho\nabla R + 2\mathrm{Ric}_\varphi(\nabla h, .).$$

Applying (3) in the above equation, leads

$$\begin{aligned}\nabla R_\varphi &= 2(k-1)\nabla\lambda + 2(k-1)\rho\nabla R + 2(\lambda g + \rho R g - \mathrm{Hess}h)\nabla h\\ &= 2(k-1)\nabla\lambda + 2(k-1)\rho\nabla R + 2\lambda\nabla h + 2\rho R\nabla h - 2\mathrm{Hess}h(\nabla h)\\ &= 2(k-1)[\nabla\lambda + \rho\nabla R] + 2\lambda\nabla h + 2\rho R\nabla h - \nabla|\nabla h|^2,\end{aligned}$$

so

$$\nabla(R_\varphi + |\nabla h|^2) = 2(k-1)[\nabla\lambda + \rho\nabla R] + 2\lambda\nabla h + 2\rho R\nabla h.$$

Using Bochner formula and (10), we find

$$\frac{1}{2}\Delta|\nabla h|^2 = |\mathrm{Hess}h|^2 + kg(\nabla\lambda, \nabla h) - g(\nabla R_\varphi, \nabla h) + \mathrm{Ric}(\nabla h, \nabla h).$$

Inserting $R_\varphi = \lambda - \alpha|\nabla\phi|^2$ and $\mathrm{Ric} = \mathrm{Ric}_\varphi + \alpha\nabla\phi\otimes\nabla\phi$ in last equation, we have

$$\begin{aligned}\frac{1}{2}\Delta|\nabla h|^2 &= |\mathrm{Hess}h|^2 + kg(\nabla\lambda, \nabla h) - 2(k-1)g(\nabla\lambda, \nabla h) - 2(k-1)\rho g(\nabla R, \nabla h)\\ &\quad -2\mathrm{Ric}_\varphi(\nabla h, \nabla h) + \mathrm{Ric}(\nabla h, \nabla h)\\ &= |\mathrm{Hess}h|^2 - (k-2)g(\nabla\lambda, \nabla h) - 2(k-1)\rho g(\nabla R, \nabla h) - \mathrm{Ric}_\varphi(\nabla h, \nabla h)\\ &\quad +\alpha|g(\nabla\varphi, \nabla h)|^2.\end{aligned}$$

□

Since the almost gradient Ricci solitons are a special kind of gradient Ricci solitons, we have the same identities for GRHBS. We just take $\nabla\lambda = 0$ to conclude the following:

Corollary 2.1. *For a GRHBS similarly we have*

$$\mathrm{divRic}_\varphi = \frac{1}{2}\nabla R_\varphi - \alpha\tau_g(\varphi)\nabla\varphi, \tag{16}$$

$$g(\nabla R_\varphi, \nabla h) = 2(k-1)\rho g(\nabla R, \nabla h) + 2\mathrm{Ric}_\varphi(\nabla h, \nabla h), \tag{17}$$

$$-\frac{1}{2}\nabla R_\varphi + \mathrm{Ric}_\varphi(\nabla h, .) + (k-1)\rho\nabla R = 0, \tag{18}$$

$$\nabla(R_\varphi + |\nabla h|^2) = 2(k-1)\rho\nabla R + 2\lambda\nabla h + 2\rho R\nabla h, \tag{19}$$

$$\frac{1}{2}\Delta|\nabla h|^2 = |\mathrm{Hess}h|^2 - 2(n-1)\rho g(\nabla R, \nabla h) - \mathrm{Ric}_\varphi(\nabla h, \nabla h) + \alpha|g(\nabla\varphi, \nabla h)|^2. \tag{20}$$

