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# GEOMETRIC PROPERTIES OF RELIABILITY POLYNOMIALS

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*Geometric modeling of multivariate reliability polynomials is based on algebraic hypersurfaces, constant level sets, rulings etc. The solved basic problems are: (i) find the reliability polynomial using the Maple and Matlab software environment; (ii) find restrictions of reliability polynomial via equi-reliable components; (iii) how should the reliability components linearly depend on time, so that the reliability of the system be linear in time? The main goal of the paper is to find geometric methods for analysing the reliability of electric systems used inside aircrafts.*

**Keywords:** reliability polynomial; ruled hypersurfaces, electric systems inside aircrafts.
**MSC2010:** 60K10, 62N05, 90B25.

## 1. Introduction

During the decades following the war, many research laboratories and universities developed and initiated programs to study life testing and reliability problems [2, 3]. Numerous such research topics focus on the study of different types of reliability systems [4, 5, 6], like serial, parallel, serial-parallel, parallel-serial, and complex, which have considerable impact on different life fields [2]-[22]. Since there exist different important available systems, the researchers attempted to find more than one method to solve these complex systems, and determine the optimal ones [2, 5, 9].

In the present work, we change the classical view, by trying to get information from the differential geometry naturally related to the stochastic models. Of particular interest is the study of reliability hypersurface and establishing the number of straight lines situated on this set. For further ideas, see [1].

## 2. Some definitions and basic terminology

We shall present first the concepts in network topology and in graph theory which are needed to calculate the network reliability [2]-[15].

**Definition 2.1.** *A graph $G = (V, E)$, where $V$ is the set of vertices (or nodes) and $E$ the set of edges (or arcs), is called a network.*

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**The network model**. We describe our system as a directed network consisting of nodes and arcs, as illustrated in Fig. 1. One node is considered as the *source* (node $A$ in the figure), and a second node is considered as a *sink* (node $D$). Each component of the network is identified as an arc passing from one node to another. The arcs are numbered for identification. A *failure of a component* is equivalent to an arc being removed or cut out from the network. The system is *successful* if there exists a valid path from the source to the sink. The system is said to be *failed* if no such path exists. The *reliability of the system* is the probability that there exist one or more successful paths from the source to the sink [11, 12].

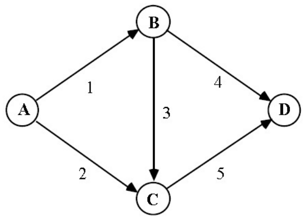


Fig. 1. A bridge network.

**Definition 2.2.** *A set of components is called a cut if, when all the components in this set fail, the system will fail, even if all other components are successful.*

**Definition 2.3.** *A cut, such that any removal of one component from it causes the resulting set do not be a cut, is called a minimal cut.*

The set of all components is a cut. In the network a minimal cut breaks all simple paths from the source to the sink. In Fig. 1, we observe that the minimal cuts are: $\{1, 2\}$, $\{1, 5\}$, $\{2, 3, 4\}$, and $\{4, 5\}$.

## 3. Complex reliability systems (network model)

We introduce a graphical network model in which it is possible to determine whether a system is working correctly by determining whether a successful path exists in the system. The system fails when no such path exists.

The system in Fig. 2 cannot be split into a group of series and parallel systems.

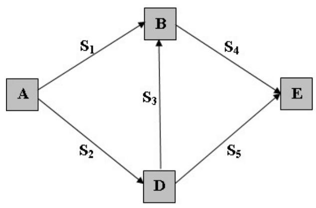


Fig. 2. A complex system (network model).

This is primarily due to the fact that the components $A$ and $D$ each allow two paths emerging from them, whereas $B$ has only one; $S_1$, $S_2$, $S_3$, $S_4$ and $S_5$ are called subsystems or arcs.

### 3.1. Minimal cut method

There exist several methods for obtaining the reliability of a complex system, as, for example, *minimal cut method*. The minimal cut method is proper for systems which are connected in the form of a bridge. When we apply this method to the system in Fig. 2, we should pursue the following steps:

a) we enumerate all the minimal cut-sets in the system;

b) the failure of all components in a minimal cut-set causes system failure;
c) this implies *parallel* connections among these components;
d) each minimal cut set determines the system failure;
e) this implies *series* connections among the minimal cut sets;
f) we draw an equivalent system and use the *parallel/seies method* to compute the system reliability.

**Theorem 3.1.** *If $S_1, S_2, S_3, S_4, S_5$ are arcs (paths) in a bridge system (fig 2), then the reliability $R_{Mc}(t)$ of all system is*

$$R_{Mc}(t) = R_1(t)R_4(t) + R_2(t)R_5(t) + R_2(t)R_3(t)R_4(t) - R_1(t)R_2(t)R_3(t)R_4(t) \quad (3.1)$$

$$-R_1(t)R_2(t)R_4(t)R_5(t) - R_2(t)R_3(t)R_4(t)R_5(t) + R_1(t)R_2(t)R_3(t)R_4(t)R_5(t).$$

*Proof.* By using minimal cut method, we have

$$\text{Minimal cut-set} = \{(S_1, S_2), (S_4, S_5), (S_2, S_4), (S_1, S_3, S_5)\},$$

and then Fig. 2 can be replaced by Fig. 3, which will represent the reliability of a parallel-series system [4], as follows:

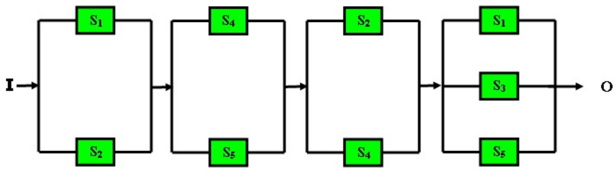


FIG. 3. A parallel-series system.

