IMPLICIT ITERATION SCHEME WITH NUMERICAL ANALYSIS FOR A FINITE FAMILY OF STRICTLY PSEUDOCONTRACTIVE MAPPINGS

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In this paper, we propose an implicit iteration scheme with perturbed mapping for a finite family of strictly pseudocontractive mappings and establish weak and strong convergence theorems. We report some preliminary computational results related to the influence of parameters of the algorithm. Results in this paper extend and improve recent results in the literature.

Keywords: Strictly pseudocontractive mapping, iteration scheme with perturbed mapping, common fixed point, strong convergence.

1. Introduction

In the past five decades, iteration processes for numerical reckoning fixed points of nonlinear mappings and their applications have been studied extensively by many authors. In 2001, Xu and Ori [24] introduced an implicit iteration process to approximate a common fixed point of a finite family of nonexpansive self mappings in a Hilbert space. In 2004, Osilike [12] further extended the iteration process of Xu and Ori to a finite family of strictly pseudocontractive self mappings in Banach spaces. In 2007, Acedo and Xu [1] proposed a parallel iterative algorithm for strictly pseudocontractive mappings in the framework of Hilbert spaces. Zeng and Yao [30] introduced in 2006 an implicit iteration process with perturbed mapping, to approximate common fixed points of a finite family of nonexpansive mappings. Ceng *et al.* [3] introduced in 2007 an implicit iteration process with perturbed mapping G, for approximating common fixed points of a finite family of continuous pseudocontractive self-mappings. Our contribution in this paper is motivated and inspired by the above described research, and proposes a new implicit iteration scheme, with perturbed mapping, to approximate fixed point of a finite family of strictly pseudocontractive self-mappings.

2. Mathematical preliminaries

Let E be a real Banach space and let $J\colon E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\,\|f\|\,;\|x\| = \|f\|\} \quad \forall\, x \in E\,,$$

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where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If E^* is strictly convex then J is single valued. We denote the single valued normalized duality mapping by j. Let $T\colon E \to E$ be a mapping. We use $F(T)$ to denote the set of fixed points of T; that is, $F(T) = \{x \in E : x = Tx\}$.

Let T be a mapping with domain $D(T)$ and range $R(T)$ in E. Mapping T is said to be λ'-strictly pseudocontractive mapping in the sense of Browder-Petryshyn [2], if there exists a constant $0 < \lambda' < 1$, $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 - \lambda' \|x - y - (Tx - Ty)\|^2 , \quad \forall\, x, y \in D(T) . \tag{2.1}$$

In Hilbert space inequality (2.1) is equivalent to the following inequality.

$$\|Tx - Ty\|^2 \le \|x-y\|^2 + \kappa \|x - y - (Tx - Ty)\|^2 ,$$

where $\kappa = (1 - 2\lambda') < 1$, and we can assume also that $\kappa \ge 0$, so that $\kappa \in [0, 1)$.

The class of strictly pseudocontractive mappings includes the class of nonexpansive mappings. A nonexapansive mapping is strictly pseudocontractive mapping for $\kappa = 0$. A mapping T is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$, for all x, y in $D(T)$. A strictly pseudocontractive mapping is pseudocontractive mapping for $\kappa = 1$. The class of strictly pseudocontractive mappings falls between the class of nonexpansive mappings and the class of pseudocontractive mappings: Karaka and Yildirim [9], Thakur *et al.* [18, 19, 20, 21, 23], Yao *et al.* [28].

Inequality (2.1) is equivalent to

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \ge \lambda' \|(I-T)x - (I-T)y\|^2. \tag{2.2}$$

It can be seen from (2.2) that every strictly pseudocontractive mapping is L-Lipschitzian with $L \ge 1 + \frac{1}{\lambda'}$.

Mapping T is pseudocontractive [2, 10], if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 , \quad \forall\, x, y \in D(T) ; \tag{2.3}$$

and T is strongly pseudocontractive, if there exists a constant $\beta \in (0,1)$, $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \beta \|x-y\|^2 , \quad \forall\, x, y \in D(T).$$

Recall that T is strongly accretive, if there exists a constant $\delta \in (0,1)$, $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \ge \delta \|x-y\|^2 , \quad \forall\, x, y \in D(T).$$

In 2001, Xu and Ori [24] introduced the implicit iteration process (2.4) to approximate a common fixed point of a finite family of nonexpansive self mappings in Hilbert space. More accurately, for arbitrary chosen $x_0 \in K$, construct x_n by the formula

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n x_n , \quad n \ge 1 , \tag{2.4}$$

where $\{\alpha_n\}$ be a sequence in $(0,1)$ and $T_n = T_{n \bmod N}$.

Zeng and Yao [30] introduced an implicit iteration process (2.5) with perturbed mapping G, to approximate common fixed points of a finite family of nonexpansive mappings. For an arbitrary initial point $x_0 \in H$, the sequence $\{x_n\}$ is generated as follows:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)[T_n x_n - \lambda_n \mu G(T_n x_n)], \quad n \ge 1 , \tag{2.5}$$

where $G\colon H \to H$ is a κ-Lipschitzian and η-strongly monotone mapping for some constants $\kappa, \eta > 0$, $\mu \in \left(0, \frac{2\eta}{\kappa^2}\right)$ is a fixed number and $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset [0,1)$.