Lemma 2.1. *For an almost RHBS* $(N^k, g, Y, \lambda, \rho, \varphi)$*, we have*

$$\begin{aligned}\frac{(1-k\rho)}{2}\Delta|Y|^2 &= (1-k\rho)|\nabla Y|^2+(k\rho-1)\mathrm{Ric}(Y,Y)+k\rho\nabla_Y \mathrm{div}Y\\ &+2\rho(1-k\rho)g(\nabla R,Y)-(k(2\rho+1)-2)g(\nabla\lambda,Y)\\ &+2\alpha(1-k\rho)g(\tau_g(\varphi)\nabla\varphi,Y)-\alpha k\rho g(\nabla|\nabla\varphi|^2,Y), \qquad (21)\end{aligned}$$

and

$$\begin{aligned}\frac{(1-k\rho)}{2}(\Delta-\nabla_Y)|Y|^2 &= (1-k\rho)|\nabla Y|^2+\lambda(k\rho-1)|Y|^2\\ &+\rho(k\rho-1)R|Y|^2+\alpha(k\rho-1)|\nabla_Y\varphi|^2\\ &+k\rho\nabla_Y\mathrm{div}Y+2\rho(1-k\rho)g(\nabla R,Y)\\ &-(k(2\rho+1)-2)g(\nabla\lambda,Y)\\ &+2\alpha(1-k\rho)g(\tau_g(\varphi)\nabla\varphi,Y)-\alpha k\rho g(\nabla|\nabla\varphi|^2,Y). \qquad (22)\end{aligned}$$

Proof. By taking divergence of (2), we have

$$2\mathrm{divRic}+\mathrm{div}(\mathcal{L}_Y g)=2\nabla\lambda+2\rho\nabla R+2\alpha\nabla(\nabla\varphi\otimes\nabla\varphi), \qquad (23)$$

and by tracing of (2), we get

$$(1-k\rho)R+\mathrm{div}Y=k\lambda+\alpha|\nabla\varphi|^2,$$

hence

$$(1-k\rho)\nabla_Y R+\nabla_Y\mathrm{div}Y=k\nabla_Y\lambda+\alpha\nabla_Y|\nabla\varphi|^2. \qquad (24)$$

From Lemma 2.3 of [5], we have

$$\mathrm{div}(\mathcal{L}_Y g)(Y)=\frac{1}{2}\Delta|Y|^2-|\nabla Y|^2+\mathrm{Ric}(Y,Y)+\nabla_Y\mathrm{div}Y. \qquad (25)$$

Using the contracted second Bianchi identity, (23) and (24), we obtain

$$\begin{aligned}\nabla_Y(\mathrm{div}Y) &= (k\rho-1)\nabla_Y R+kg(\nabla\lambda,Y)+\alpha g(\nabla|\nabla\varphi|^2,Y)\\ &= 2(k\rho-1)\mathrm{divRic}(Y)+kg(\nabla\lambda,Y)+\alpha g(\nabla|\nabla\varphi|^2,Y)\\ &= (k\rho-1)\big(-\mathrm{div}(\mathcal{L}_Y g)(Y)+2\rho g(\nabla R,Y)+2g(\nabla\lambda,Y)\\ &\quad +2\alpha g(\nabla(\nabla\varphi\otimes\nabla\varphi),Y)\big)+kg(\nabla\lambda,Y)+\alpha g(\nabla|\nabla\varphi|^2,Y).\end{aligned}$$

Placing (25) in above equation, we arrive at

$$\begin{aligned}\nabla_Y(\mathrm{div}Y) &= (1-k\rho)(\frac{1}{2}\Delta|Y|^2-|\nabla Y|^2+\mathrm{Ric}(Y,Y)+\nabla_Y\mathrm{div}Y)\\ &+2\rho(k\rho-1)g(\nabla R,Y)+(k(2\rho+1)-2)g(\nabla\lambda,Y)\\ &+2\alpha(k\rho-1)g(\nabla(\nabla\varphi\otimes\nabla\varphi),Y)+\alpha g(\nabla|\nabla\varphi|^2,Y).\end{aligned}$$