We shall assume that $R_i(t)$ represents the reliability of the $S_i$-th component (probability that the component $S_i$ to be functional on whole interval $[0, t]$) in a cut set $M_{Cj}$, $j \in \{1, 2, 3, 4\}$. Therefore, there exist four possibilities of cut sets, and their representation shows as a *parallel-series* system, as shown in Fig. 3, and any failure which occurs in a cut set that will cause the system failure.

A symbolic expression for reliability of such a complex system is evaluated by applying Boolean Function (BF) Technique. The probability that each cut set $M_{Cj}$ fails is

$$M_{C1}(t) = 1 - [(1 - R_1(t))(1 - R_2(t))]$$

$$M_{C2}(t) = 1 - [(1 - R_4(t))(1 - R_5(t))]$$

$$M_{C3}(t) = 1 - [(1 - R_2(t))(1 - R_4(t))]$$

$$M_{C4}(t) = 1 - [(1 - R_1(t))(1 - R_3(t))(1 - R_5(t))].$$

That is why, the reliability of the system is

$$R_{Mc}(t) = M_{C1}(t)M_{C2}(t)M_{C3}(t)M_{C4}(t),$$

where the computations have probabilistic-boolean sense, i.e., $R_i^2(t)$ is formally replaced by $R_i(t)$. We find the expression (3.1) which is the pullback of the reliability polynomial (see also [4, 10] and the Section 4). □

## 4. **Reliability-Geometry transfer**

The basic ingredient is a vector of probabilities $(R_1(t), R_2(t), R_3(t), R_4(t), R_5(t))$ and associated relation with $R_{Mc}(t)$. From the pullback we go to the polynomial equation and conversely.

Here, geometric modeling means: (i) to change probability functions into variables whose values are in the interval $[0, 1]$, and then to variables in the interval $(-\infty, \infty)$, (ii) to analyse and identify a body of techniques that can model certain classes of piecewise parametric surfaces, subject to particular conditions of shape and smoothness, and (iii) to come back into the context of probability variables, reinterpreting the geometric results; the stochastic results are read within the (uni-, bi-,..., and six-dimensional) unit cube $[0, 1], [0, 1]^2$, ..., $[0, 1]^6$.

Via geometric interpretation, we find new properties of any reliability polynomial.

### 4.1. **Multivariate reliability polynomial**

The real multivariate polynomial

$$R = R_1R_4 + R_2R_5 + R_2R_3R_4 - R_1R_2R_3R_4 \tag{3.2}$$

$$-R_1R_2R_4R_5 - R_2R_3R_4R_5 + R_1R_2R_3R_4R_5$$

is an extension of the reliability polynomial. It is linear affine in each variable.

The critical points of the polynomial (3.2) determine a *variety* described by the system

$$\frac{\partial R}{\partial R_1} = 0, \frac{\partial R}{\partial R_2} = 0, \frac{\partial R}{\partial R_3} = 0, \frac{\partial R}{\partial R_4} = 0, \frac{\partial R}{\partial R_5} = 0.$$

To solve this system, we can use Maple or Matlab procedures

$$\begin{aligned} \text{solve}(\{ \quad & x_4 - x_2x_3x_4 - x_2x_4x_5 + x_2x_3x_4x_5 = 0, \\ & x_5 + x_3x_4 - x_1x_3x_4 - x_1x_4x_5 - x_3x_4x_5 + x_1x_3x_4x_5 = 0, \\ & x_2x_4 - x_1x_2x_4 - x_2x_4x_5 + x_1x_2x_4x_5 = 0, \\ & x_1 + x_2x_3 - x_1x_2x_3 - x_1x_2x_5 - x_2x_3x_5 + x_1x_2x_3x_5 = 0, \\ & x2 - x_1x_2x_4 - x_2x_3x_4 + x_1x_2x_3x_4 = 0 \\ & \}, \ \ [x_1, x_2, x_3, x_4, x_5]) \end{aligned}$$

which lead to, e.g., the obvious solutions $\{(0, 0, x_3, 0, 0) \mid x_3 \in \mathbb{R}\}$, which proves the nontrivial compatibility of the system.

**Theorem 4.1.** *All critical points of multivariate polynomial* (3.2) *are saddle points.*

*Proof.* We compute the second order differential, which turns out to be non-definite.