It is clear that if $\lambda_n = 0$, $n \geq 1$, then the implicit iteration scheme (2.5) reduces to the implicit iteration process (2.4).

Ceng *et al.* [3] introduced an implicit iteration process (2.6) with perturbed mapping G, for approximation of common fixed points of finite family of continuous pseudocontractive self-mapping. For an arbitrary chosen initial point $x_0 \in E$, the sequence $\{x_n\}$ is generated as follows:

$$x_n = \alpha_n(x_{n-1} - \lambda_n G(x_{n-1})) + (1-\alpha_n)T_n x_n, \quad n \geq 1, \tag{2.6}$$

where $G\colon E \to E$ be a mapping which is both λ'-strictly pseudocontractive and δ-strongly accretive with $\lambda' + \delta \geq 1$, and $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset [0,1)$.

Clearly, if $\lambda_n = 0$, for all $n \geq 1$, then the implicit iteration scheme (2.6) reduces to the implicit iteration process (2.4).

Motivated and inspired by the above described contribution, we propose a new implicit iteration scheme with perturbed mapping, to approximate fixed point of a finite family of κ_i-strictly pseudocontractive self-mappings $\{T_i\}_{i=1}^N$ as we will explain below.

Let E be a real Banach space and $G\colon E \to E$ be a perturbed mapping which is both λ'-strictly pseudocontractive and δ-strongly accretive with $\lambda' + \delta \geq 1$. For an arbitrary initial point $x_0 \in E$, the sequence $\{x_n\}$ is generated by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)\Big[\beta_n\left(x_n - \lambda G(x_n)\right) + \gamma_n \sum_{i=1}^{N} \mu_i T_i x_n\Big], \tag{2.7}$$

where $\{\mu_i\}_{i=1}^N$ is a sequence of weights satisfying $\sum_{i=1}^N \mu_i = 1$, $\{\alpha_n\} \subset (0,1), \{\gamma_n\} \subset (0,1]$, $\{\beta_n\} \subset [0,1)$ and $\lambda \in [0,1)$.

In this paper, we establish some strong and weak convergence theorems for a finite family of strictly pseudocontractive mappings using implicit iteration scheme (2.7). Last but not least, in Section 4 we illustrate our results on concrete examples and numerically compute the fixed point. The results obtained in this paper improve and extend the results of Xu and Ori [24], Ceng *et al.* [3], Chen *et al.* [5], Dewangan *et al.* [7, 8], Thakur *et al.* [22], Yang *et al.* [25], Yao *et al.* [26, 27, 29], Zeng and Yao [30], Zhou [31] and some other results in this direction.

To introduce our results, we need other definitions and results. In this respect, recall that the norm of Banach space E is said to be Gáteaux differentiable (or E is said to be smooth) if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.8}$$

exists for each x, y on the unit sphere $S = \{x \in E : \|x\| = 1\}$ of E. Moreover, if for each y in S the limit defined by (2.8) is uniformly attained for $x \in S$, we say that the norm of E is uniformly Gâteaux differentiable. The norm of E is said to be Fréchet differentiable if, for each $x \in S$, the limit (2.8) is attained uniformly for $y \in S$. The norm of E is said to be uniformly Fréchet differentiable (or is said to be uniformly smooth) if the limit (2.8) attained uniformly for $(x, y) \in S \times S$. We know that if E is smooth then the normalized duality mapping J is single valued and continuous from the strong topology to the weak* topology.

Definition 2.1 ([11])**.** A Banach space E is said to satisfy *Opial's condition* if whenever $\{x_n\}$ is a sequence in E which converges weakly to x, as $n \to \infty$, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \text{for all } \; y \in E, \; y \neq x.$$

Lemma 2.1 ([14])**.** *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty closed convex subset of E and $T\colon K \to K$ be a strictly pseudocontractive mapping in the terminology of Browder-Petryshyn. Then $(I - T)$ is demiclosed at zero, i.e. $\{x_n\} \in D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{(I - T)x_n\}$ converges strongly to 0, then $x - Tx = 0$.*

Lemma 2.2 ([13])**.** *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be three sequences of nonnegative real numbers satisfying the following condition:*

$$a_{n+1} \leq (1 + b_n)a_n + c_n \quad \textit{for all } n \geq n_0\,,$$

where n_0 is some nonnegative integer, $\sum_{n=0}^{\infty} b_n < \infty$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then

(i) *$\lim_{n\to\infty} a_n$ exists;*

(ii) *if, in addition, there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $a_n \to 0$ as $n \to \infty$.*

Lemma 2.3 ([16])**.** *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n \quad \textit{for all } \; n \geq 1.$$

If $\sum_{n=0}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.4 ([6])**.** *Let K be a nonempty closed convex subset of a smooth Banach space E and let $T\colon K \to K$ be a strictly pseudocontractive mapping. Then the fixed point set $F(T)$ is a closed convex subset of E.*

Lemma 2.5 ([3])**.** *Let E be a smooth Banach space and $G\colon E \to E$ be both λ'-strictly pseudocontractive and δ-strongly accretive with $\lambda' + \delta \geq 1$. Then $I - G$ is nonexpansive mapping.*

Proposition 2.1. *Let E be a smooth Banach space and $G\colon E \to E$ be both λ'-strictly pseudocontractive and δ-strongly accretive with $\lambda' + \delta \geq 1$. Then $S_\lambda = (I - \lambda G)\colon E \to E$ is a pseudocontractive mapping, for $0 \leq \lambda < 1$.*

