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CONVERGENCE THEOREMS FOR FAMILIES OF *Q*-HOMEOMORPHISMS ON RIEMANN AND KLEIN SURFACES

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Acest articol de sinteză prezintă rezultatele noastre recente care generalizează criterii de normalitate şi compacitate pentru familii de aplicaţii K - cvasiconforme (K – QC) şi K – cvasiregulate (K – QR), pe suprafeţe riemanienne sau Klein la cazurile în care: a) deformarea este majorată de o funcţie din clasa BMO_{loc} sau FMO_{loc}, b) aplicaţiile sunt ring FMO_{loc} - homeomorfisme (sau - QR aplicaţii), c) aplicaţiile sunt FLD – homeomorfisme.

This survey paper presents our recent results which generalize the normality and compactness properties of families of K – quasiconformal (K – QC) and K – quasiregular (K - QR) mappings on Riemann or Klein surfaces to the cases: a) of mappings whose distortion is dominated by a BMO_{loc} - or by a FMO_{loc} – function, b) when instead of QC (or QR) mappings we work with ring FMO_{loc} - homeomorphisms (or with QR mappings), c) when the mappings are of finite length distortion(FLD).

Keywords: quasiconformal (QC), quasiregular (QR), BMO_{loc} – QC, BMO_{loc} – QR, FMO_{loc} , FLD, ring $Q(p)$ – homeomorphism, Riemann surface, Klein surface.

1. Introduction

Conformal mappings and analytic functions of one complex variable were first treated in the work of H. Grötzsch, M. A. Lavrentiev, L. V. Ahlfors, O. Teichmüller and others and then generalised to the spaces $\mathbf{R^n}$, $n \geq 2$ in the theory of quasiconformal (QC) and quasiregular (QR) mappings in the monographs by O. Lehto and K. I. Virtanen [1], L. V. Ahlfors, C.Andreian Cazacu [2] for $n = 2$, J. Väisälä [3], P. Caraman, Yu. G. Reshetnyak, M. Vuorinen and others for $n \geq 3$. Later on, the dilatation of the QC mappings was dominated by a given measurable function Q, which can be BMO (bounded mean oscillation), FMO (finite mean oscillation), L^1_{loc}, etc. A complete list of the works with this subject can be found in [4] and [5]. My aim is to present in a general frame the results obtained both in joint with C. Andreian Cazacu [6, 7, 8, 9] and alone [10, 11, 12, 13, 14, 15, 16]

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about convergence in the 2 – dimensional case. In the following, *D* and *D'* will designate domains in **C** and $g : D \to D'$ a mapping.

$$K(z,g) = \frac{1+|\mu(z)|}{1-|\mu(z)|}, \tag{1}$$

is the (*maximal*) *dilatation* of function g and $\mu : D \to \mathbf{C}$ the *complex dilatation*, i.e. a measurable function with $|\mu(z)| \le 1$ a.e. solution of the Beltrami equation

$$\frac{\partial g}{\partial \bar{z}} = \mu(z)\frac{\partial g}{\partial z}. \tag{2}$$

The classical geometric definition of *K* – QC mappings due to H. Grötzsch for *n* = 2 is

$$M(\Gamma)/K \le M(g\Gamma) \le KM(\Gamma), \tag{3}$$

for every path family Γ in *D*, where the (conformal) modulus of Γ is

$$M(\Gamma) = \inf_{\rho \in adm\Gamma} \iint_C \rho^2(z)dm(z), \tag{4}$$

with *dm*(*z*) the Lebesgue measure in **C** and $\rho : \mathbf{C} \to [\,0, \infty\,]$ a Borel function called *admissible* for Γ ($\rho \in$ adm Γ) with the property

$$\int_\gamma \rho ds \ge 1 \tag{5}$$

for each $\gamma \in \Gamma$. Here $K(z,g) \le K$ a.e.