It follows that the extrema points of interest are only on the boundary of a compact set, as for example $[0, 1]^6$. Consequently the significant optimization problems involving the previous polynomial are of the type min max, max min or optimizations with constraints. Particularly, we can find solutions in the 6-dimensional interval $[0, 1]^6$. □

An example for locating such solutions, using Maple, is described below:

> with(Optimization);

> Minimize$(x1 * x4 + x2 * x5 + x2 * x3 * x4 - x1 * x2 * x3 * x4 - x1 * x2 * x4 * x5$
$-x2 * x3 * x4 * x5 + x1 * x2 * x3 * x4 * x5, 0. <= x1 - 2 * x2,$
assume = nonnegative);

$$[0., [x1 = 0.684438040345821397e-1, x2 = 0., x3 = 1., x4 = 0.,$$
$$x5 = .913400576368876061]];$$

$$[0.0, [x1 = 0.0684438040345821397, x2 = 0.0, x3 = 1.0, x4 = 0.0,$$
$$x5 = 0.913400576368876061]$$

$$> \text{Maximize}(x1 * x4 + x2 * x5 + x2 * x3 * x4 - x1 * x2 * x3 * x4$$
$$-x1 * x2 * x4 * x5 - x2 * x3 * x4 * x5 + x1 * x2 * x3 * x4 * x5,$$
$$x1 = 0..1, x2 = 0..1, x3 = 0..1, x4 = 0..1, x5 = 0..1, \text{location})$$

### 4.2. Restrictions of reliability polynomial via equi-reliable components

**Theorem 4.2.** *There are* $1, 15, 50, 60, 120$ *diagonal polynomials induced by* (3.2), *corresponding to* $1, ..., 5$ *variables.*

*Proof.* These restrictions are counted as:

1 variable: 1 polynomial.

2 variables: if one variable is $x$, and the other four are $y$, we have five polynomials; if two variables are $x$ and the other three are $y$, we have $C_5^2 = 10$ polynomials.

3 variables: if one variable is $x$, another is $y$ and the other three are $z$, the we have 20 polynomials; if one variable is $x$, other two are $y$, and other two $z$, then we have 30 polynomials.

4 variables: if one variable is $x$, another is $y$, another is $z$ and the other two are $w$, then we have 60 polynomials.

5 variables: the number of polynomials is permutations of 5, i.e. 120.

For example, if we take $R_1 = R_2 = ... = R_5 = x$, with independent identical units, we get one diagonal (univariate) polynomial

$$y = 2x^2 + x^3 - 3x^4 + x^5. \tag{4.1}$$

□

**Proposition 4.1.** *The graph of the restriction of the polynomial*

$$y = 2x^2 + x^3 - 3x^4 + x^5$$

*to* $[0, 1]$ *looks like "standard logistic sigmoid function graph" and particularly like "stress-strain curve for low-carbon steel" .*

In the case of two variables, the following particular cases appear:

(i) the substitutions $R_1 = x, R_2 = R_3 = R_4 = R_5 = y$ produce the diagonal polynomial (there are 5 such polynomials):

$$P(x, y) = xy - 2xy^3 + xy^4 + y^2 + y^3 - y^4;$$

(ii) the substitutions $R_1 = R_2 = x, R_3 = R_4 = R_5 = y$ produce the diagonal polynomial (there are $C_5^2 = 10$ such polynomials)

$$Q(x, y) = x^2y^3 - 2x^2y^2 - xy^3 + xy^2 + 2xy.$$

### 4.3. Straight lines contained in the reliability hypersurface

The graph of the multivariate polynomial (3.2) is a hypersurface in $R^6$, called *"reliability hypersurface"*. Our aim is to solve the following problem:

**Problem.** How should the components $R_i$ linearly depend on time, so that the reliability of the system be linear in time? Geometrically, this means to find all the straight lines which are contained in the "reliability hypersurface".

**Theorem 4.3.** *The family of straight lines in the "reliability hypersurface" depends on at least five and at most six parameters.*

*Proof.* Let us find the number of essential parameters such that the family of straight lines

$$R_1 = a_1t + b_1, R_2 = a_2t + b_2, R_3 = a_3t + b_3,$$

$$R_4 = a_4t + b_4, R_5 = a_5t + b_5, R(S) = a_6t + b_6,$$

with $a_1^2 + ... + a_6^2 > 0$, are included in reliability hypersurface. Here, $t$ is a parameter on the line. All reasonings remain similar if we replace $a_it + b_i$ by $a_ie^{-b_it}$, $a_i$, $b_i > 0$.