Since $\nabla(\nabla\varphi\otimes\nabla\varphi)=\tau_g(\varphi)\nabla\varphi+\frac{1}{2}\nabla|\nabla\varphi|^2$, the proof of (21) is complete. By (2), we have

$$\mathrm{Ric}(Y,Y)=-\frac{1}{2}(\mathcal{L}_Y g)(Y,Y)+\lambda|Y|^2+\rho R|Y|^2+\alpha|\nabla_Y\varphi|^2. \qquad (26)$$

Using (26) in first equation of this lemma, we deduce

$$\begin{aligned}\frac{(1-k\rho)}{2}\Delta|Y|^2 &= (1-k\rho)|\nabla Y|^2+(k\rho-1)\big(-\frac{1}{2}(\mathcal{L}_Y g)(Y,Y)+\lambda|Y|^2\\ &+\rho R|Y|^2+\alpha|\nabla_Y\varphi|^2\big)+k\rho\nabla_Y\mathrm{div}Y\\ &+2\rho(1-k\rho)g(\nabla R,Y)-(k(2\rho+1)-2)g(\nabla\lambda,Y)\\ &+2\alpha(1-k\rho)g(\tau_g(\varphi)\nabla\varphi,Y)-\alpha k\rho g(\nabla|\nabla\varphi|^2,Y).\end{aligned}$$

Since $\mathcal{L}_Y g(Y,Y)=\frac{1}{2}\nabla_Y|Y|^2$, we infer

$$\begin{aligned}\frac{(1-k\rho)}{2}\Delta|Y|^2 &= (1-k\rho)|\nabla Y|^2+\frac{(1-k\rho)}{2}\nabla_Y|Y|^2+\lambda(k\rho-1)|Y|^2\\ &+\rho(k\rho-1)R|Y|^2+\alpha(k\rho-1)|\nabla_Y\varphi|^2+k\rho\nabla_Y\mathrm{div}Y\\ &+2\rho(1-k\rho)g(\nabla R,Y)-(k(2\rho+1)-2)g(\nabla\lambda,Y)\\ &+2\alpha(1-k\rho)g(\tau_g(\varphi)\nabla\varphi,Y)-\alpha k\rho g(\nabla|\nabla\varphi|^2,Y),\end{aligned}$$

this proves (22). □

Now we prove Theorem 1.1 here.

Proof of Theorem 1.1. Taking divergence of (12), we have

$$\Delta R_\varphi+\Delta|\nabla h|^2-2(k-1)\Delta\lambda-\rho\Delta R-2<\nabla\lambda,\nabla h>-2\lambda\Delta h=0. \tag{27}$$

Plugging (13) into (27), we obtain

$$\begin{aligned}\Delta R_\varphi+2|\mathrm{Hess}h|^2-2(k-2)<\nabla\lambda,\lambda h>-4(k-1)\rho<\nabla R,\nabla h>-2\mathrm{Ric}_\varphi(\nabla h,\nabla h)\\ +2\alpha|<\nabla\varphi,\nabla h>|^2-2(k-1)\Delta\lambda-\rho\Delta R-2<\nabla\lambda,\nabla h>-2\lambda\Delta h=0,\end{aligned}$$

and by (10), we get

$$\begin{aligned}&\Delta R_\varphi+2|\mathrm{Hess}h|^2-2(k-2)<\nabla\lambda,\nabla h>-4(k-1)\rho<\nabla R,\nabla h>\\ &+2(k-1)<\nabla\lambda,\nabla h>+2(k-1)\rho<\nabla R,\nabla h>-<\nabla R_\varphi,\nabla h>\\ &+2\alpha|<\nabla\varphi,\nabla h>|^2-2(k-1)\Delta\lambda-\rho\Delta R-2<\nabla\lambda,\nabla h>-\lambda\Delta h=0.\end{aligned}$$

Hence

$$\begin{aligned}\Delta R_\varphi+2|\mathrm{Hess}h|^2-2k\rho<\nabla R,\nabla h>-<\nabla R_\varphi,\nabla h>+2\alpha|<\nabla\varphi,\nabla h>|^2\\ -2(k-1)\Delta\lambda-\rho\Delta R-2\lambda\Delta h=0.\end{aligned} \tag{28}$$