Proof. Since $S_\lambda(x) = x - \lambda G(x)$, using Lemma 2.5 we have

$$\begin{aligned}
\langle S_\lambda(x) - S_\lambda(y), j(x-y)\rangle &= \langle x - \lambda G(x) - y + \lambda G(y), j(x-y)\rangle\,,\\
&= \langle (1-\lambda)x + \lambda(I-G)x - (1-\lambda)y - \lambda(I-G)y, j(x-y)\rangle\,,\\
&\leq (1-\lambda)\|x-y\|^2 + \lambda\,\langle (I-G)x - (I-G)y, j(x-y)\rangle\,,\\
&\leq (1-\lambda)\|x-y\|^2 + \lambda\,\|(I-G)x - (I-G)y\|\,\|x-y\|\,,\\
&\leq (1-\lambda)\|x-y\|^2 + \lambda\,\|x-y\|\,\|x-y\|\,,\\
&= \|x-y\|^2\,,
\end{aligned}$$

hence S_λ is a pseudocontractive mapping. □

Lemma 2.6 ([31]). *Let E be a smooth Banach space and K be a nonempty convex subset of E. $T_i\colon K \to K$ is a κ_i-strictly pseudocontractive mapping for some $0 < \kappa_i < 1$. Assume that $\{\mu_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \mu_i = 1$. Set $S = \sum_{i=1}^N \mu_i T_i$, then*

(i) *$S : K \to K$ is a κ-strictly pseudocontractive mapping with $\kappa = \min\{\kappa_i : 1 \le i \le N\}$.*

(ii) *If $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, then $F(S) = \mathcal{F}$.*

Lemma 2.7 ([4]). *If $J\colon E \to 2^{E^*}$ is a normalized duality mapping, then for all $x, y \in E$,*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \;\; \textit{for all} \;\; j(x+y) \in J(x+y).$$

Throughout the paper, we shall denote strong (respectively weak) convergence of the sequence $\{x_n\}$ to x by $x_n \to x$ (respectively $x_n \rightharpoonup x$) and for the weak ω-limit set of a sequence $\{x_n\}$ in E, we shall use the following notation

$$\omega_\omega(x_n) = \left\{x \in E; \left\{x_{n_j}\right\} \rightharpoonup x, \text{ for some subsequence } \{n_j\} \text{ of } \{n\}\right\}.$$

3. Weak and strong convergence theorems

Lemma 3.1. *Let E be a real smooth Banach space E, $G\colon E \to E$ be both λ'-strictly pseudocontractive and δ-strongly accretive with $\lambda' + \delta \ge 1$ and $T_i\colon E \to E$ be a finite family of κ_i-strictly pseudocontractive mappings, where $i \in \{1, 2, \dots, N\}$ such that $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences satisfying the conditions $0 < a \le \alpha_n \le b < 1$, $0 < a \le \gamma_n \le 1$, $\beta_n + \gamma_n = 1$ and $\sum_{n=1}^\infty \beta_n < \infty$. Let $\{x_n\}$ be a sequence generated by* (2.7). *Then*

(a) $\lim_{n\to\infty} \|x_n - p\|$ *exists, for all $p \in \mathcal{F}$;*

(b) $\lim_{n\to\infty} d(x_n, \mathcal{F})$ *exists, where* $d(x_n, \mathcal{F}) = \inf_{p\in\mathcal{F}} \|x_n - p\|$.

Proof. Let $S = \sum_{i=1}^N \mu_i T_i$, where $\mu_i > 0$ for all $1 \le \mu_i \le N$ such that $\sum_{i=1}^N \mu_i = 1$. Then by Lemma 2.6, S is a κ-strictly pseudocontractive mapping with $\kappa = \min\{\kappa_i : 1 \le i \le N\}$ and $F(S) = \mathcal{F}$.

Suppose $p \in \mathcal{F}$, and using Proposition 2.1 and Lemma 2.6, we have

$$\begin{aligned}
\|x_n - p\|^2 &= \langle \alpha_n x_{n-1} + (1-\alpha_n)[\beta_n(x_n - \lambda G(x_n)) + \gamma_n S x_n] - p, j(x_n - p)\rangle, \\
&= \alpha_n \langle x_{n-1} - p, j(x_n - p)\rangle + (1-\alpha_n)\langle [\beta_n S_\lambda(x_n) + \gamma_n S x_n] - p, j(x_n - p)\rangle, \\
&= \alpha_n \langle x_{n-1} - p, j(x_n - p)\rangle + (1-\alpha_n)\langle \beta_n(S_\lambda(x_n) - p) + \gamma_n(Sx_n - p), j(x_n - p)\rangle, \\
&= \alpha_n \langle x_{n-1} - p, j(x_n - p)\rangle + (1-\alpha_n)\beta_n \langle S_\lambda(x_n) - S_\lambda(p), j(x_n - p)\rangle \\
&\quad - (1-\alpha_n)\beta_n \langle \lambda G(p), j(x_n - p)\rangle + (1-\alpha_n)\gamma_n \langle (Sx_n - p), j(x_n - p)\rangle, \\
&\le \alpha_n \|x_{n-1} - p\| \|x_n - p\| + (1-\alpha_n)\beta_n \|x_n - p\|^2 \\
&\quad + (1-\alpha_n)\beta_n \lambda \|G(p)\| \|x_n - p\| + (1-\alpha_n)\gamma_n \|x_n - p\|^2 \\
&\quad - (1-\alpha_n)\gamma_n \kappa \|x_n - p - (Sx_n - p)\|^2, \\
&\le \alpha_n \|x_{n-1} - p\| \|x_n - p\| + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad + (1-\alpha_n)\beta_n \lambda \|G(p)\| \|x_n - p\|.
\end{aligned} \tag{3.1}$$