First, we compute the following products:

$$\begin{aligned}
R_1R_4 =& a_1a_4t^2 + (a_1b_4 + b_1a_4)t + b_1b_4,\\
R_2R_5 =& a_2a_5t^2 + (a_2b_5 + b_2a_5)t + b_2b_5,\\
R_2R_3R_4 =& (a_2a_3a_4)t^3 + (a_2a_3b_4 + a_2a_4b_3 + a_3a_4b_2)t^2 + (a_2b_3b_4 + a_3b_2b_4\\
&+ a_4b_2b_3)t + b_2b_3b_4,\\
R_1R_2R_3R_4 =& (a_1a_2a_3a_4)t^4 + (a_1a_2a_3b_4 + a_1a_2a_4b_3 + a_1a_3a_4b_2 + a_2a_3a_4b_1)t^3\\
&+ (a_1a_2b_3b_4 + a_1a_3b_2b_4 + a_1a_4b_2b_3 + a_2a_3b_1b_4 + a_2a_4b_1b_3\\
&+ a_3a_4b_1b_2)t^2 + (a_1b_2b_3b_4 + a_2b_1b_3b_4 + a_3b_1b_2b_4 + a_4b_1b_2b_3)t + b_1b_2b_3b_4,\\
R_1R_2R_4R_5 =& (a_1a_2a_4a_5)t^4 + (a_1a_2a_4b_5 + a_1a_2a_5b_4 + a_1a_4a_5b_2 + a_2a_4a_5b_1)t^3\\
&+ (a_1a_2b_4b_5 + a_1a_4b_2b_5 + a_1a_5b_2b_4 + a_2a_4b_1b_5 + a_2a_5b_1b_4\\
&+ a_4a_5b_1b_2)t^2 + (a_1b_2b_4b_5 + a_2b_1b_4b_5 + a_4b_1b_2b_5 + a_5b_1b_2b_4)t + b_1b_2b_4b_5,\\
R_2R_3R_4R_5 =& (a_2a_3a_4a_5)t^4 + (a_2a_3a_4b_5 + a_2a_3a_5b_4 + a_2a_4a_5b_3 + a_3a_4a_5b_2)t^3\\
&+ (a_2a_3b_4b_5 + a_2a_4b_3b_5 + a_2a_5b_3b_4 + a_3a_4b_2b_5 + a_3a_5b_2b_4 + a_4a_5b_2b_3)t^2\\
&+ (a_2b_3b_4b_5 + a_3b_2b_4b_5 + a_4b_2b_3b_5 + a_5b_2b_3b_4)t + b_2b_3b_4b_5,\\
R_1R_2R_3R_4R_5 =& (a_1a_2a_3a_4a_5)t^5 + (a_1a_2a_3a_4b_5 + a_1a_2a_3a_5b_4 + a_1a_2a_4a_5b_3\\
&+ a_1a_3a_4a_5b_2 + a_2a_3a_4a_5b_1)t^4 + (a_1a_2a_3b_4b_5 + a_1a_2a_4b_3b_5\\
&+ a_1a_2a_5b_3b_4 + a_1a_3a_4b_2b_5 + a_1a_3a_5b_2b_4 + a_1a_4a_5b_2b_3\\
&+ a_2a_3a_4b_1b_5 + a_2a_3a_5b_1b_4 + a_2a_4a_5b_1b_3 + a_3a_4a_5b_1b_2)t^3\\
&+ (a_1a_2b_3b_4b_5 + a_1a_3b_2b_4b_5 + a_1a_4b_2b_3b_5 + a_1a_5b_2b_3b_4\\
&+ a_2a_3b_1b_4b_5 + a_2a_4b_1b_3b_5 + a_2a_5b_1b_3b_4 + a_3a_4b_1b_2b_5\\
&+ a_3a_5b_1b_2b_4 + a_4a_5b_1b_2b_3)t^2 + (a_1b_2b_3b_4b_5 + a_2b_1b_3b_4b_5\\
&+ a_3b_1b_2b_4b_5 + a_4b_1b_2b_3b_5 + a_5b_1b_2b_3b_4)t + b_1b_2b_3b_4b_5.
\end{aligned}$$

By replacement, ordering by powers of $t$ and identifying, we obtain a system whose solutions describe the number of straight lines situated on the reliability hypersurface. We write the system ordering by the coefficients of the powers of degree from zero to five, relative to $t$:

$$\begin{aligned}
b_6 =& b_1b_4 + b_2b_5 + b_2b_3b_4 - b_1b_2b_3b_4 - b_1b_2b_4b_5 - b_2b_3b_4b_5 + b_1b_2b_3b_4b_5,\\
a_6 =& a_1b_4 + a_4b_1 + a_2b_5 + a_5b_2 + a_2b_3b_4 + a_3b_2b_4 + a_4b_2b_3 - a_1b_2b_3b_4\\
&- a_2b_1b_3b_4 - a_3b_1b_2b_4 - a_4b_1b_2b_3 - a_1b_2b_4b_5 - a_2b_1b_4b_5 - a_4b_1b_2b_5\\
&- a_5b_1b_2b_4 - a_2b_3b_4b_5 - a_3b_2b_4b_5 - a_4b_2b_3b_5 - a_5b_2b_3b_4 + a_1b_2b_3b_4b_5\\
&+ a_2b_1b_3b_4b_5 + a_3b_1b_2b_4b_5 + a_4b_1b_2b_3b_5 + a_5b_1b_2b_3b_4;\\
0 =& a_1a_4 + a_2a_5 + a_2a_3b_4 + a_2a_4b_3 + a_3a_4b_2 - a_1a_2b_3b_4 - a_1a_3b_2b_4 - a_1a_4b_2b_3\\
&- a_2a_3b_1b_4 - a_2a_4b_1b_3 - a_3a_4b_1b_2 - a_1a_2b_4b_5 - a_1a_4b_2b_5 - a_1a_5b_2b_4\\
&- a_2a_4b_1b_5 - a_2a_5b_1b_4 - a_4a_5b_1b_2 - a_2a_3b_4b_5 - a_2a_4b_3b_5 - a_2a_5b_3b_4
\end{aligned}$$