Using the identity $|\mathrm{Hess}h-\frac{1}{k}g\Delta h|^2=|\mathrm{Hess}h|^2-\frac{1}{k}(\Delta h)^2$, equation (28) becomes

$$\begin{aligned}\Delta R_\varphi+2|\mathrm{Hess}h-\frac{1}{k}g\Delta h|^2-<\nabla R_\varphi,\nabla h>-2(k-1)\Delta\lambda-2\Delta h(\lambda-\frac{1}{k}\Delta h)\\ -2k\rho<\nabla R,\nabla h>-\rho\Delta R+2\alpha|<\nabla\varphi,\nabla h>|^2=0.\end{aligned}$$

Notice that $\Delta h=k\lambda-R_\varphi$ so $\frac{1}{k}R_\varphi=\lambda-\frac{1}{k}\Delta h$, thus

$$\begin{aligned}\Delta R_\varphi+2|\mathrm{Hess}h-\frac{1}{k}g\Delta h|^2-<\nabla R_\varphi,\nabla h>-2(k-1)\Delta\lambda-\frac{2}{k}R_\varphi\Delta h\\ -2k\rho<\nabla R,\nabla h>-\rho\Delta R+2\alpha|<\nabla\varphi,\nabla h>|^2=0.\end{aligned}$$

Integrating over N, one has

$$\begin{aligned}2\int_N |\mathrm{Hess}h - \frac{1}{k}g\Delta h|^2 dv &= \frac{k-2}{k}\int_N < \nabla R_\varphi, \nabla h > dv - 2\alpha\int_N | < \nabla\varphi, \nabla h > |^2 dv \\ &+2k\rho\int_N < \nabla R, \nabla h > dv + \rho\int_N \Delta R dv.\end{aligned}$$

Since $\nabla R_\varphi = \nabla R - \alpha\nabla|\nabla\phi|^2$, we obtain

$$\begin{aligned}2\int_N |\mathrm{Hess}h - \frac{1}{k}g\Delta h|^2 dv &= \frac{k-2}{k}\int_N < \nabla R, \nabla h > dv \\ &-\frac{k-2}{k}\int_N \alpha < \nabla|\nabla\varphi|^2, \nabla h > dv \\ &-2\alpha\int_N | < \nabla\varphi, \nabla h > |^2 dv \\ &+2k\rho\int_N < \nabla R, \nabla h > dv + \rho\int_N \Delta R dv.\end{aligned}$$

□

We have next Lemma just like Lemma 3.1 in [8].

Lemma 2.2. *Let $(N^k, g, Y, \lambda, \rho, \alpha)$ be an almost RHBS with $K \geq 3$. If $Y \neq 0$ is $\mathcal{L}_Y g = 2\psi g$, thus we get*

$$(1-k\rho)\nabla_j R = \nabla_j(\alpha|\nabla\varphi|^2) - k\alpha\nabla_j\nabla_i\varphi\nabla_j\varphi + k(\frac{1}{2}-\rho)\nabla_j R. \tag{29}$$

If $\nabla\varphi$ is constant, and N is connected, then R and $\lambda - \psi$ are constant.

Proof. Since Y is conformal, we get the following

$$\mathrm{Ric} - \alpha\nabla\varphi\otimes\nabla\varphi = (\lambda + \rho R - \psi)g,$$

and in local coordinates, we conclude

$$R_{ij} - \alpha\nabla_i\varphi\nabla_j\varphi = (\lambda + \rho R - \psi)g_{ij}. \tag{30}$$

Taking the divergence, we obtain

$$\nabla_i R_{ij} - \alpha\nabla_i\nabla_i\varphi\nabla_j\varphi = \nabla_i(\lambda + \rho R - \psi)g_{ij},$$

and hence

$$(\frac{1}{2}-\rho)\nabla_j R - \alpha\nabla_i\nabla_i\varphi\nabla_j\varphi = \nabla_j(\lambda - \psi). \tag{31}$$