By Theorem 1.1, p. 24 in [1], or Theorem 34.3 in [3], a homeomorphism *g* is QC in the geometric sense if and only if

$$M(g\Gamma) \le KM(\Gamma), \tag{6}$$

for some $K \in [1, \infty)$ and for every path family Γ in *D*, that means that it is sufficient that

$$\sup \frac{M(g\Gamma)}{M(\Gamma)} < \infty, \tag{7}$$

where the supremum is taken over all path families Γ in *D* for which *M* (Γ) and *M* (*g*Γ) are not simultaneously 0 or ∞.

Taking into account the relation between (3) and (6), O. Martio gives a natural extension of the definition of QC mappings (see [4], chap. 4, p. 81, [1], p. 221, or [5], p. 551).

Let *Q*: $D \to [\,1, \infty\,]$ be a measurable function. We say that a homeomorphism

$$f: D \to \hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$$

is a *Q* (*z*) – *homeomorphism*, if

$$M(f\ \Gamma) \le \iint_D Q(z)\rho^2(z)dm(z) \tag{8}$$

for Γ family of paths in D and $\rho \in$ adm Γ. This concept is related in a natural way to the theory of moduli with weights, see [17, 18]. We deal with some subclasses of Q (z) – homeomorphisms, which are introduced in [19 - 26], $K(z,g) \le Q(z)$, Q in $\mathrm{BMO_{loc}}$ or $\mathrm{FMO_{loc}}$, and apply some of convergence results obtained there in the plane to Riemann and Klein surfaces. For the definitions see [4], chap. 5, 7, 8, 12.

2. Convergence theorems

Starting with the classical theorems on normality and compactness of meromorphic functions, which have been extended to QC or QR mappings ([1], II, 5, p. 71), many authors generalized this topic on other classes of mappings, e. g. G. David, [27], P. Tukia, [28], V. Ryazanov, [20], V. Ryazanov, U. Srebro and E. Yakubov [21, 22, 23, 24, 25, 26].

Let Y and Y' be Riemann or Klein homeomorphic surfaces, ($\hat{Y}$, Π, Y) and ($\hat{Y}'$, Π', Y') their universal coverings, where Π: $\hat{Y} \to Y$ and Π': $\hat{Y}' \to Y'$. Here $\hat{Y}$ and $\hat{Y}'$ are either **C**, Δ – the unit disc or $\hat{\mathbf{C}}$ with the corresponding metric: Euclidean, hyperbolic or spherical. The metrics on $\hat{Y}$ and $\hat{Y}'$ induce by Π and Π' the metrics of Y and Y'. The convergence is always taken with respect to these metrics. Let $Q : Y \to \mathbf{R}$, be a function defined on Y such that the composition $Q_1 = Q \circ \Pi$: $\hat{Y} \to \mathbf{R}$, be a $\mathrm{BMO_{loc}}$ or $\mathrm{FMO_{loc}}$ function. Then, we say that a mapping $g : Y \to Y'$ is a Q (p) – QC (or QR) mapping if the lifting $\hat{g} : \hat{Y} \to \hat{Y}'$ is a $Q_1(z)$ – QC (or QR) mapping (cf. [29], p. 9). In this paper, homeomorphism (embedding) means homeomorphism onto (into, respectively) and QC – mapping is a general name for both cases.

$$K(p,g) \le Q(p) \tag{9}$$

holds if and only if $K(z, \hat{g}) \le Q_1(z)$.

Let be $p_0 \in Y$ and $p_0' \in Y'$ two arbitrary but fixed points of Y and Y' respectively. We consider the family G of mappings $g : Y \to Y'$ normalized by the condition:

$$g(p_0) = p'_0 . \tag{10}$$

Theorem 1. (*i*) *If Y' is non conformally equivalent to either* **C** *or* $\hat{\mathbf{C}}$, *then G is normal if*:

1. $Q \in \mathrm{BMO_{loc}}(Y)$ *and g from G is a Q* (p) – QC *mapping* ;
2. $Q \in \mathrm{FMO_{loc}}(Y)$ *and g from G is a ring Q* (p) *– homeomorphism*;
3. $g \in G$ *is a* FLD *– homeomorphism*, $Q \in \mathrm{FMO}(Y)$, *with* (9).