If $\|x_n - p\| = 0$, the result follows. Next, let $\|x_n - p\| > 0$. Then from (3.1), we have

$$\begin{aligned}\|x_n - p\| &\leq \|x_{n-1} - p\| + \frac{(1-\alpha_n)}{\alpha_n}\beta_n\lambda \|G(p)\|, \\ &\leq \|x_{n-1} - p\| + \frac{(1-a)}{a}\beta_n\lambda \|G(p)\|. \end{aligned} \tag{3.2}$$

Using condition $\sum_{n=1}^{\infty}\beta_n < \infty$ and by Lemma 2.3, we get that $\lim_{n\to\infty}\|x_n - p\|$ exists.

Since $\lim_{n\to\infty}\|x_n - p\|$ exists, so $\{x_n\}$ is bounded. Since G is $L-$Lipschitzian with $L \geq 1 + \frac{1}{\lambda'}$, so $\{G(x_n)\}$ is also bounded. Let $\|G(x_n)\| < M$, for all $n \geq 1$, for some $M > 0$. Then by equation (3.2)

$$\begin{aligned}\|x_n - p\| &\leq \|x_{n-1} - p\| + \frac{(1-a)}{a}\beta_n\lambda \|G(p) - G(x_{n-1}) + G(x_{n-1})\|, \\ &\leq \|x_{n-1} - p\| + \frac{(1-a)}{a}\beta_n\lambda \|G(p) - G(x_{n-1})\| + \frac{(1-a)}{a}\beta_n\lambda \|G(x_{n-1})\|, \\ &\leq \|x_{n-1} - p\| + \frac{(1-a)}{a}\beta_n\lambda L \|x_{n-1} - p\| + \frac{(1-a)}{a}\beta_n\lambda M, \\ &= \left[1 + \frac{(1-a)}{a}\beta_n\lambda L\right]\|x_{n-1} - p\| + \frac{(1-a)}{a}\beta_n\lambda M.\end{aligned}$$

Taking the infimum over all $p \in \mathcal{F}$, we have

$$d(x_n, \mathcal{F}) \leq \left[1 + \frac{(1-a)}{a}\beta_n\lambda L\right] d(x_{n-1}, \mathcal{F}) + \frac{(1-a)}{a}\beta_n\lambda M.$$

By applying Lemma 2.2, we have $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists.
This completes the proof. □

Lemma 3.2. *Let E be a real reflexive and smooth Banach space, G and T_i are as in Lemma* 3.1 *and all conditions of Lemma* 3.1 *are satisfied. Let $S = \sum_{i=1}^{N}\mu_i T_i \colon E \to E$ be a κ-strictly pseudocontractive mapping. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in E$ by* (2.7). *Then $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$.*

Proof. Since E is smooth, the normalized duality mapping J is single-valued. Also, since the mappings $T_i \colon E \to E$ are κ_i-strictly pseudocontractive for each $i \in \{1, 2, \ldots, N\}$, we deduce from (2.2) that

$$\langle (I - T_i)x - (I - T_i)y, j(x-y)\rangle \geq \kappa_i \|(I - T_i)x - (I - T_i)y\|^2,$$

for all $x, y \in E$, where $\kappa_i \in (0,1)$, for $i \in \{1, 2, \ldots, N\}$.

Put $\kappa = \min_{1\leq i\leq N}\{\kappa_i\}$. Then $\kappa \in (0,1)$ and for each $i \in \{1, 2, \ldots, N\}$

$$\langle (I - T_i)x - (I - T_i)y, j(x-y)\rangle \geq \kappa \|(I - T_i)x - (I - T_i)y\|^2,$$

for all $x, y \in E$. Since $S = \sum_{i=1}^{N}\mu_i T_i \colon E \to E$ is a κ-strictly pseudocontractive mapping with $\kappa = \min\{\kappa_i : 1 \leq i \leq N\}$, so we have

$$\langle (I - S)x - (I - S)y, j(x-y)\rangle \geq \kappa \|(I - S)x - (I - S)y\|^2, \text{ for all } x, y \in E. \tag{3.3}$$

Moreover, it is easy to see that each T_i $(1 \leq i \leq N)$ is β-Lipschitzian with $\beta \geq 1 + \frac{1}{\kappa}$. Hence all the conditions of Lemma 3.1 are satisfied. Thus, by Lemma 3.1 we conclude that

$\lim_{n\to\infty} \|x_n - p\|$ exists, for all $p \in \mathcal{F}$. Consequently, $\{x_n\}$ and $\{G(x_n)\}$ are bounded, due to the fact that G is L-Lipschitzian with $L \geq 1 + \frac{1}{\lambda'}$. Using equation (2.7), we have