On the other hand, taking trace and derivation of (30), we get

$$(1-k\rho)\nabla_j R - \nabla_j(\alpha|\nabla\varphi|^2) = k\nabla_j(\lambda - \psi). \tag{32}$$

According to (31) and (32), we have

$$\begin{aligned}(1-k\rho)\nabla_j R &= \nabla_j(\alpha|\nabla\varphi|^2) + k\nabla_j(\lambda-\psi) \\ &= \nabla_j(\alpha|\nabla\varphi|^2) + k(\frac{1}{2}-\rho)\nabla_j R - k\alpha\nabla_i\nabla_i\varphi\nabla_j\varphi.\end{aligned}$$

So if $\nabla\varphi$ is constant, and N be connected, then both R and $\lambda - \psi$ are constant. □

Corollary 2.2. *With the same assumptions as in Lemma 2.2, we infer*
(1) If N is compact, then $\mathcal{L}_Y g = 0$, and hence N is a trivial RHBS.
(2) If N is non-compact, complete, and a GRHBS, then either N is isometric to the Euclidean space, or $\mathcal{L}_Y g = 0$.

In fact, we can say that this corollary is the generalization of Theorem 3 in [2], and Theorem 1.3 in [8]. Now, we can prove Theorem 1.2 by Lemma 2.1.

Proof of Theorem 1.2. Integrating from both sides of (21) on compact N, we have

$$\int_N [(1-k\rho)|\nabla Y|^2 + (k\rho - 1)\mathrm{Ric}(Y,Y) + k\rho\nabla_Y \mathrm{div}Y + 2\rho(1-k\rho)g(\nabla R, Y)$$
$$-(k(2\rho+1)-2)g(\nabla\lambda, Y) + 2\alpha(1-k\rho)g(\tau_g(\varphi)\nabla\varphi, Y) - \alpha n\rho g(\nabla|\nabla\varphi|^2, Y)]dv = 0,$$

and by the fact that $\rho \neq \frac{1}{k}$, we get

$$\begin{aligned}\int_N |\nabla Y|^2 &= \int_N [\mathrm{Ric}(Y,Y) + \frac{k\rho}{k\rho-1}\nabla_Y \mathrm{div}Y - 2\rho g(\nabla R, Y) \\ &\quad -\frac{(k(2\rho+1)-2)}{k\rho-1}g(\nabla\lambda, Y) - 2\alpha g(\tau_g(\varphi)\nabla\varphi, Y) \\ &\quad -\frac{\alpha k\rho}{k\rho-1}g(\nabla|\nabla\varphi|^2, Y)]dv,\end{aligned}$$

so if

$$\int_N [\mathrm{Ric}(Y,Y) + \frac{k\rho}{k\rho-1}\nabla_Y \mathrm{div}Y - 2\rho g(\nabla R, Y) - \frac{(k-(2\rho+1)-2)}{k\rho-1}g(\nabla\lambda, Y)$$
$$-2\alpha g(\tau_g(\varphi)\nabla\varphi, Y) - \frac{\alpha k\rho}{k\rho-1}g(\nabla|\nabla\varphi|^2, Y)]dv \leq 0$$

then Y is Killing vector field. Consequently $(N^k, g, Y, \lambda, \rho, \varphi)$ is trivial. ☐

By Proposition 2.2, we can prove Theorem 1.3 as follows:

Proof of Theorem 1.3. Taking divergence of (12), we conclude

$$\begin{aligned}\Delta R_\varphi + \Delta|\nabla h|^2 - 2(k-1)\Delta\lambda - 2 < \nabla\lambda, \nabla h > -2\rho < \nabla R, \nabla h > \\ -2\lambda\Delta h - 2\rho R\Delta h - 2(k-1)\rho\Delta R = 0.\end{aligned} \tag{33}$$