(*ii*) *if Y is non conformally equivalent to* **C** *or* Δ, *and* $\{ g_n \}$ *is a sequence in G*, *which l. uniformly converges to* g_0, *then* g_0 *is a*:

1. *$Q(p)$ – QC embedding, if* $g_n \in G$ *is a* $Q\ (p)$ – QC *mapping* $Q \in$ BMO$_{\text{loc}}$(Y);
2. *ring* $Q(p)$ – *embedding, if* $g_n \in G$ *is a ring* $Q\ (p)$ *– homeomorphism* $Q \in$ FMO$_{\text{loc}}$(Y).

(*iii*) *Let* $\{ g_n \}$ *be a sequence of* FLD *– homeomorphisms of D into* **C** *with* a. e. $K(z, g_n) \le Q(z) \in L^1_{loc}(D)$, *converging l. uniformly to a mapping* $g_0 : D \to$ **C**. *Then* g_0 *is either an* FLD *– embedding or* $g_0 \equiv$ const. *in* D.

(*iv*) *G is closed in the following cases*:

1. *if Y' is compact,* $g \in G$ *is a* $Q\ (p)$ – QC *mapping* $Q \in$ BMO$_{\text{loc}}$(Y);
2. *if Y' is non conformally equivalent to* **C**, $g \in G$ *is a* $Q\ (p)$ – QC *mapping* $Q \in$ BMO(Y) (*or ring* $Q\ (p)$ *– homeomorphism* $Q \in$ FMO(Y)) .
3. *if* $Y \ne$ **C** *and there exists a function* $C \in$ BMO$_{\text{loc}}$(Y') (FMO(Y') *respectively*) *such that* g^{-1} *is a* $C\ (p')$ – QC *mapping* (*ring* $C\ (p')$ *– homeomorphism*) *for every* $g \in G$.

These results have been extended [12, 14] to the families *G'* and *G"* of BMO$_{\text{loc}}$(Y) $Q\ (p)$ – QC mapping (FMO$_{\text{loc}}$(Y) ring $Q\ (p)$ – homeomorphism) $g : Y \to Y'$ which map a given compact subset $M \subset Y$ into, respectively onto, another given compact subset $M' \subset Y'$.

Remark 1. The follwing example shows that the conclusion of Theorem 1, does not hold in the case excluded by its hypotheses.

Example 1. Take $D = \left\{ z \in C / |z| < \frac{1}{n} e^{-\frac{\alpha}{4}} \right\}$, $n > 2$, $f_n(z) = \frac{1}{n} \exp\left(-\sqrt{\frac{\alpha}{2} \log \frac{1}{|z|}} \right) \frac{z}{|z|}$. Using a formula from [1], p. 220 for the complex dilatation, we deduce that $K(z, f_n) = \sqrt{\frac{8}{\alpha} \log \frac{1}{|z|}}$.

Choosing $Q(z) = \sqrt{\frac{8}{\alpha}} \log \frac{1}{|z|} \in$ BMO (**C**) [29], Corollary 2 to Lemma 3, we can infer that f_n is a $Q\ (z)$ – QC embedding. For $g_n(z) = \begin{cases} f_n(z), z \in D \\ \frac{1}{n} z|z|, z \in C \setminus D \end{cases}$, we get a sequence $\{g_n\}$ of $Q_1(z)$ – QC mappings, where $Q_1(z)$ = max$\{Q(z),2\} \in$ BMO (**C**) by [29], p.2. This shows that $\{g_n\}$ converges l. u. in **C** to a constant $g_0 \equiv 0$, hence G is not closed.

Example 2. $Y' = \Delta$, Y = C or Δ. Let $\{r_n\}$ be an increasing sequence of numbers $0 < r_n < n$ tending to infinity if Y = C or $0 < r_n < 1$ tending to 1 if $Y = \Delta$. Define $g_n : Y \to \Delta$ by

$$g_n = \begin{cases} \dfrac{z}{n}, |z| < r_n \\ \dfrac{G_n(|z|)z}{|z|}, |z| > r_n \end{cases}$$

with a) $G_n(|z|) = \dfrac{|z|}{|z| + n - r_n}$ if Y = C and b) $G_n(|z|) = \dfrac{r_n}{n} + \left[\dfrac{(n - r_n)(|z| - r_n)}{n(1 - r_n)}\right]$ if Y = Δ.