$$
\begin{aligned}
& - a_3a_4b_2b_5 - a_3a_5b_2b_4 - a_4a_5b_2b_3 + a_1a_2b_3b_4b_5 + a_1a_3b_2b_4b_5 + a_1a_4b_2b_3b_5 \\
& + a_1a_5b_2b_3b_4 + a_2a_3b_1b_4b_5 + a_2a_4b_1b_3b_5 + a_2a_5b_1b_3b_4 + a_3a_4b_1b_2b_5 \\
& + a_3a_5b_1b_2b_4 + a_4a_5b_1b_2b_3; \\
0 =& a_2a_3a_4 - a_1a_2a_3b_4 - a_1a_2a_4b_3 - a_1a_3a_4b_2 - a_2a_3a_4b_1 - a_1a_2a_4b_5 - a_1a_2a_5b_4 \\
& - a_1a_4a_5b_2 - a_2a_4a_5b_1 - a_2a_3a_4b_5 - a_2a_3a_5b_4 - a_2a_4a_5b_3 - a_3a_4a_5b_2 \\
& + a_1a_2a_3b_4b_5 + a_1a_2a_4b_3b_5 + a_1a_2a_5b_3b_4 + a_1a_3a_4b_2b_5 + a_1a_3a_5b_2b_4 \\
& + a_1a_4a_5b_2b_3 + a_2a_3a_4b_1b_5 + a_2a_3a_5b_1b_4 + a_2a_4a_5b_1b_3 + a_3a_4a_5b_1b_2; \\
0 =& a_1a_2a_3a_4b_5 - a_1a_2a_4a_5 - a_2a_3a_4a_5 - a_1a_2a_3a_4 + a_1a_2a_3a_5b_4 + a_1a_2a_4a_5b_3 \\
& + a_1a_3a_4a_5b_2 + a_2a_3a_4a_5b_1; \qquad 0 = a_1a_2a_3a_4a_5.
\end{aligned}
$$

Starting from the last equation, at least one of the numbers $a_i, i = 1, ..., 5$ must be zero (number of cases: $C_5^1 + C_5^2 + C_5^3 + C_5^4 = 30$). So the straight-lines are parallel to some hyperplane of coordinates. The first equation shows that at $t = 0$, the point $(b_1, ..., b_6)$ is on the "reliability hypersurface". This remark requires the following procedure: we choose arbitrarily $b_1, ..., b_5$, and compute $b_6$. We replace the values $b_1, ..., b_5$ in the remaining equations. If the new system, in unknown $(a_1, ..., a_6)$, has a solution with at least non-zero component, then there exists one straight line passing through the point $(b_1, ..., b_6)$ and lying on the "reliability hypersurface". Explicitly, after solving the algebraic system, we have the following cases:

**Case 1** ($a_1 = 0$):

$$
\begin{aligned}
b_6 =& b_1b_4 + b_2b_5 + b_2b_3b_4 - b_1b_2b_3b_4 - b_1b_2b_4b_5 - b_2b_3b_4b_5 + b_1b_2b_3b_4b_5, \\
a_6 =& a_4b_1 + a_2b_5 + a_5b_2 + a_2b_3b_4 + a_3b_2b_4 + a_4b_2b_3 - a_2b_1b_3b_4 - a_3b_1b_2b_4 \\
& - a_4b_1b_2b_3 - a_2b_1b_4b_5 - a_4b_1b_2b_5 - a_5b_1b_2b_4 - a_2b_3b_4b_5 - a_3b_2b_4b_5 \\
& - a_4b_2b_3b_5 - a_5b_2b_3b_4 + a_2b_1b_3b_4b_5 + a_3b_1b_2b_4b_5 + a_4b_1b_2b_3b_5 + a_5b_1b_2b_3b_4, \\
0 =& a_2a_5 + a_2a_3b_4 + a_2a_4b_3 + a_3a_4b_2 - a_2a_3b_1b_4 - a_2a_4b_1b_3 - a_3a_4b_1b_2 \\
& - a_2a_4b_1b_5 - a_2a_5b_1b_4 - a_4a_5b_1b_2 - a_2a_3b_4b_5 - a_2a_4b_3b_5 - a_2a_5b_3b_4 \\
& - a_3a_4b_2b_5 - a_3a_5b_2b_4 - a_4a_5b_2b_3 + a_2a_3b_1b_4b_5 + a_2a_4b_1b_3b_5 + a_2a_5b_1b_3b_4 \\
& + a_3a_4b_1b_2b_5 + a_3a_5b_1b_2b_4 + a_4a_5b_1b_2b_3, \\
0 =& - a_2a_3a_4 - a_2a_3a_4b_1 - a_2a_4a_5b_1 - a_2a_3a_4b_5 - a_2a_3a_5b_4 - a_2a_4a_5b_3 - a_3a_4a_5b_2 \\
& + a_2a_3a_4b_1b_5 + a_2a_3a_5b_1b_4 + a_2a_4a_5b_1b_3 + a_3a_4a_5b_1b_2, \qquad 0 = a_2a_3a_4a_5(b_1 - 1).
\end{aligned}
$$