Replacing (13) in (33), we have

$$\begin{aligned}0 &= \Delta R_\varphi + 2|\mathrm{Hess}h|^2 - 2(k-1) < \nabla\lambda, \nabla h > -2(2k-1)\rho < \nabla R, \nabla h > \\ &\quad -2\mathrm{Ric}_\varphi(\nabla h, \nabla h) + 2\alpha| < \nabla\varphi, \nabla h > |^2 - 2(k-1)\Delta\lambda \\ &\quad -2\lambda\Delta h - 2\rho R\Delta h - 2(k-1)\rho\Delta R.\end{aligned}$$

Substituting (10) in last equation, we arrive at

$$\begin{aligned}0 &= \Delta R_\varphi + 2|\mathrm{Hess}h|^2 - 2k\rho < \nabla R, \nabla h > - < \nabla R_\varphi, \nabla h > +2\alpha| < \nabla\varphi, \nabla h > |^2 \\ &\quad -2(k-1)\Delta\lambda - 2\lambda\Delta h - 2\rho R\Delta h - 2(k-1)\rho\Delta R.\end{aligned}$$

Using identity $\Delta h = k\lambda - R_\varphi$, we infer

$$\begin{aligned} 0 &= \Delta R_\varphi + 2|\lambda g + \rho R g - \mathrm{Ric}_\varphi|^2 - < \nabla R_\varphi, \nabla h > -2k\rho < \nabla R, \nabla h > -2(k-1)\Delta\lambda \\ &\quad -2\lambda(k\lambda - R_\varphi) - 2\rho R(k\lambda - R_\varphi) + 2\alpha| < \nabla\varphi, \nabla h > |^2 - 2(k-1)\rho\Delta R \\ &= \Delta R_\varphi + 2|\mathrm{Ric}_\varphi|^2 + 2k\rho R^2 + 2k\rho\lambda R - 2\lambda R_\varphi - 2\rho R R_\varphi - < \nabla R_\varphi, \nabla h > \\ &\quad -2k\rho < \nabla R, \nabla h > -2(k-1)\Delta\lambda + 2\alpha| < \nabla\varphi, \nabla h > |^2 - 2(k-1)\rho\Delta R. \end{aligned}$$

Based on the condition $\Delta\lambda \leq 0$, which was stated in the assumptions of the theorem, and inequality $|\mathrm{Ric}_\varphi|^2 \geq \frac{1}{k} R_\varphi^2$, we conclude

$$\begin{aligned} 0 &\geq \Delta R_\varphi + \frac{2}{k} R_\varphi^2 + 2k\rho\lambda R - 2\lambda R_\varphi - 2\rho R R_\varphi - \nabla R_\varphi(\nabla h) - 2k\rho < \nabla R, \nabla h > \\ &\quad -2(k-1)\rho\Delta R. \end{aligned}$$

Hence, if $\lambda = 0$, and R is constant, by compactness of N, there is a point belong to N named q such that R_φ adopts its minimum. Applying maximum principle at this point, we have $\Delta R_\varphi(q) \geq 0$, and $\nabla R_\varphi(q) = 0$, next by $R_{min}(R_{min} - k\rho R) \leq 0$, we get the results. □

Now we prove the last important result of this paper.

Proof of Theorem 1.4. Integrating of (13), leads

$$\begin{aligned} \frac{1}{2}\int_N \Delta|\nabla h|^2 dv &= \int_N |\mathrm{Hess} h|^2 dv - (k-2)\int_N < \nabla\lambda, \nabla h > dv \\ &\quad -2(k-1)\rho\int_N < \nabla R, \nabla h > dv - \int_N \mathrm{Ric}_\varphi(\nabla h, \nabla h > dv \\ &\quad +\alpha\int N| < \nabla\varphi, \nabla h > |^2 dv. \end{aligned}$$

Since N is compact, we get

$$\begin{aligned} \int_N |\mathrm{Hess} h|^2 dv &= \int_N (\mathrm{Ric}_\varphi(\nabla h, \nabla h) + (k-2) < \nabla\lambda, \nabla h > +2(k-1)\rho < \nabla R, \nabla h >) dv \\ &\quad -\alpha\int_N | < \nabla\varphi, \nabla h > |^2 dv. \end{aligned} \tag{34}$$