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)\left[\beta_n\left(x_n - \lambda G(x_n)\right) + \gamma_n S x_n\right],$$

and

$$x_{n-1} = \frac{1}{\alpha_n} x_n + \left(1 - \frac{1}{\alpha_n}\right)\left[\beta_n\left(x_n - \lambda G(x_n)\right) + \gamma_n S x_n\right], \tag{3.4}$$

then

$$\begin{aligned}
x_n - x_{n-1} &= \left(1 - \frac{1}{\alpha_n}\right) x_n - \left(1 - \frac{1}{\alpha_n}\right)\left[\beta_n\left(x_n - \lambda G(x_n)\right) + \gamma_n S x_n\right],\\
&= \left(1 - \frac{1}{\alpha_n}\right)(1-\beta_n) x_n - \left(1 - \frac{1}{\alpha_n}\right)\gamma_n S x_n + \left(1 - \frac{1}{\alpha_n}\right)\beta_n \lambda G(x_n),\\
&= \left(1 - \frac{1}{\alpha_n}\right)\gamma_n (x_n - S x_n) + \left(1 - \frac{1}{\alpha_n}\right)\beta_n \lambda G(x_n).
\end{aligned}$$

Now,

$$\begin{aligned}
\langle x_n - x_{n-1}, j(x_n - p)\rangle &= \left(1 - \frac{1}{\alpha_n}\right)\gamma_n \langle x_n - S x_n, j(x_n - p)\rangle\\
&\quad + \left(1 - \frac{1}{\alpha_n}\right)\beta_n \lambda \langle G(x_n), j(x_n - p)\rangle.
\end{aligned} \tag{3.5}$$

By Lemma 2.7 and using (3.3) and (3.5), we have

$$\begin{aligned}
\|x_n - p\|^2 &= \|x_{n-1} - p + x_n - x_{n-1}\|^2,\\
&\leq \|x_{n-1} - p\|^2 + 2\langle x_n - x_{n-1}, j(x_n - p)\rangle,\\
&\leq \|x_{n-1} - p\|^2 + 2\left(1 - \frac{1}{\alpha_n}\right)\gamma_n \langle x_n - S x_n - (p - Sp), j(x_n - p)\rangle\\
&\quad + 2\left(1 - \frac{1}{\alpha_n}\right)\beta_n \lambda \langle G(x_n), j(x_n - p)\rangle,\\
&\leq \|x_{n-1} - p\|^2 - 2\left(\frac{1-\alpha_n}{\alpha_n}\right)\gamma_n \langle (I - S)x_n - (I - S)p, j(x_n - p)\rangle\\
&\quad - 2\left(\frac{1-\alpha_n}{\alpha_n}\right)\beta_n \lambda \langle G(x_n), j(x_n - p)\rangle,\\
&\leq \|x_{n-1} - p\|^2 - 2\left(\frac{1-\alpha_n}{\alpha_n}\right)\gamma_n \kappa \|x_n - S x_n\|^2\\
&\quad + 2\left(\frac{1-\alpha_n}{\alpha_n}\right)\beta_n \lambda \|G(x_n)\| \|x_n - p\|.
\end{aligned} \tag{3.6}$$

Thus from (3.6) and using conditions $0 < a \leq \alpha_n \leq b < 1$, $0 < a \leq \gamma_n \leq 1$, we have

$$\begin{aligned} 2\left(\frac{1-b}{b}\right) a\kappa \left\|x_n - Sx_n\right\|^2 &\leq 2\left(\frac{1-\alpha_n}{\alpha_n}\right)\gamma_n\kappa\left\|x_n - Sx_n\right\|^2, \\ &\leq \left\|x_{n-1}-p\right\|^2 - \left\|x_n - p\right\|^2 \\ &\quad + 2\left(\frac{1-\alpha_n}{\alpha_n}\right)\beta_n\lambda\left\|G(x_n)\right\|\left\|x_n - p\right\|, \\ &\leq \left\|x_{n-1}-p\right\|^2 - \left\|x_n - p\right\|^2 \\ &\quad + 2\left(\frac{1-a}{a}\right)\beta_n\lambda M\left\|x_n - p\right\|. \end{aligned} \tag{3.7}$$

Since $\lim_{n\to\infty}\|x_n - p\|$ exists and $\sum_{n=1}^{\infty}\beta_n < \infty$, it follows that

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{3.8}$$

This completes the proof. □

We now establish a weak convergence result:

Theorem 3.1. *Let E be a uniformly convex Banach space satisfying Opial's condition. Let G, T_i and S are as in Lemma* 3.2 *and all conditions of Lemma* 3.2 *are satisfied. Then the sequence $\{x_n\}$ generated by* (2.7) *converges weakly to a member of $\mathcal{F}$.*

Proof. Every uniformly convex Banach space is reflexive, also $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in \mathcal{F}$ by Lemma 3.1, thus $\{x_n\}$ is bounded. Hence $\{x_n\}$ is a bounded sequence in reflexive space, therfore by Eberlein's theorem $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to some $q \in E$. Now, we prove that $\{x_n\}$ has a unique weak subsequential limit in $\mathcal{F}$. To prove this, let x^* be the weak limit of another subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$. By Lemma 3.2,

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0$$

and $I - S$ is demiclosed with respect to zero by Lemma 2.1 and so we obtain $q \in \mathcal{F}$. Again, in the same way, we can prove that $x^* \in \mathcal{F}$. Since E satisfies Opial's condition, it follows from a standard argument that $q = x^*$. Thus $\{x_n\}$ converges weakly to a member of $\mathcal{F}$, and the proof is now complete. □