i) ($a_1 = 0$ and $b_1 = 1$):

$$
\begin{aligned}
& b_6 = b_4 + b_2b_5 - b_2b_4b_5, \qquad a_6 = a_4 + a_2b_5 + a_5b_2 - a_2b_4b_5 - a_4b_2b_5 - a_5b_2b_4, \\
& 0 = a_2a_5 - a_2a_4b_5 - a_2a_5b_4 - a_4a_5b_2, \qquad 0 = a_2a_4a_5.
\end{aligned}
$$

In this case, for an arbitrary point $(b_1 = 1, b_2, b_3, b_4, b_5, b_6)$, the solution $(a_1 = 0, a_2, a_3, a_4, a_5, a_6)$ depends on six parameters (a family of straight lines). All the foregoing straight lines are in the plane $R_1 = 1$. In this case the "reliability hypersurface" is a fiber bundle (ruled hypersurface). ii) ($a_1 = a_2 = 0$):

$$
\begin{aligned}
b_6 =& b_1b_4 + b_2b_5 + b_2b_3b_4 - b_1b_2b_3b_4 - b_1b_2b_4b_5 - b_2b_3b_4b_5 + b_1b_2b_3b_4b_5, \\
a_6 =& a_4b_1 + a_5b_2 + a_3b_2b_4 + a_4b_2b_3 - a_3b_1b_2b_4 - a_4b_1b_2b_3 - a_4b_1b_2b_5 - a_5b_1b_2b_4 \\
& - a_3b_2b_4b_5 - a_4b_2b_3b_5 - a_5b_2b_3b_4 + a_3b_1b_2b_4b_5 + a_4b_1b_2b_3b_5 + a_5b_1b_2b_3b_4, \\
0 =& a_3a_4b_2 - a_3a_4b_1b_2 - a_4a_5b_1b_2 - a_3a_4b_2b_5 - a_3a_5b_2b_4 - a_4a_5b_2b_3 \\
& + a_3a_4b_1b_2b_5 + a_3a_5b_1b_2b_4 + a_4a_5b_1b_2b_3, \qquad 0 = a_3a_4a_5b_2(b_1 - 1).
\end{aligned}
$$

iii) ($a_1 = a_3 = 0$):

$$
\begin{aligned}
b_6 =& b_1b_4 + b_2b_5 + b_2b_3b_4 - b_1b_2b_3b_4 - b_1b_2b_4b_5 - b_2b_3b_4b_5 + b_1b_2b_3b_4b_5, \\
a_6 =& a_4b_1 + a_2b_5 + a_5b_2 + a_2b_3b_4 + a_4b_2b_3 - a_2b_1b_3b_4 - a_4b_1b_2b_3
\end{aligned}
$$

$$- a_2b_1b_4b_5 - a_4b_1b_2b_5 - a_5b_1b_2b_4 - a_2b_3b_4b_5 - a_4b_2b_3b_5 - a_5b_2b_3b_4$$
$$+ a_2b_1b_3b_4b_5 + a_4b_1b_2b_3b_5 + a_5b_1b_2b_3b_4,$$
$$0 =a_2a_5 + a_2a_4b_3 - a_2a_4b_1b_3 - a_2a_4b_1b_5 - a_2a_5b_1b_4 - a_4a_5b_1b_2 - a_2a_4b_3b_5$$
$$- a_2a_5b_3b_4 - a_4a_5b_2b_3 + a_2a_4b_1b_3b_5 + a_2a_5b_1b_3b_4 + a_4a_5b_1b_2b_3,$$
$$0 =a_2a_4a_5(b_1b_3 - b_3 - b_1).$$

iv) $(a_1a_4 = 0)$:

$$b_6 =b_1b_4 + b_2b_5 + b_2b_3b_4 - b_1b_2b_3b_4 - b_1b_2b_4b_5 - b_2b_3b_4b_5 + b_1b_2b_3b_4b_5,$$
$$a_6 =a_2b_5 + a_5b_2 + a_2b_3b_4 + a_3b_2b_4 - a_2b_1b_3b_4 - a_3b_1b_2b_4 - a_2b_1b_4b_5 - a_5b_1b_2b_4$$
$$- a_2b_3b_4b_5 - a_3b_2b_4b_5 - a_5b_2b_3b_4 + a_2b_1b_3b_4b_5 + a_3b_1b_2b_4b_5 + a_5b_1b_2b_3b_4,$$
$$0 =a_2a_5 + a_2a_3b_4 - a_2a_3b_1b_4 - a_2a_5b_1b_4 - a_2a_3b_4b_5 - a_2a_5b_3b_4 - a_3a_5b_2b_4$$
$$+ a_2a_3b_1b_4b_5 + a_2a_5b_1b_3b_4 + a_3a_5b_1b_2b_4, \qquad 0 = a_2a_3a_5b_4(b_1 - 1).$$