So, the second integration of RHS is nonnegative, and by assumption, we know $\mathrm{Hess} h = 0$. Therefore ∇h is a Killing vector field, and h is constant. □

Proposition 2.3. *Let $(N^k, g, h, \lambda, \rho, \varphi)$ be an almost GRHBS with $k \geq 2$, and $\mathrm{Ric}_\varphi \geq 0$ ($\mathrm{Ric}_\varphi \leq 0$). Then R_φ is constant, iff $\mathrm{Ric}_\varphi(\nabla h, \nabla h) = 0$, $< \nabla\lambda, \nabla h > \leq 0$ and $< \nabla R, \nabla h > \leq 0$.*

Proof. By Proposition 2.2, we obtain

$$\frac{1}{2}\nabla R_\varphi(\nabla h) = (k-1) < \nabla\lambda, \nabla h > +(k-1)\rho < \nabla R, \nabla h > +\mathrm{Ric}_\varphi(\nabla h, \nabla h),$$

by assumptions we conclude $\nabla R_\varphi = 0$, so R_φ is constant. □

Conclusion

Many authors have obtained some concepts and results of Ricci soliton for Ricci-Harmonic and Ricci Bourguignon solitons. Since Ricci-Harmonic Bourguignon solitons are generalizations of both Ricci-Harmonic and Ricci Bourguignon solitons, it is important to study the structure of these solitons. Also, it is interesting to investigate the conditions that cause these solitons have special vector fields, like Killing and conformal vector fields. Moreover, the study of the conditions that infer triviality results for all kinds of solitons is an important topic in their geometry. So, in this paper, we tried to investigate the important properties in the geometry of this new solitons. There are several points that require further study. For instance, as a generalization, one can define Ricci-harmonic Bourguignon solitons on multiply warped products. It is also interesting to replace Ricci with $*$-Ricci (this notion was introduced by Tachibana in [11]) in equation (2).

REFERENCES

[1] *A. Abolarinwa*, Basic structural equations for almost Ricci-Harmonic soliton and applications, Differential Geometry - Dynamical Systems, **21**, 2019, 1-13.

[2] *C. Aquino, A. Barros, E. Ribeiro Jr*, Some applications of the Hodge-de Rham decomposition to Ricci solitons, Results Math, **60**, 2011, 245-254.

[3] *S. Azami*, Ricci-Bourguignon flow coupled with harmonic map flow, International Jornal of Mathematics, **30**(**10**), 1950049, 2019.

[4] *A. Barros and E. Ribeiro Jr*, Some characterizations for compact almost Ricci solitons, Proc. Amer. Math. Soc, **140**, 2012, 1033-1040.

[5] *A. Barros, R. Batista and E. Rieberio Jr*, Rigidity of gradient almost Ricci solitons, Illinois J. Math, **56**, 2012, 1267-1279.

[6] *A. Barros, J. N. Gomes, E. Rieberio Jr*, A note on Rigidity of the almost Ricci solitons, Arch. Math, **100**, 2013, 481-490.

[7] *J. Bourguignon*, Ricci curvature and Einstein metrics, Global Differential Geometry and Global Analysis, 1981, 42-63.

[8] *S. Dwivedi*, Some results on Ricci-Bourguignon and Ricci-Bourguignon almost solitons, Journal of Geometry and Physics, **168**, 2021.

[9] *F. E. Feitosa, A. A. Freitas Filho, J. N. Gomes, R. S. Pina*, Gradient almost Ricci soliton warped product, Journal of Geometry and Physics, **143**, 2019, 22-32.

[10] *S. Pigola, M. Rigoli, M. Rimoldi, A. G. Setti*, Ricci almost solitons, Ann. Sc. Norm. Super. Pisa Cl. Sci, (**5**), 2011, 757-799.

[11] *S. Tachibana*, On almost-analytic vectors in almost Kahlerian manifolds. Tohoku Math. J. **11**, 1959, 247-265.