Theorem 3.2. *Let E, G, T_i and S are as in Lemma* 3.1 *and all conditions of Lemma* 3.1 *are satisfied. Then the sequence $\{x_n\}$ generated by* (2.7) *converges strongly to a member of $\mathcal{F}$ if and only if* $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Proof. The necessity of condition is obvious. Thus, we will only prove the sufficiency.
Let $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Then by Lemma 3.1, we have

$$\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0\,.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in E.
From equation (3.2), we have

$$\begin{aligned}
\|x_{n+m}-p\| &\le \|x_{n+m-1}-p\| + \frac{(1-a)}{a}\beta_{n+m}\lambda \|G(p)\|\,,\\
&\le \|x_{n+m-2}-p\| + \frac{(1-a)}{a}\beta_{n+m-1}\lambda \|G(p)\| + \frac{(1-a)}{a}\beta_{n+m}\lambda \|G(p)\|\,,\\
&\vdots\\
&\le \|x_n-p\| + \left[\frac{(1-a)}{a}\beta_{n+1}\lambda + \frac{(1-a)}{a}\beta_{n+2}\lambda + \cdots + \frac{(1-a)}{a}\beta_{n+m}\lambda\right]\|G(p)\|\,,\\
&\le \|x_n-p\| + \frac{(1-a)}{a}\lambda \|G(p)\| \sum_{i=n+1}^{n+m}\beta_i\,. \qquad (3.9)
\end{aligned}$$

Now, using (3.9), we have

$$\begin{aligned}
\|x_{n+m}-x_n\| &\le \|x_{n+m}-p\| + \|x_n-p\|\,,\\
&\le 2\|x_n-p\| + \frac{(1-a)}{a}\lambda\|G(p)\|\sum_{i=n+1}^{n+m}\beta_i,\\
&\le 2\|x_n-p\| + \frac{(1-a)}{a}\lambda\|G(p)-G(x_n)\|\sum_{i=n+1}^{n+m}\beta_i + \frac{(1-a)}{a}\lambda\|G(x_n)\|\sum_{i=n+1}^{n+m}\beta_i,\\
&\le \left(2+\frac{(1-a)}{a}\lambda L\sum_{i=n+1}^{n+m}\beta_i\right)\|x_n-p\| + \frac{(1-a)}{a}\lambda M\sum_{i=n+1}^{n+m}\beta_i\,.
\end{aligned}$$

Taking the infimum over all $p\in\mathcal{F}$, we obtain

$$\begin{aligned}
\|x_{n+m}-x_n\| &\le \left(2+\frac{(1-a)}{a}\lambda L\sum_{i=n+1}^{n+m}\beta_i\right)d(x_n,\mathcal{F}) + \frac{(1-a)}{a}\lambda M\sum_{i=n+1}^{n+m}\beta_i\\
&\to 0 \ \text{ as } n\to\infty\,.
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since E is complete, therefore $\{x_n\}$ converges to some $q\in E$. Since S is a strictly pseudocontractive mapping, by Lemma 2.4, $F(S)$ is closed. Again by Lemma 2.6, we have $F(S)=\mathcal{F}$, so $\mathcal{F}$ is closed and hence $q\in\mathcal{F}$, and his completes the proof. □

We recall the following definitions:

Definition 3.1 ([15])**.** A mapping $T\colon E\to E$ with $F(T)\neq\emptyset$ is said to satisfy condition (A) on E if there exists a nondecreasing function $f\colon[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(r)>r$ for all $r\in(0,\infty)$ such that for all $x\in E$,

$$\|x-Tx\|\ge f(d(x,F(T)))\,,$$

where $d(x,F(T))=\inf_{x^*\in F(T)}\|x-x^*\|$.

Definition 3.2 ([17])**.** A finite family $T_i\colon E\to E$ of self mappings, where $i=\{1,2,\ldots,N\}$ with $\mathcal{F}=\bigcap_{i=1}^N F(T_i)\neq\emptyset$ is said to satisfy condition (BS) on E if there exist f and d as in Definition 3.1, such that

$$\|x-Sx\|\ge f(d(x,\mathcal{F}))\quad\text{for all } x\in E$$

where $S = \sum_{i=1}^{N} \mu_i T_i$ and $\{\mu_i\}_{i=1}^{N}$ is a sequence of positive number such that $\sum_{i=1}^{N} \mu_i = 1$.

Theorem 3.3. *Let E, G, T_i and S are as in Lemma 3.2 and all conditions of Lemma 3.2 are satisfied and let finite family of T_i satisfies condition (BS). Then the sequence $\{x_n\}$ generated by (2.7) converges strongly to a member of $\mathcal{F}$.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists and $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists, also by Lemma 3.2, we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0\,.$$

Since finite family of T_i satisfies *condition* (BS), so we have

$$\lim_{n\to\infty} f(d(x_n, \mathcal{F})) = 0\,.$$

By the nature of f and the fact that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists, we have

$$\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0\,,$$

thus there exists a sequence say $\{x_j^*\}$ in $\mathcal{F}$ and subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\left\|x_{n_j} - x_j^*\right\| \le \frac{1}{2^j}\,, \quad \text{for } j \ge 1\,.$$

From (3.2), we have

$$\left\|x_{n_{j+1}} - x_j^*\right\| \le \left\|x_{n_j} - x_j^*\right\| + \frac{(1-a)}{a}\beta_{n_{j+1}}\lambda \|G(p)\|$$