v) $(a_1a_5 = 0)$:

$$b_6 =b_1b_4 + b_2b_5 + b_2b_3b_4 - b_1b_2b_3b_4 - b_1b_2b_4b_5 - b_2b_3b_4b_5 + b_1b_2b_3b_4b_5,$$
$$a_6 =a_4b_1 + a_2b_5 + a_2b_3b_4 + a_3b_2b_4 + a_4b_2b_3 - a_2b_1b_3b_4 - a_3b_1b_2b_4$$
$$- a_4b_1b_2b_3 - a_2b_1b_4b_5 - a_4b_1b_2b_5 - a_2b_3b_4b_5 - a_3b_2b_4b_5 - a_4b_2b_3b_5$$
$$+ a_2b_1b_3b_4b_5 + a_3b_1b_2b_4b_5 + a_4b_1b_2b_3b_5,$$
$$0 =a_2a_3b_4 + a_2a_4b_3 + a_3a_4b_2 - a_2a_3b_1b_4 - a_2a_4b_1b_3 - a_3a_4b_1b_2 - a_2a_4b_1b_5$$
$$- a_2a_3b_4b_5 - a_2a_4b_3b_5 - a_3a_4b_2b_5 + a_2a_3b_1b_4b_5 + a_2a_4b_1b_3b_5 + a_3a_4b_1b_2b_5,$$
$$0 =a_2a_3a_4(1 - b_1 - b_5 + b_1b_5).$$

In case v), for an arbitrary point $(b_1, b_2, b_3, b_4, b_5, b_6)$, the solution $(a_1 = 0, a_2, a_3, a_4, a_5 = 0, a_6)$ depends on six parameters (a family of straight lines). In this case the "reliability hypersurface" is a fiber bundle (ruled hypersurface). The situations ii)-iv) are similar.

**Case 2.** $(a_1 = 0, a_2 = 0)$:

$$b_6 =b_1b_4 + b_2b_5 + b_2b_3b_4 - b_1b_2b_3b_4 - b_1b_2b_4b_5 - b_2b_3b_4b_5 + b_1b_2b_3b_4b_5,$$
$$a_6 =a_4b_1 + a_5b_2 + a_3b_2b_4 + a_4b_2b_3 - a_3b_1b_2b_4 - a_4b_1b_2b_3 - a_4b_1b_2b_5 - a_5b_1b_2b_4$$
$$- a_3b_2b_4b_5 - a_4b_2b_3b_5 - a_5b_2b_3b_4 + a_3b_1b_2b_4b_5 + a_4b_1b_2b_3b_5 + a_5b_1b_2b_3b_4,$$
$$0 =a_3a_4b_2 - a_3a_4b_1b_2 - a_4a_5b_1b_2 - a_3a_4b_2b_5 - a_3a_5b_2b_4 - a_4a_5b_2b_3$$
$$+ a_3a_4b_1b_2b_5 + a_3a_5b_1b_2b_4 + a_4a_5b_1b_2b_3, \qquad 0 = a_3a_4a_5b_2(b_1 - 1).$$

i) $(a_1 = 0, a_2 = 0$ and $b_1 = 1)$:

$$b_6 =b_4 + b_2b_5 - b_2b_4b_5, \qquad a_6 =a_4 + a_5b_2 - a_4b_2b_5 - a_5b_2b_4,$$
$$0 =a_2a_5 - a_2a_4b_5 - a_2a_5b_4 - a_4a_5b_2, \qquad 0 =a_4a_5b_2.$$

ii) $(a_1 = 0, a_2 = 0, a_3 = 0)$:

$$b_6 =b_1b_4 + b_2b_5 + b_2b_3b_4 - b_1b_2b_3b_4 - b_1b_2b_4b_5 - b_2b_3b_4b_5 + b_1b_2b_3b_4b_5,$$
$$a_6 =a_4 + a_5b_2 - a_4b_2b_5 - a_5b_2b_4,$$
$$0 =a_4b_1 + a_5b_2 + a_4b_2b_3 - a_4b_1b_2b_3 - a_4b_1b_2b_5 - a_5b_1b_2b_4 - a_4b_2b_3b_5 - a_5b_2b_3b_4$$
$$+ a_4b_1b_2b_3b_5 + a_5b_1b_2b_3b_4, \qquad 0 = a_4a_5(b_1b_2b_3 - b_2b_3 - b_1b_2).$$

iii) $(a_1 = 0, a_2 = 0, a_4 = 0)$:

$$b_6 =b_1b_4 + b_2b_5 + b_2b_3b_4 - b_1b_2b_3b_4 - b_1b_2b_4b_5 - b_2b_3b_4b_5 + b_1b_2b_3b_4b_5,$$
$$a_6 =a_5b_2 + a_3b_2b_4 - a_3b_1b_2b_4 - a_5b_1b_2b_4 - a_3b_2b_4b_5 - a_5b_2b_3b_4$$
$$+ a_3b_1b_2b_4b_5 + a_5b_1b_2b_3b_4, \qquad 0 = a_3a_5b_2b_4(b_1 - 1).$$