The function $Q(z) = 1$ for $|z| \le r_1$ and $Q(z) = \max\{K_{g_1}(z), ..., K_{g_{N-1}}(z)\}$ for $r_{N-1} < |z| \le r_N, N \ge 2$, is BMO$_{\text{loc}}$(Y), since it is l. bounded. Thus g_n is a $Q(z)$ – qc homeomorphism and the sequence $\{g_n\}$ tends to 0 l. u. in Y. Indeed, in any disk $B(0, r)$, $r < r_n$, $g_n(z) = \dfrac{z}{n}$ for $n \ge N$.

By using [23], Corollary 5.6, p. 15, a normality criterion was proved for BMO$_{\text{loc}}$ – QR (or for ring FMO$_{\text{loc}}$ – QR) mappings in [13] and [15].

The BMO – QR solutions of (2) are obtained by Stoilow's factorization theorem, which says that every open and discrete mapping g: $D \to \hat{\mathbf{C}}$ admits a representation

$$g = h \circ \varphi \tag{11}$$

where φ is an embedding and h a meromorphic function in $\varphi(D)$ ([30], V, 5, p. 120). The conclusion of Theorem 1 does not hold for the class of $Q(p)$ – QR mappings, as follows from

Example 3. Consider the sequence $g_n(z) = n\left(\dfrac{1}{\log|z|} e^{i \arg z} + \dfrac{1}{\alpha}\right) - 1, \alpha > 2$ of BMO – QR mappings defined on the set $D = \{z \in C / |z| < e^{-1}, -(\pi/3) < \arg z < (2\pi/3)\}$. The preimage of -1 is $e^{-\alpha}$ and the limit mapping is $g_0(z) = \begin{cases} -1, z = e^{-\alpha} \\ \infty, z \ne e^{-\alpha} \end{cases}$.

Theorem 2. *Let X and X' be two homeomorphic Riemann surfaces, X' is non conformally equivalent to either* **C** *or* $\hat{\mathbf{C}}$, $z_j \in X, \zeta_j \in X', j = 0,1, z_0 \neq z_1, \zeta_0 \neq \zeta_1$ *and* $Q \in$ BMO$_{loc}(X)$ ($Q \in$ FMO$_{loc}(X)$). *If Φ is a family of* $Q(z)$ – QC (*ring* $Q(z)$ – QR) *mappings* $f : X \to X'$ *such that* $f^{-1}(\zeta_j) = z_j, j = 0,1$, *then Φ is normal*.

Remark 2. The conclusion of Theorem 2 is not true if either (*i*) *X'* = $\hat{\mathbf{C}}$ or (*ii*) *X'* = **C**, as follows from

Example 4. (*i*) If f_n: $\hat{\mathbf{C}} \to \hat{\mathbf{C}}$, $f_n(z) = nz$ then f_n are $Q(z)$ – QR maps such that $f_n^{-1}(0) = 0$, $f_n^{-1}(\infty) = \infty$, but $f_0(z) = \begin{cases} 0, z = 0 \\ \infty, z \neq 0 \end{cases}$ hence $f_0(z)$ being discontinuous, the family of $Q(z)$ – QR maps from $\hat{\mathbf{C}}$ to $\hat{\mathbf{C}}$ which having 0 and ∞ as fixed points, is not normal.

(*ii*)Take f_n: **C** → **C**, $f_n(z) = ze^{n(1-z)}$, then $f_n^{-1}(0) = 0$, $f_n^{-1}(1) = 1$ yielding $f_0(z) = \begin{cases} 1, z = 1 \\ \infty, z \neq 1 \end{cases}$.

The proof method is to lift the problem to the universal coverings, obtain the results there, and then factorize, using [31] , Proposition 1 and 2, p. 173 .

3. Conclusions

Theorems 1 and 2 open prospects of generalizations to wider classes, e. g. to strong ring $Q(z)$ – homeomorphisms or super $Q(z)$ – homeomorphisms.

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