This implies that

$$\left\|x_{j+1}^* - x_j^*\right\| \le \left\|x_{j+1}^* - x_{n_{j+1}}\right\| + \left\|x_{n_{j+1}} - x_j^*\right\| \le \frac{1}{2^{(j+1)}} + \frac{1}{2^j} + \frac{(1-a)}{a}\beta_{n_{j+1}}\lambda \|G(p)\|$$

Hence $\{x_j^*\}$ is a Cauchy sequence and so converges to some x^* in K. Since $\mathcal{F}$ is closed, x^* is in $\mathcal{F}$ and since $\lim_{n\to\infty} \|x_n - x^*\|$ exists. $\{x_n\}$ converges strongly to x^*.
This completes the proof. □

Let E be a Banach space. A mapping $T\colon E \to E$ is said to be semicompact, if for any bounded sequence $\{x_n\}$ in E such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to x^* \in K$ as $j \to +\infty$.

We now establish a strong convergence result using semicompact condition.

Theorem 3.4. *Let E, G, T_i and S are as in Lemma 3.2 and all conditions of Lemma 3.2 are satisfied and let $S = \sum_{i=1}^{N} \mu_i T_i$ be semicompact. Then the sequence $\{x_n\}$ generated by (2.7) converges strongly to a member of $\mathcal{F}$.*

Proof. Since $S = \sum_{i=1}^{N} \mu_i T_i$ is semicompact, we see that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x^*$. Notice that

$$\left\|x^* - \sum_{i=1}^{N} \mu_i T_i x^*\right\| \le \|x^* - x_{n_k}\| + \left\|x_{n_k} - \sum_{i=1}^{N} \mu_i T_i x_{n_k}\right\| + \left\|\sum_{i=1}^{N} \mu_i T_i x_{n_k} - \sum_{i=1}^{N} \mu_i T_i x^*\right\|$$

Since $S=\sum_{i=1}^N \mu_i T_i$ is Lipschitz continuous, we see from (3.8) that $x^* \in F(\sum_{i=1}^N \mu_i T_i)=\mathcal{F}$. From Lemma 3.1, we know that $\lim_{n\to\infty}\|x_n-p\|$ exists for each $p\in\mathcal{F}$. In view of $x_{n_k}\to x^*$, we find that

$$\lim_{n\to\infty}\|x_n-x^*\|=0.$$

This completes the proof. □

4. **Numerical experiments**

We now consider an example to illustrate the theoretical result:

Example 4.1. Let $\mathbb{R}$ denote the set of real numbers with the usual norm. Let $T_m\colon \mathbb{R}\to\mathbb{R}$, $m=1,2,\dots,N$ be a finite family of mappings defined by $T_m x=-2mx$.
Then

$$|x-T_m x-(y-T_m y)|^2=(2m+1)^2|x-y|^2\,,$$

and

$$\langle x-T_m x-(y-T_m y),\, x-y\rangle=(2m+1)|x-y|^2\,.$$

Then by (2.2), each T_m is strictly pseudocontractive mapping with unique fixed point $x^*=0$.

Let $\mu_r=\binom{N-1}{r-1}\frac{(N-1)^{r-1}}{N^{N-1}}$, clearly $\sum_{r=1}^N \mu_r=1$, then by Lemma 2.6 we see that $S=\sum_{r=1}^N \mu_r T_r$ is strictly pseudocontractive mapping and $\mathcal{F}=\bigcap_{r=1}^N F(T_r)=F(S)=\{0\}$. Now,

$$Sx=\sum_{r=1}^N \mu_r T_r x=-\sum_{r=1}^N 2r\mu_r\, x=-\left(2\sum_{r=1}^N r\binom{N-1}{r-1}\frac{(N-1)^{r-1}}{N^{N-1}}\right)x\,. \tag{4.1}$$

Set $G=I$ identity mapping, and let $\{x_n\}$ be the sequence defined by (2.7), then

$$x_n=\alpha_n x_{n-1}+(1-\alpha_n)\left[\beta_n(1-\lambda)x_n+\gamma_n S x_n\right]\,,$$

using (4.1), we have

$$x_n=\alpha_n x_{n-1}+(1-\alpha_n)\left[\beta_n(1-\lambda)-\gamma_n\left(2\sum_{r=1}^N r\binom{N-1}{r-1}\frac{(N-1)^{r-1}}{N^{N-1}}\right)\right]x_n\,,$$

after simplification, we get

$$x_n=\frac{\alpha_n}{\left[1-(1-\alpha_n)\left(\beta_n(1-\lambda)-\gamma_n\left(2\sum_{r=1}^N r\binom{N-1}{r-1}\frac{(N-1)^{r-1}}{N^{N-1}}\right)\right)\right]}\,x_{n-1}\,.$$

In order to examine the influence of parameters involved in the algorithm (2.7), we take the following set of parameters:

i) $\alpha_n=\dfrac{1}{n+4},\quad \beta_n=\dfrac{1}{(n+2)^3},$

ii) $\alpha_n=\dfrac{1}{3n+5},\quad \beta_n=\dfrac{1}{(2n+1)^6},$

iii) $\alpha_n=\dfrac{1}{2n+9},\quad \beta_n=\dfrac{1}{(n+4)^{5/2}},$

iv) $\alpha_n=\dfrac{1}{7n+8},\quad \beta_n=\dfrac{1}{(3n+2)^5},$

v) $\alpha_n=\dfrac{1}{5n+1},\quad \beta_n=\dfrac{1}{(n+1)^{7/2}}.$

Using the above set of parameters, we now examine influence of number of members in the finite family. We set initial point $x_0 = 30$ and take $\lambda = 0.5$. The stopping criterion is $\|x_n - x^*\| \leq 10^{-15}$. The respective number of iterations for different values of N are reported in Table-1.