**Case 3.** $(a_1 = 0, a_2 = 0, a_3 = 0)$:

$$\begin{aligned} b_6 =& b_1b_4 + b_2b_5 + b_2b_3b_4 - b_1b_2b_3b_4 - b_1b_2b_4b_5 - b_2b_3b_4b_5 + b_1b_2b_3b_4b_5, \\ a_6 =& a_4b_1 + a_5b_2 + a_4b_2b_3 - a_4b_1b_2b_3 - a_4b_1b_2b_5 - a_5b_1b_2b_4 - a_4b_2b_3b_5 \\ & - a_5b_2b_3b_4 + a_4b_1b_2b_3b_5 + a_5b_1b_2b_3b_4, \qquad 0 = a_4a_5(b_1b_2b_3 - b_2b_3 - b_1b_2). \end{aligned}$$

i) $(a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0)$:

$$\begin{aligned} b_6 =& b_1b_4 + b_2b_5 + b_2b_3b_4 - b_1b_2b_3b_4 - b_1b_2b_4b_5 - b_2b_3b_4b_5 + b_1b_2b_3b_4b_5, \\ a_6 =& a_5b_2(1 - b_1b_4 - b_3b_4 + b_1b_3b_4). \end{aligned}$$

ii) $(a_1 = 0, a_2 = 0, a_3 = 0, a_5 = 0)$:

$$\begin{aligned} b_6 =& b_1b_4 + b_2b_5 + b_2b_3b_4 - b_1b_2b_3b_4 - b_1b_2b_4b_5 - b_2b_3b_4b_5 + b_1b_2b_3b_4b_5, \\ a_6 =& a_4b_1 + a_4b_2b_3 - a_4b_1b_2b_3 - a_4b_1b_2b_5 - a_4b_2b_3b_5 + a_4b_1b_2b_3b_5. \end{aligned}$$

**Case 4.** $(a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0)$:

$$\begin{aligned} b_6 =& b_1b_4 + b_2b_5 + b_2b_3b_4 - b_1b_2b_3b_4 - b_1b_2b_4b_5 - b_2b_3b_4b_5 + b_1b_2b_3b_4b_5, \\ a_6 =& a_5b_2(1 - b_1b_4 - b_3b_4 + b_1b_3b_4). \end{aligned}$$

The rest of cases are similar. For each case, using the Jacobian matrix and its rank, we count the number of essential parameters. □

### 4.4. Returning to the probability framework

In order to return to the probability ansatz, we must assume that the coefficients $a_i, b_i, i = 1, ..., 6$, satisfy the conditions imposed by the assumption that each function $a_it+b_i$, $i = 1, ..., 6$, is a probability, i.e., $0 \leq a_it + b_i \leq 1, i = 1, ..., 6$. If $a_k = 0$, then $0 \leq b_k \leq 1$. We further assume that all $a_i$ are different from zero. (i) If $a_i > 0$, then we find the intervals $I_i : -\frac{b_i}{a_i} \leq t \leq \frac{1-b_i}{a_i}, i = 1, ..., 6$. (ii) If there exists $a_k < 0$, then a non-void interval is $I_k : \frac{1-b_k}{a_k} \leq t \leq -\frac{b_k}{a_k}$.

Suppose we have a non-void intersection $I = \cap I_i$. Consequently, the significant parts from probabilistic point of view are segments of straight lines included in the interval $[0, 1]^6$.

**Theorem 4.4.** *Let us consider the vector of probabilities $(R_1(t), R_2(t), R_3(t), R_4(t), R_5(t))$. The most plausible situation is that which imposes a maximum number of parameters in the family of straight lines on the "reliability hypersurface".*

*Proof.* In this case we have maximum degrees of freedom (number of parameters). □

**Remark 4.5.** We can reiterate the process, by replacing this time the affine framework with an exponential or a logarithmic one.

### 4.5. Equi-reliable hypersurfaces

We further consider in $\mathbb{R}^5$ the constant level algebraic hypersurfaces of the multivariate polynomial (3.2) (the *"equi-reliable hypersurfaces"*):

$$\begin{aligned} c =& R_1R_4 + R_2R_5 + R_2R_3R_4 - R_1R_2R_3R_4 \\ & - R_1R_2R_4R_5 - R_2R_3R_4R_5 + R_1R_2R_3R_4R_5. \end{aligned} \tag{4.2}$$

**Open problem.** How many straight lines are included in each "equi-reliable hypersurface"? As an example, the constant level zero hypersurface contains the linear varieties $OR_3R_4R_5 : R_1 = 0, R_2 = 0$; $OR_1R_3R_5 : R_2 = 0, R_4 = 0$; $OR_2R_3 : R_1 = 0, R_4 = 0, R_5 = 0$. Indeed, we have

$$R_{Mc} = R_1(R_4 - R_2R_3R_4 - R_2R_4R_5 + R_2R_3R_4R_5) + R_2(R_5 + R_3R_4 - R_3R_4R_5).$$

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