Next, we tested the algorithm for different initial points and different set of parameters. The parameters λ, N are fixed to 0.5 and 20 respectively. Each iteration starts with a particular chosen x_0 and stops whenever $\|x_n - x^*\| \leq 10^{-15}$. Respective numbers of iterations are given in Table-2.

Now, we test the algorithm for influence of λ. Set $x_0 = 30$, $N = 20$ and the stopping criterion is $\|x_n - x^*\| \leq 10^{-15}$. Values of λ are chosen from $(0, 1)$. Findings are reported in Table-3.

Table-1 indicates that the new algorithm is quite efficient. Table-2 shows that the algorithm is very stable and effective no matter what initial point is chosen. Table-3 shows that the convergence is oblivious to the choice of λ.

5. **Conclusion**

In this part, we proposed a new implicit iteration scheme, with perturbed mapping, to approximate fixed points of finite families of strictly pseudocontractive self-mappings. We established some strong and weak convergence theorems for the iterates of a finite family of strictly pseudocontractive mappings using our implicit iteration scheme. We illustrated our results on concrete examples and numerically compute the fixed point. The results obtained in this part improve and extend the results of Xu and Ori [24], Ceng *et al.* [3], Chen *et al.* [5], Yang *et al.* [25], Zeng and Yao [30], Zhou [31] and some other results in this direction.

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Table 1. The influence of number of members in the family

Parameters			Number of iterations for different values of N					
			$N=5$	$N=10$	$N=50$	$N=100$	$N=500$	$N=1000$
$\alpha_n=\frac{1}{n+4}$	$\beta_n=\frac{1}{(n+2)^3}$	$\gamma_n=1-\beta_n$	9	8	6	6	5	5
$\alpha_n=\frac{1}{3n+5}$	$\beta_n=\frac{1}{(2n+1)^6}$	$\gamma_n=1-\beta_n$	8	7	6	5	5	4
$\alpha_n=\frac{1}{2n+9}$	$\beta_n=\frac{1}{(n+4)^{5/2}}$	$\gamma_n=1-\beta_n$	8	7	6	5	5	4
$\alpha_n=\frac{1}{7n+8}$	$\beta_n=\frac{1}{(3n+2)^5}$	$\gamma_n=1-\beta_n$	7	6	5	5	4	4
$\alpha_n=\frac{1}{5n+1}$	$\beta_n=\frac{1}{(n+1)^{7/2}}$	$\gamma_n=1-\beta_n$	8	7	6	5	5	4

Table 2. The influence of initial point

Parameters			Number of iterations for different initial points					
			$x_0=-50$	$x_0=-25$	$x_0=-5$	$x_0=5$	$x_0=25$	$x_0=50$
$\alpha_n=\frac{1}{n+4}$	$\beta_n=\frac{1}{(n+2)^3}$	$\gamma_n=1-\beta_n$	7	7	7	7	7	7
$\alpha_n=\frac{1}{3n+5}$	$\beta_n=\frac{1}{(2n+1)^6}$	$\gamma_n=1-\beta_n$	7	7	6	6	7	7
$\alpha_n=\frac{1}{2n+9}$	$\beta_n=\frac{1}{(n+4)^{5/2}}$	$\gamma_n=1-\beta_n$	7	6	6	6	6	7
$\alpha_n=\frac{1}{7n+8}$	$\beta_n=\frac{1}{(3n+2)^5}$	$\gamma_n=1-\beta_n$	6	6	6	6	6	6
$\alpha_n=\frac{1}{5n+1}$	$\beta_n=\frac{1}{(n+1)^{7/2}}$	$\gamma_n=1-\beta_n$	7	6	6	6	6	7

Table 3. The influence of parameter λ

Parameters			Number of iterations for different value of λ					
			$\lambda=0.90$	$\lambda=0.75$	$\lambda=0.60$	$\lambda=0.45$	$\lambda=0.30$	$\lambda=0.15$
$\alpha_n=\frac{1}{n+4}$	$\beta_n=\frac{1}{(n+2)^3}$	$\gamma_n=1-\beta_n$	7	7	7	7	7	7
$\alpha_n=\frac{1}{3n+5}$	$\beta_n=\frac{1}{(2n+1)^6}$	$\gamma_n=1-\beta_n$	7	7	7	7	7	7
$\alpha_n=\frac{1}{2n+9}$	$\beta_n=\frac{1}{(n+4)^{5/2}}$	$\gamma_n=1-\beta_n$	6	6	6	6	6	6
$\alpha_n=\frac{1}{7n+8}$	$\beta_n=\frac{1}{(3n+2)^5}$	$\gamma_n=1-\beta_n$	6	6	6	6	6	6
$\alpha_n=\frac{1}{5n+1}$	$\beta_n=\frac{1}{(n+1)^{7/2}}$	$\gamma_n=1-\beta_n$	6	6	6	6	6	6